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Multivariate Hausdorff operators on the spaces $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$

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Dedicated to Professor S. M. Nikol'skiĭ on his hundredth birthday with admiration and deep respect

Abstract. A multivariate Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu, c, A)$ is defined in terms of a σ-finite Borel measure μ on $\mathbb{R}^n$, a Borel measurable function c on $\mathbb{R}^n$, and an $n \times n$ matrix A whose entries are Borel measurable functions on $\mathbb{R}^n$ and such that A is nonsingular μ-a.e. The operator $\mathcal{H}^* := \mathcal{H}\ (\mu, c|\det A^{-1}|,\ A^{-1})$ is the adjoint to $\mathcal{H}$ in a well-defined sense. Our goal is to prove sufficient conditions for the boundedness of these operators on the real Hardy space $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$. Our main tool is proving commuting relations among $\mathcal{H}$, $\mathcal{H}^*$, and the Riesz transforms $\mathcal{R}_j$. We also prove commuting relations among $\mathcal{H}$, $\mathcal{H}^*$, and the Fourier transform.

1. Introduction

The notion of Hausdorff (as well as quasi Hausdorff) operators with respect to a positive Borel measure on the unit interval $[0, 1]$ was introduced by HARDY [3, Ch. XI]. In [1], multivariate Hausdorff operators with respect to complex Borel measures on $\mathbb{R}^n$ were introduced in a more general framework.

To go into details, let μ be a σ-finite complex Borel measure defined on $\mathbb{R}^n$; let $c : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function; and let $A = [a_{jk}]$ be an $n \times n$ matrix whose entries $a_{jk} : \mathbb{R}^n \to \mathbb{C}$ are Borel measurable functions and such that A is nonsingular μ-a.e. We shall take it for granted that these assumptions on μ, c, and A are satisfied throughout this paper; in particular, in each of our Theorems 1, 2, and 3.

We shall consider Lebesgue–Stieltjes integrals with respect to the measure μ. The reader may consult the books [4, Ch. 9] by KAMKE and [7, Ch. 3] by SAKS.

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In [1], first we defined the multivariate Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu, c, A)$ acting on the continuous functions $f \in \mathbb{R}^n \to \mathbb{C}$, in symbol: $f \in C(\mathbb{R}^n)$, by setting

$$\mathcal{H}f(x) := \int c(s) f(A(s)x)\, d\mu(s), \tag{1.1}$$

provided that the integral on the right exists as a Lebesgue–Stieltjes integral. Here and in the sequel (if it is not indicated otherwise), the integral is taken over the whole space $\mathbb{R}^n$, and the variables s, x (and t later on) are points of (or vectors in) $\mathbb{R}^n$. Second, we extended the operator $\mathcal{H}$ from $C(\mathbb{R}^n)$ to the whole space $L^p(\mathbb{R}^n)$, where $1 \le p < \infty$, by making use of the so-called "density" argument familiar in Functional Analysis. This approach does not apply to $L^\infty(\mathbb{R}^n)$, since $C(\mathbb{R}^n)$ is not dense in it. Therefore, by $L^\infty(\mathbb{R}^n)$ we mean $C(\mathbb{R}^n)$ in the sequel if μ is not absolutely continuous with respect to the Lebesgue measure.

On the other hand, if μ is absolutely continuous with respect to the Lebesgue measure, then $\mathcal{H}f$ can be defined immediately by (1.1) for any Borel measurable function f belonging to $L^p(\mathbb{R}^n)$ for some $1 \le p \le \infty$, provided that the integral on the right-hand side of (1.1) exists. Indeed, it is well known that for any $f \in L^p(\mathbb{R}^n)$, there exists a Borel measurable function $f_1 \in L^p(\mathbb{R}^n)$ such that $f(x) = f_1(x)$ at almost every $x \in \mathbb{R}^n$ with respect to the Lebesgue measure.

The operator $\mathcal{H}^*$ adjoint to $\mathcal{H}$ (in the sense of (2.4) below) is given by

$$\mathcal{H}^* f(x) := \int c(s) |\det A^{-1}(s)| f(A^{-1}(s)x)\, d\mu(s),$$

provided that the integral on the right exists. Clearly, $\mathcal{H}^*$ is also a Hausdorff operator corresponding to the triplet $\mu(s)$, $c(s)|\det A^{-1}(s)|$, $A^{-1}(s)$; that is $\mathcal{H}^*(f) := \mathcal{H}(\mu, c|\det A^{-1}|, A^{-1})$.

The reader is referred to [1] for more details.

We make a last remark about practical notation. If there are two measures, μ and μ_a (see (5.4) in Section 5), then we use the abbreviations $\mathcal{H}(\mu) := \mathcal{H}(\mu, c, A)$ and $\mathcal{H}(\mu_a) := \mathcal{H}(\mu_a, c, A)$ to distinguish the corresponding Hausdorff operators. The notations $\mathcal{H}^*(\mu)$ and $\mathcal{H}^*(\mu_a)$, or $\mathcal{H}(c)$ and $\mathcal{H}(c_1)$ (see Lemma 3 in Section 3) are used in the same sense.

2. Main results

We recall that the real Hardy space $H^1(\mathbb{R}^n)$ consists of those functions $f \in L^1(\mathbb{R}^n)$ whose Riesz transforms $\mathcal{R}_j f$ also belong to $L^1(\mathbb{R}^n)$ for $j = 1, 2, \dots, n$; and $H^1(\mathbb{R}^n)$ is endowed with the norm

$$\|f\|_{H^1} := \|f\|_1 + \sum_{j=1}^{n} \|\mathcal{R}_j f\|_1$$

(see, e.g. [8, pp. 26, 123-124] by Stein). As is usual, we set

$$\|f\|_p := \Big(\int |f(x)|^p \, dx \Big)^{1/p} \quad \text{for} \ \ 1 \le p < \infty,$$

and

$$\|f\|_\infty := \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{R}^n\}.$$

Let $1 \le p \le \infty$ and denote by p^* the exponent conjugate to p; that is, let $1/p + 1/p^* = 1$ with the agreement that $1/\infty := 0$. In [1], the following sufficient conditions were proved for the boundedness of the operators $\mathcal{H}$ and $\mathcal{H}^*$ on $L^p(\mathbb{R}^n)$.

Lemma 1. *If*

$$k_p := \int |c(s)||\det A^{-1}(s)|^{1/p} d|\mu|(s) < \infty \tag{2.1}$$

for some $1 \le p \le \infty$, *where* $|\mu|$ *denotes the total variation of* μ, *then* $\mathcal{H}$ *is bounded on* $L^p(\mathbb{R}^n)$:

$$\|\mathcal{H} f\|_p \le k_p \|f\|_p; \tag{2.2}$$

and $\mathcal{H}^*$ *is bounded on* $L^{p^*}(\mathbb{R}^n)$:

$$\|\mathcal{H}^* f\|_{p^*} \le k_p \|f\|_{p^*}. \tag{2.3}$$

In [1], we also proved that the operators $\mathcal{H}$ and $\mathcal{H}^*$ are adjoint of each other in the sense that if (2.1) is satisfied for some $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^n)$, and $g \in L^{p^*}(\mathbb{R}^n)$, then

$$\int [\mathcal{H} f(x)] g(x) \, dx = \int f(x) [\mathcal{H}^* g(x)] \, dx. \tag{2.4}$$

Our first main result reads as follows.

Theorem 1. *Assume* $A(s) := \operatorname{diag}(a(s), \dots, a(s))$, *where* $a(s) : \mathbb{R}^n \to \mathbb{C}$ *is a Borel measurable function and* $a(s) \ne 0$ μ-*a.e.*

(i) *If* $k_1 < \infty$, *then* $\mathcal{H}$ *is bounded on* $H^1(\mathbb{R}^n)$:

$$\|\mathcal{H} f\|_{H^1} \le k_1 \|f\|_{H^1}. \tag{2.5}$$

(ii) *If* $k_\infty < \infty$, *then* $\mathcal{H}^*$ *is bounded on* $H^1(\mathbb{R}^n)$:

$$\|\mathcal{H}^* f\|_{H^1} \le k_\infty \|f\|_{H^1}. \tag{2.6}$$

We recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of those locally integrable functions f on $\mathbb{R}^n$ for which the quasinorm

$$\|f\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx$$

is finite, where the supremum is extended over all balls $B \subset \mathbb{R}^n$ of volume $|B|$ and

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy.$$

The next theorem follows from Theorem 1 by a familiar duality argument.

Theorem 2. *Assume $A(s)$ is the same as in Theorem* 1.

(i) *If $k_\infty < \infty$, then $\mathcal{H}$ is bounded on* $\mathrm{BMO}(\mathbb{R}^n)$:

$$\|\mathcal{H}f\|_{\mathrm{BMO}} \le k_\infty \|f\|_{\mathrm{BMO}}. \tag{2.7}$$

(ii) *If $k_1 < \infty$, then $\mathcal{H}^*$ is bounded on* $\mathrm{BMO}(\mathbb{R}^n)$:

$$\|\mathcal{H}^* f\|_{\mathrm{BMO}} \le k_1 \|f\|_{\mathrm{BMO}}. \tag{2.8}$$

The most common examples of Hausdorff operators correspond to the following triplets: μ is the ordinary Lebesgue measure supported on the unit cube $[0,1]^n$ of $\mathbb{R}^n$, $c(s) \equiv 1$, and $A(s) := \mathrm{diag}(s_1, s_2, \ldots, s_n)$, and called the multivariate Cesàro operator, while the adjoint is called the multivariate Copson operator. There are various kinds of mixed Cesàro–Copson operators, as well. In these examples the diagonal entries of the matrix A are different, and thus Theorems 1 and 2 do not apply.

However, in the case when

$$A(s) := \mathrm{diag}(|s|, \ldots, |s|), \qquad |s| := \Big(\sum_{j=1}^n s_j^2\Big)^{1/2},$$

Theorems 1 and 2 apply and provide the following corollaries:

(i) The 'radial' Cesàro operator

$$\mathcal{H}f(x) := \int_{[0,1]^n} f(|s|x_1, \ldots, |s|x_n)\, ds_1 \cdots ds_n =$$
$$= \omega_{n-1} \int_0^1 f(rx_1, \ldots, rx_n) r^{n-1}\, dr, \qquad x \in \mathbb{R}^n,$$

is bounded on $\mathrm{BMO}(\mathbb{R}^n)$, where

$$\omega_{n-1} := 2\pi^{n/2}/\Gamma(n/2), \qquad n = 1, 2, \ldots,$$

is the surface area of the unit sphere $\Sigma_{n-1} := \{s \in \mathbb{R}^n : |s| = 1\}$, and Γ is the familiar gamma function.

(ii) The 'radial' Copson operator

$$\mathcal{H}^* f(x) := \int_{[0,1]^n} \frac{1}{|s|^n} f\Big(\frac{x_1}{|s|}, \dots, \frac{x_n}{|s|}\Big)\, ds_1 \cdots ds_n =$$
$$= \omega_{n-1} \int_0^1 f\Big(\frac{x_1}{r}, \dots, \frac{x_n}{r}\Big) \frac{dr}{r}, \qquad x \in \mathbb{R}^n,$$

is bounded on $H^1(\mathbb{R}^n)$.

It is worthy of mention that in the particular case when f is rotationally invariant, that is, when $f(x_1, \dots, x_n)$ depends only on $R := |x|$, which we denote also by $f(R)$, we find

$$\mathcal{H} f(x) = \omega_{n-1} \int_0^1 f(Rr) r^{n-1}\, dr = \frac{\omega_{n-1}}{R^n} \int_0^R \rho^{n-1} f(\rho)\, d\rho$$

and

$$\mathcal{H}^* f(x) = \omega_{n-1} \int_0^1 f\Big(\frac{R}{r}\Big) \frac{dr}{r} = \omega_{n-1} \int_R^\infty \frac{f(\rho)}{\rho}\, d\rho, \qquad x \in \mathbb{R}^n.$$

The reader is referred to [1, especially Section 5] for more details.

We note that for $n = 1$, the above statements (i) and (ii) were proved in [6], where the term 'harmonic Copson operator' was used in place of 'Cesàro operator', and the term 'harmonic Cesàro operator' was used in place of 'Copson operator'. We also note that Part (ii) in Theorem 1 was proved in [5] when $n = 1$ and μ is absolutely continuous with respect to the Lebesgue measure.

Furthermore, we note that the radial Cesàro operator $\mathcal{H}$ in (i) is not bounded on $H^1(\mathbb{R}^n)$, and the radial Copson operator $\mathcal{H}^*$ in (ii) is not bounded on $\mathrm{BMO}(\mathbb{R}^n)$.

3. Auxiliary results

We recall that the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(t) := (2\pi)^{-n/2} \int f(x) e^{-it\cdot x}\, dx, \qquad \text{where } \ t \cdot x := \sum_{j=1}^n t_j x_j$$

is the familiar inner product of the vectors $t = (t_1, \dots, t_n)$ and $x = (x_1, \dots, x_n)$. It is well known that $\widehat{f} \in C(\mathbb{R}^n)$ and by the Riemann–Lebesgue lemma

$$\widehat{f}(t) \to 0 \quad \text{ as } \ \max\{|t_j| : j = 1, 2, \dots, n\} \to \infty.$$

In [1], we proved the following commuting relations among $\mathcal{H}$, $\mathcal{H}^*$, and the Fourier transform.

Lemma 2. *Assume the matrix* $A(s)$ *is symmetric* μ*-a.e. and* $f \in L^1(\mathbb{R}^n)$.

(i) *If* $k_1 < \infty$, *then*

$$(\mathcal{H}f)^\wedge(t) = \mathcal{H}^*\widehat{f}(t), \qquad t \in \mathbb{R}^n. \tag{3.1}$$

(ii) *If* $k_\infty < \infty$, *then*

$$(\mathcal{H}^*f)^\wedge(t) = \mathcal{H}\widehat{f}(t), \qquad t \in \mathbb{R}^n. \tag{3.2}$$

Now, we shall prove similar commuting relations among $\mathcal{H}$, $\mathcal{H}^*$, and the Riesz transforms $\mathcal{R}_j$.

Lemma 3. *Assume* $A(s)$ *is the same as in Theorem* 1, $f \in H^1(\mathbb{R}^n)$, $c_1(s) := c(s)\operatorname{sign} a(s)$, *and* $1 \le j \le n$.

(i) *If* $k_1 < \infty$, *then*

$$\mathcal{H}(c)\mathcal{R}_j f = \mathcal{R}_j\mathcal{H}(c_1)f. \tag{3.3}$$

(ii) *If* $k_\infty < \infty$, *then*

$$\mathcal{H}^*(c)\mathcal{R}_j f = \mathcal{R}_j\mathcal{H}^*(c_1)f. \tag{3.4}$$

It is plain that $k_p(c) = k_p(c_1)$ for all $1 \le p \le \infty$.

Proof of Lemma 3. It hinges on the fact that the Riesz transform $\mathcal{R}_j$ can be defined on $L^1(\mathbb{R}^n)$ in terms of the following multiplier transformation:

$$(\mathcal{R}_j f)^\wedge(t) = -i\frac{t_j}{|t|}\widehat{f}(t), \qquad t \in \mathbb{R}^n, \tag{3.5}$$

where the Fourier transform on the left is understood in the sense of tempered distributions. (See, e.g., [9, pp. 19–30] by Stein and Weiss.)

(i) Suppose $k_1 < \infty$ and $f \in H^1(\mathbb{R}^n)$. Since $\mathcal{R}_j f \in L^1(\mathbb{R}^n)$, by (2.2), we have $\mathcal{H}(c)\mathcal{R}_j f \in L^1(\mathbb{R}^n)$. By (3.1) and (3.5), we may proceed as follows

$$\begin{aligned}(\mathcal{H}(c)\mathcal{R}_j f)^\wedge(t) &= \mathcal{H}^*(c)(\mathcal{R}_j f)^\wedge(t) := \\ &:= \int c(s)|\det A^{-1}(s)|(\mathcal{R}_j f)^\wedge(A^{-1}(s)t)\, d\mu(s) = \\ &= \int c(s)|\det A^{-1}(s)|(-i)\frac{[A^{-1}(s)t]_j}{|A^{-1}(s)t|}\widehat{f}(A^{-1}(s)t)\, d\mu(s),\end{aligned} \tag{3.6}$$

where $[A^{-1}(s)t]_j$ means the jth component of the vector $A^{-1}(s)t$ in $\mathbb{R}^n$. By assumption,

$$\frac{[A^{-1}(s)t]_j}{|A^{-1}(s)t|} = \frac{t_j}{|t|}\operatorname{sign} a^{-1}(s).$$

Thus, by (3.6) we conclude that

$$(3.7)\qquad (\mathcal{H}(c)\mathcal{R}_j f)^\wedge(t) = -i\frac{t_j}{|t|}\mathcal{H}^*(c_1)\widehat{f}(t), \qquad o \neq t \in \mathbb{R}^n.$$

On the other hand, $\mathcal{H}^*(c_1)f \in L^1(\mathbb{R}^n)$, due to (2.3). By (3.5) and (3.1), we have

$$(3.8)\qquad (\mathcal{R}_j\mathcal{H}(c_1)f)^\wedge(t) = -i\frac{t_j}{|t|}(\mathcal{H}(c_1)f)^\wedge(t) = -i\frac{t_j}{|t|}\mathcal{H}^*(c_1)\widehat{f}(t),$$

where the first equality is understood in the distributional sense.

Combining (3.7) and (3.8) yields

$$(\mathcal{H}(c)\mathcal{R}_j f)^\wedge(t) = (\mathcal{R}_j\mathcal{H}(c_1)f)^\wedge(t),$$

also in the distributional sense. By the uniqueness of the Fourier transform, we conclude (3.3).

The proof of (3.4) is analogous to that of (3.3), while making use of (3.2) in place of (3.1). □

In the proof of Theorem 2, we shall need the following result, which is a folklore in Operator Theory (see, e.g., [2, p. 172] by Friedman).

Lemma 4. *If B_1 and B_2 are Banach spaces and $\mathcal{L}$ is a bounded linear operator from B_1 into B_2, then an adjoint operator $\mathcal{L}^*$ can be uniquely defined from the dual of B_2 into the dual of B_1 such that*

$$\|\mathcal{L}^*\|_{B_2^*\to B_1^*} = \|\mathcal{L}\|_{B_1\to B_2},$$

where B^* denotes the Banach space dual of B.

In case $B_1 = B_2$, the shorter notation $\|\mathcal{L}\|_{B_1}$ is used.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. (i) Suppose $k_1 < \infty$ and $f \in H^1(\mathbb{R}^n)$. Since $c_1(s)\operatorname{sign} a(s) = c(s)$ and $\mathcal{R}_j f \in L^1(\mathbb{R}^n)$, by (3.3) and (2.2) we have

$$\|\mathcal{R}_j(\mathcal{H}(c)f)\|_1 = \|\mathcal{H}(c_1)(\mathcal{R}_j f)\|_1 \le$$
$$\le k_1(c_1)\|\mathcal{R}_j f\|_1 = k_1(c)\|\mathcal{R}_j f\|_1, \qquad j = 1, 2, \ldots, n.$$

Combining inequalities just obtained with (2.2) yields

$$\|\mathcal{H}f\|_{H^1} := \|\mathcal{H}f\|_1 + \sum_{j=1}^n \|\mathcal{R}_j(\mathcal{H}f)\|_1 \le k_1(\|f\|_1 + \sum_{j=1}^n \|\mathcal{R}_j f\|_1) =: k_1\|f\|_{H^1}.$$

This proves (2.5).

(ii) The proof of (2.6) is analogous to that of (2.5), while making use of (2.3) in place of (2.2). □

P r o o f o f T h e o r e m 2. (i) Suppose $k_\infty < \infty$ and $f \in \mathrm{BMO}(\mathbb{R}^n)$. We recall (see, e.g., [8, p. 141] by Stein) that there exists a constant γ_n such that the inequality

$$\int_B |f(x)|\,dx \le \gamma_n \|f\|_{\mathrm{BMO}}\ r^n \ln(r+2) \tag{4.1}$$

holds for all $f \in \mathrm{BMO}(\mathbb{R}^n)$ and balls B with radius $r > 0$ and center at the origin.

Given a ball B with radius r, we consider an arbitrary function $g \in L^\infty(B)$ with $\|g\|_\infty \le 1$. By Fubini's theorem, we obtain

$$\begin{aligned} \Big|\int_B g(x)\mathcal{H}f(x)\,dx\Big| &\le \int_B |g(x)|\,dx \Big|\int c(s) f(A(s)x)\,d\mu(s)\Big| \le \\ &\le \int |c(s)|\,d|\mu|(s) \int_B |g(x)||f(A(s)x)|\,dx \le \\ &\le \|g\|_\infty \int |c(s)|\,d|\mu|(s) \int_B |f(A(s)x)|\,dx. \end{aligned} \tag{4.2}$$

Taking into account that $f(A(s)x) = f(a(s)x_1, \dots, a(s)x_n)$ also belongs to $\mathrm{BMO}(\mathbb{R}^n)$ and

$$\|f\|_{\mathrm{BMO}} = \|f(A(s))\|_{\mathrm{BMO}},$$

provided $a(s) \ne 0$, from (4.1) and (4.2) it follows that

$$\begin{aligned} \Big|\int_B g(x)\mathcal{H}f(x)\,dx\Big| &\le \gamma_n r^n \ln(r+2) \int |c(s)|\|f(A(s))\|_{\mathrm{BMO}}\,d|\mu|(s) = \\ &= \gamma_n \|f\|_{\mathrm{BMO}} k_\infty r^n \ln(r+2) < \infty. \end{aligned}$$

Since this inequality is valid for any $g \in L^\infty(B)$ with $\|g\|_\infty \le 1$ and the radius r of the ball B can be arbitrarily large, the reverse Hölder's inequality implies that $\mathcal{H}f(x)$ is locally integrable on $\mathbb{R}^n$.

Now, it remains to recall that the Banach space dual of $H^1(\mathbb{R}^n)$ is isomorphic to $\mathrm{BMO}(\mathbb{R}^n)$ (modulo constant) (see, e.g., [8, p. 142] by Stein), (2.7) follows immediately from (2.6) via Lemma 4.

(ii) The validity of (2.8) follows from (2.5) in an analogous way. □

5. Commuting relations involving Fourier transform

It is of interest to extend the validity of Lemma 2 when $f \in L^p(\mathbb{R}^n)$ for some $1 < p \le 2$. We recall that the Fourier transform $\widehat{f}$ of a function $f \in L^p(\mathbb{R}^n)$ for some $1 < p \le 2$ is defined as the limit in $L^{p^*}(\mathbb{R}^n)$-norm of the truncated function

$$f_a(x) := f(x)\chi_{Q_a}(x) \quad \text{as} \ \ a \to \infty,$$

where $p^* := p/(p-1)$ and

$$Q_a := \Big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \ |x_j| \le a, \ j = 1, 2, \ldots, n \Big\}$$

is the cube with sides $2a$ and centered at the origin. In symbol, we write

$$\widehat{f} := L^{p^*}\text{-}\lim_{a\to\infty} f_a.$$

As is well known, $\widehat{f} : L^p(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$ is a bounded linear operator:

$$\|\hat{f}\|_{p^*} \le \gamma_{p,n} \|f\|_p, \qquad \gamma_{p,n} := (2\pi)^{n[(1/p^*)-(1/p)]/2}, \qquad 1 \le p \le 2. \tag{5.1}$$

(See, e.g. [10, Vol. 2, p. 254] by Zygmund.)

The following theorem expresses commuting relations among $\mathcal{H}$, $\mathcal{H}^*$, and the Fourier transform, and it may be useful in other contexts.

Theorem 3. *Assume the matrix $A(s)$ is symmetric μ-a.e. and $f \in L^p(\mathbb{R}^n)$ for some $1 < p \le 2$.*

(i) *If $\det A^{-1}(s)$ is locally bounded μ-a.e. and $k_p < \infty$, then*

$$(\mathcal{H}f)^\wedge(t) = \mathcal{H}^* \widehat{f}(t) \qquad a.e. \tag{5.2}$$

(ii) *If $\det A(s)$ is locally bounded μ-a.e. and $k_{p^*} < \infty$, then*

$$(\mathcal{H}^* f)^\wedge(t) = \mathcal{H} \widehat{f}(t) \qquad a.e. \tag{5.3}$$

Proof. (i) We shall prove (5.2) in details. Let $0 < a \in \mathbb{R}^1$ and denote by μ_a the restriction of the measure μ to the cube Q_a; that is, let

$$\mu_a(S) := \mu(S \cap Q_a) \tag{5.4}$$

for every Borel measurable subset S of $\mathbb{R}^n$. By assumption, $\det A^{-1}(s)$ is locally bounded μ-a.e., and consequently,

$$k_1(\mu_a) := \int |c(s)||\det A^{-1}(s)|\, d|\mu_a|(s) \le$$
$$\le k_p(\mu) \operatorname*{ess\,sup}_{\mu-\text{a.e.}} \{|\det A^{-1}(s)|^{1/p^*} : s \in Q_a\} < \infty.$$

By (3.1), for all $g \in L^1(\mathbb{R}^n)$ we have

$$(\mathcal{H}(\mu_a)g)^\wedge(t) = \mathcal{H}^*(\mu_a)\hat{g}(t), \qquad t \in \mathbb{R}^n.$$

Since $(L^1 \cap L^p)(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we conclude, for all $f \in L^p(\mathbb{R}^n)$ and $a > 0$,

$$(\mathcal{H}(\mu_a)f)^\wedge = \mathcal{H}^*(\mu_a)\widehat{f} \quad \text{in} \ \ L^{p^*}(\mathbb{R}^n). \tag{5.5}$$

Since $k_p < \infty$ and $f \in L^p(\mathbb{R}^n)$, it follows from (2.2) that $\mathcal{H}f \in L^p(\mathbb{R}^n)$. By (5.1) and (2.2), we have

$$\|(\mathcal{H}(\mu)f)^\wedge - (\mathcal{H}(\mu_a)f)^\wedge\|_{p^*} \le \gamma_{p,n} \|\mathcal{H}(\mu)f - \mathcal{H}(\mu_a)f\|_p = \tag{5.6}$$

$$= \gamma_{p,n} \|\mathcal{H}(\mu - \mu_a) f\|_p \le \gamma_{p,n} k_p(\mu - \mu_a) \|f\|_p =:$$
$$=: \gamma_{p,n} \|f\|_p \int_{\mathbb{R}^n \setminus Q_a} |c(s)| |\det A^{-1}(s)|^{1/p} \, d|\mu|(s).$$

Clearly, the integral over $\mathbb{R}^n \setminus Q_a$ tends to 0 as $a \to \infty$. Thus, by (5.6) we see that

$$L^{p^*}\text{-}\lim_{a\to\infty} (\mathcal{H}(\mu_a) f)^\wedge = (\mathcal{H}(\mu) f)^\wedge. \tag{5.7}$$

On the other hand, applying Lemma 1 to $\widehat{f} \in L^{p^*}(\mathbb{R}^n)$ gives that $\mathcal{H}^*(\mu)\widehat{f}$, $\mathcal{H}^*(\mu_a)\widehat{f} \in L^{p^*}(\mathbb{R}^n)$. By (2.3) and (5.1), we find (cf. (5.6)) that

$$\|\mathcal{H}^*(\mu)\widehat{f} - \mathcal{H}^*(\mu_a)\widehat{f}\|_{p^*} \le k_p(\mu - \mu_a) \|\widehat{f}\|_{p^*} \le$$
$$\le \gamma_{p,n} k_p(\mu - \mu_a) \|f\|_p \to 0 \quad \text{as} \quad a \to \infty.$$

This means that

$$L^{p^*}\text{-}\lim_{a\to\infty} \mathcal{H}^*(\mu_a)\widehat{f} = \mathcal{H}^*(\mu)(f). \tag{5.8}$$

Combining (5.5), (5.7) and (5.8) yields (5.2).

(ii) The proof of (5.3) runs along the same lines as that of (5.2). □

References

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Многомерные операторы Хаусдорфа на пространствах $H^1(\mathbb{R}^n)$ и $\mathrm{BMO}(\mathbb{R}^n)$

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Многомерный оператор Хаусдорфа $\mathcal{H} = \mathcal{H}(\mu, c, A)$ определяется в терминах σ-конечной борелевской меры μ на $\mathbb{R}^n$, функция c на $\mathbb{R}^n$, измеримой по Борелю, и матрицы A вида $n \times n$, элементы которой — функции, измеримые по Борелю на $\mathbb{R}^n$, причем предполагается, что A невырождена μ-почти всюду. Оператор $\mathcal{H}^* := \mathcal{H}(\mu, c|\det A^{-1}|, A^{-1})$ сопряжен с $\mathcal{H}$ в обычном смысле. Цель работы — найти достаточные условия ограниченности этих операторов на вещественном пространстве Харди $H^1(\mathbb{R}^n)$ и на $\mathrm{BMO}(\mathbb{R}^n)$. Для этого устанавливаются коммутационные соотношения между $\mathcal{H}$, $\mathcal{H}^*$ и преобразованиями Рисса $\mathcal{R}_j$. Мы устанавливаем также коммутационные соотношения между $\mathcal{H}$, $\mathcal{H}^*$ и преобразованием Фурье.

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