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Series representations for γ and other mathematical constants

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Abstract. We present series representations for some mathematical constants, like γ, π, $\log 2$, $\zeta(3)$. In particular, we prove that the following representation for Euler's constant is valid:

$$\gamma = \sum_{r=1}^{\infty} \sum_{s=1}^{r} \binom{r-1}{s-1} (-1)^{r-s} 2^s \left(\frac{1}{s} + \log \frac{s}{s+1} \right).$$

1. Introduction

Many remarkable series, product, integral, and continued fraction representations for e, π, and other mathematical constants are given in the literature. An excellent account on this subject can be found in FINCH's monograph [4]. The author describes in detail the history of numerous constants and their interrelationships. We also refer the interested reader to the recently published research paper [1] and the references therein.

It is the aim of this note to present new series representations for powers of π, $\log 2$, Euler's constant

$$\gamma = \lim_{n\to\infty} (H_n - \log n) = \int_0^1 \left(\frac{1}{\log t} + \frac{1}{1-t} \right) dt = 0.57721\ldots,$$

where H_n denotes the n-th harmonic number,

$$H_n = 1 + 1/2 + \cdots + 1/n\,,$$

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Apéry's constant

$$\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}=\frac{1}{2}\int_0^1\frac{(\log t)^2}{1-t}\,dt=1.20205\ldots,$$

and Catalan's constant

$$G=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}=-\int_0^1\frac{\log t}{1+t^2}\,dt=0.91596\ldots.$$

Our work has been inspired by an interesting paper of Amore [2], who modified a method of Flajolet and Vardi to obtain series representations for π and G. A key role in our proofs plays the following formula for the logarithmic derivative of Euler's gamma function,

$$\psi=\Gamma'/\Gamma,$$

which is due to Ramanujan:

$$\psi\Big(\frac{z}{x}\Big)-\psi(y)+\log x=\int_0^1\Big(\frac{t^{y-1}}{1-t}-\frac{xt^{z-1}}{1-t^x}\Big)\,dt\qquad(x,y,z>0). \tag{1}$$

A proof of (1) is given in [3].

2. Series representations for $\psi(z)-\psi(y)$

In this section we offer two series representations for $\psi(z)-\psi(y)$, where $y,z>0$. Let $\lambda<1/2$. From (1) we get

$$\begin{aligned}\psi(z)-\psi(y)&=\int_0^1\frac{t^{y-1}-t^{z-1}}{1-t}\,dt=\frac{1}{1-\lambda}\int_0^1\frac{t^{y-1}-t^{z-1}}{1-\frac{t-\lambda}{1-\lambda}}\,dt=\\&=\frac{1}{1-\lambda}\int_0^1(t^{y-1}-t^{z-1})\sum_{k=0}^{\infty}\Big(\frac{t-\lambda}{1-\lambda}\Big)^k\,dt=\\&=\sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\int_0^1(t^{y-1}-t^{z-1})(t-\lambda)^k\,dt=\\&=\sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\int_0^1(t^{y-1}-t^{z-1})\sum_{j=0}^{k}\binom{k}{j}t^j(-\lambda)^{k-j}\,dt=\\&=\sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\int_0^1(t^{y-1}-t^{z-1})t^j\,dt=\\&=(z-y)\sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\frac{1}{(j+y)(j+z)}.\end{aligned} \tag{2}$$

We denote by $(x)_n$ the Pochhammer symbol, which is defined by

$$(x)_0 = 1, \quad (x)_n = x(x+1)(x+2)\cdots(x+n-1) \ \ (n \in \mathbb{N}).$$

Let $0 < \mu < 2$. Using (1) leads to

$$\begin{aligned}\psi(z) - \psi(y) &= \int_0^1 \frac{(1-t)^{y-1} - (1-t)^{z-1}}{t}\,dt = \\ &= \mu \int_0^1 \frac{(1-t)^{y-1} - (1-t)^{z-1}}{1-(1-\mu t)}\,dt = \\ &= \mu \int_0^1 [(1-t)^{y-1} - (1-t)^{z-1}] \sum_{k=0}^{\infty} (1-\mu t)^k\,dt = \\ &= \mu \sum_{k=0}^{\infty} \int_0^1 [(1-t)^{y-1} - (1-t)^{z-1}](1-\mu t)^k\,dt = \\ &= \mu \sum_{k=0}^{\infty} \int_0^1 [(1-t)^{y-1} - (1-t)^{z-1}] \sum_{j=0}^{k} \binom{k}{j} (-\mu t)^j\,dt = \\ &= \mu \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mu^j \int_0^1 [(1-t)^{y-1} - (1-t)^{z-1}] t^j\,dt.\end{aligned}$$

We have

$$\int_0^1 (1-t)^{y-1} t^j\,dt = \frac{\Gamma(y)\Gamma(j+1)}{\Gamma(y+j+1)} = \frac{j!}{(y)_{j+1}}\,.$$

Thus,

$$\psi(z) - \psi(y) = \tag{3}$$
$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j \mu^{j+1} (k-j+1)_j \Big(\frac{1}{(y)_{j+1}} - \frac{1}{(z)_{j+1}} \Big).$$

In the next section we show that (2) and (3) can be applied to obtain series representations for several well-known constants.

3. Mathematical constants

I. *Euler's constant*. We set $y = 1$ in (2) and integrate both sides with respect to z from 1 to $x > 0$. Since

$$\psi(1) = -\gamma,$$

this yields

$$\log\Gamma(x) = \gamma(1-x) + \sum_{k=0}^{\infty} \frac{1}{(1-\lambda)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} \Big(\frac{x-1}{j+1} + \log\frac{j+1}{j+x} \Big).$$

The special case $x = 2$ gives

$$\gamma = \sum_{k=0}^{\infty} \frac{1}{(1-\lambda)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} \Big(\frac{1}{j+1} + \log \frac{j+1}{j+2} \Big).$$

And, if λ tends to $1/2$, then this gives

$$\gamma = \sum_{r=1}^{\infty} \sum_{s=1}^{r} \binom{r-1}{s-1} (-1)^{r-s} 2^s \Big(\frac{1}{s} + \log \frac{s}{s+1} \Big).$$

II. π *and* $\log 2$. Using

$$\psi(1/4) = -\gamma - 3\log 2 - \frac{\pi}{2},$$

we conclude from (2) (with $y = 1$, $z = 1/4$) that

$$\frac{\pi}{6} = -\log 2 + \sum_{k=0}^{\infty} \frac{1}{(1-\lambda)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} \frac{1}{(j+1)(4j+1)}.$$

And, if λ tends to $1/2$, then we have

$$\frac{\pi}{12} = -\frac{1}{2} \log 2 + \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{2^j}{(j+1)(4j+1)}. \tag{4}$$

Applying (2) (with $y = 1$, $z = 1/2$) and

$$\psi(1/2) = -\gamma - 2\log 2$$

leads to

$$2 \log 2 = \sum_{k=0}^{\infty} \frac{1}{(1-\lambda)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} \frac{1}{(j+1)(2j+1)}.$$

We let λ tend to $1/2$. This gives

$$\log 2 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{2^j}{(j+1)(2j+1)}. \tag{5}$$

Combining (4) and (5) yields

$$\pi = 6 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{2^j}{(j+1)(2j+1)(4j+1)}. \tag{6}$$

We have

$$\psi\Big(\frac{1}{3}\Big) = -\frac{1}{6}\sqrt{3}\pi - \frac{3}{2}\log 3 - \gamma$$

and

$$\psi\Big(\frac{2}{3}\Big) = \frac{1}{6}\sqrt{3}\pi - \frac{3}{2}\log 3 - \gamma.$$

Using (2) (with $y = 1/3,\ z = 2/3,\ \lambda \to 1/2$) we obtain the following counterpart of (6):

$$\pi = 6\sqrt{3}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(3j+1)(3j+2)}\,.$$

III. $\zeta(3)$, G, *and powers of* π. Let $n \geq 1$ be an integer. By differentiation with respect to z we get from (2) (with $y = 1$):

$$\psi^{(n)}(z) = (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\frac{1}{(j+z)^{n+1}}. \tag{7}$$

Since

$$\psi^{(n)}(1/2) = (-1)^{n+1}n!(2^{n+1}-1)\zeta(n+1),$$

we conclude from (7) that

$$\zeta(n+1) = \frac{1}{2^{n+1}-1}\sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\frac{1}{(j+1/2)^{n+1}}\,.$$

If λ tends to $1/2$, then this gives

$$\zeta(n+1) = \frac{2}{1-2^{-(n+1)}}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(2j+1)^{n+1}}\,. \tag{8}$$

The special cases $n = 1, 2, 3$ lead to the following:

$$\pi^2 = 16\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(2j+1)^2}\,, \tag{9}$$

$$\zeta(3) = \frac{16}{7}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(2j+1)^3}\,, \tag{10}$$

$$\pi^4 = 192\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(2j+1)^4}\,. \tag{11}$$

We have

$$\psi^{(n)}(1) = (-1)^{n+1}n!\zeta(n+1),$$

so that (7) implies

$$\zeta(n+1) = \sum_{k=0}^{\infty}\frac{1}{(1-\lambda)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\frac{1}{(j+1)^{n+1}}\,.$$

If λ tends to 1/2, then this gives

$$\zeta(n+1) = 2\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(j+1)^{n+1}}\,. \tag{12}$$

For $n = 1, 2, 3$ we find the following striking companions of (9)–(11):

$$\pi^2 = 12\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(j+1)^2}\,,$$

$$\zeta(3) = 2\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(j+1)^3}\,,$$

$$\pi^4 = 180\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^j}{(j+1)^4}\,.$$

The formula

$$\zeta(2n) = (-1)^{n+1}2^{2n-1}\frac{B_{2n}}{(2n)!}\pi^{2n},$$

where B_k denotes the k-th Bernoulli number, reveals that (8) and (12) can be used to obtain series representations for π^{2n} with $n \geq 3$.

From (3) we find by differentiation with respect to z:

$$\psi'(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^j\mu^{j+1}(k-j+1)_j\sum_{i=0}^{j}\frac{1}{z+i}\prod_{i=0}^{j}\frac{1}{z+i}\,.$$

If $z = 1$ and $\mu = 1$, then we obtain

$$\pi^2 = 6\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{H_{j+1}}{j+1}\,.$$

We have

$$\psi'(1/4) = \pi^2 + 8G \quad \text{and} \quad \psi'(3/4) = \pi^2 - 8G. \tag{13}$$

Applying (7) (with $n = 1$ and $\lambda \to 1/2$) gives

$$\pi^2 + 8G = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^{j+1}}{(j+1/4)^2}$$

and

$$\pi^2 - 8G = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{2^{j+1}}{(j+3/4)^2}\,.$$

These lead to

$$G = 16 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{2j+1}{(4j+1)^2(4j+3)^2}$$

and

$$\pi^2 = 32 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{16j^2+16j+5}{(4j+1)^2(4j+3)^2} .$$

From (7) (with $n = 2$ and $\lambda \to 1/2$) it follows that

$$\psi''\Big(\frac{1}{4}\Big) = -2 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{2^{j+1}}{(j+1/4)^3}$$

and

$$\psi''\Big(\frac{3}{4}\Big) = -2 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{2^{j+1}}{(j+3/4)^3} .$$

Using the formulas

(14) $\qquad \psi''(1/4) = -2(\pi^3 + 28\zeta(3)) \quad$ and $\quad \psi''(3/4) = 2(\pi^3 - 28\zeta(3))$

yields

$$\zeta(3) = \frac{64}{7} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{(2j+1)(16j^2+16j+7)}{(4j+1)^3(4j+3)^3}$$

and

$$\pi^3 = 128 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{48j^2+48j+13}{(4j+1)^3(4j+3)^3} .$$

Formulas (13) and (14) are given in [5], where it is also shown that similar expressions exist for $\psi^{(2n)}(1/4)$ and $\psi^{(2n)}(3/4)$. These expressions and (7) lead to series representations for $\zeta(2n+1)$ and π^{2n+1} with $n \geq 2$.

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Представление в виде рядов γ и других математических констант

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Получено представление в виде рядов некоторых математических констант, таких как π, $\log 2$, $\zeta(3)$. В частности, установлено следующее представление для постоянной Эйлера:

$$\gamma = \sum_{r=1}^{\infty} \sum_{s=1}^{r} \binom{r-1}{s-1} (-1)^{r-s} 2^s \left(\frac{1}{s} + \log \frac{s}{s+1} \right).$$