Acta Math. Hungar.
https://doi.org/10.1007/s10474-024-01464-5

ON CERTAIN CLASSES OF FIRST BAIRE FUNCTIONALS

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(Received January 16, 2024; revised June 18, 2024; accepted June 26, 2024)

Abstract. We investigate first Baire functionals on the dual ball of a separable Banach space X which are pointwise limit of a sequence of X whose closed span does not contain any copy of ℓ_1 (or has separable dual). We propose an example of a $C(K)$ space where not all such first Baire functionals exhibit this behavior. As an application, we study a quantitative version, in terms of descriptive set theory, of family a separable Banach spaces with this peculiarity.

1. Introduction

It is well-known that, for a separable Banach space X, the unit ball of its dual, endowed with the weak* topology, is a compact metrizable space. If we denote $K = (B_{X^*}, w^*)$, then X can be naturally identified with a closed subspace of $C(K)$, and X^{**} as a closed subspace of $A_\infty(K)$, the Banach space of all bounded affine functions on K in the sup norm. Many properties of the geometry of X can be deduced from these natural identifications. For instance, X is reflexive if and only if $X^{**} \subseteq C(K)$.

Let us denote by $B_1(X) \subseteq X^{**}$ the class of bounded first Baire functionals on K (i.e., the pointwise limits on K of a bounded sequence in X), and by $DBSC(X) \subseteq B_1(X)$ the subclass of differences of bounded semicontinuous functions on K. To emphasize the influence of such spaces on the geometry of X, it is worth recalling the following theorem.

THEOREM 1. (a) $X^{**} = B_1(X)$ *if and only if* ℓ_1 *does not embed isomorphically into* X;

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Key words and phrases: first Baire function, analytic and coanalytic sets.
Mathematics Subject Classification: primary 46B20, secondary 54E52.

Published online: 16 September 2024

(b) *$DBSC(X) = C(K)$ if and only if c_0 does not embed isomorphically into X.*

The first statement is a classical theorem of E. Odell and H.P. Rosenthal ([15,18]), while part (b) was proved earlier by C. Bessaga and A. Pelczynski ([1]). Generalizations of $B_1(X)$ to spaces where K is not compact metric space, with applications to Banach space theory, have been developed in [9]. Theorem 1(a) was refined by J. Bourgain [5], where an ordinal was introduced, now known as *Bourgain's index*. H.P. Rosenthal asked whether, for a fixed $x^{**} \in B_1(X)$, there always exist a sequence $(x_n)_n \subseteq X$ such that $x^{**} = w^*\text{-}\lim_{n\to+\infty} x_n$ and either ℓ_1 does not embed in the closed subspace generated by $(x_n)_n$ or the subspace has a separable dual (see [5,6]). It was shown in [6] that the answer is positive if x^{**} is a difference of bounded semi-continuous functions, but negative in general.

In Bourgain's example, the extreme situation occurs: there exists a Banach space $\mathcal{X}_B$ such that $B_1^0(\mathcal{X}_B) = B_1^1(\mathcal{X}_B) = \mathcal{X}_B$ and $B_1(\mathcal{X}_B) \neq \mathcal{X}_B$. A Banach space X is called *weakly sequentially complete* if $B_1(X) = X$. In connection with the above questions, it is worth mentioning the following (see e.g., [17]).

Theorem 2. *Let X be an infinite dimensional Banach space.*

a. *If X has the Schur property, then X is weakly sequentially complete and has no reflexive subspaces.*

b. *If X is weakly sequentially complete and has no reflexive subspaces, then X is hereditarily ℓ_1 (i.e., any infinite dimensional subspace of X contains ℓ_1).*

For the aforementioned Bourgain's example, the space $\mathcal{X}_B$ provides also an example of Banach space where the converse of Theorem 2(b) does not hold. Since the example in [6] was briefly sketched, we will provide it here in more detail for the reader's convenience. Moreover, following Bourgain's idea, we also provide an example showing that the converse of Theorem 2(a) does not hold either.

At this point, it is convenient to introduce the following notations.

Definition 3. Let X be a separable infinite dimensional Banach space.

- $B_1^0(X)$ consists of those $x^{**} \in X^{**}$ such that x^{**} is the pointwise limit on K of a sequence $(x_n)_n$ in X (i.e.; $\lim_n x^*(x_n) = x^{**}(x^*)$ for all $x^* \in K$), and the closed span $\overline{\text{span}}\{x_n : n \in \mathbb{N}\}$ has separable dual.
- $B_1^1(X)$ consists of those $x^{**} \in X^{**}$ such that x^{**} is the pointwise limit on K of a sequence $(x_n)_n$ in X, and the closed span $\overline{\text{span}}\{x_n : n \in \mathbb{N}\}$ has no ℓ_1-subspace.

From the definitions above, the following inclusions are clear:

$$X \subseteq B_1^0(X) \subseteq B_1^1(X) \subseteq B_1(X) \subseteq X^{**}.$$

For the strict inclusions $B_1^0(X) \subsetneq B_1(X)$ and $B_1^1(X) \subsetneq B_1(X)$, Bourgain's example is very extremal, making it difficult to describe a family of separable Banach spaces where these inclusions are strict. To overcome this obstacle, we provide a $C(K)$ space in which the phenomenon above holds. For such a space, it becomes almost standard to derive the descriptive set nature of the family of separable Banach spaces X, endowed with the Effros–Borel structure, such that $B_1^0(X)$ and/or $B_1^1(X) \subsetneq B_1(X)$. It is shown that a quantitative version of Bourgain's theorem holds: namely, this subfamily is not Borel in the family of all separable Banach spaces. The idea of studying the descriptive set nature of a family of separable Banach spaces has already been used by various authors (see [2–4,13] for example). It is worth mentioning that in most of these constructions, the Pelczynski universal space [14] was used; here, we consider a different universal space instead.

To avoid any confusion, in the following, we assume that $0 \notin \mathbb{N}$, which is relevant when we consider any sum starting at index 1.

2. Preliminaries

We recall that a topological space X is called a *Polish space* if it admits a compatible metric d such that (X, d) is a separable complete metric space. A measure space (X, Σ), where X is a set and Σ a σ-algebra of subsets of X, is called *standard Borel* if there is a Polish topology on this set whose Borel σ-algebra coincides with Σ. A continuous image of one Polish space into another is called an *analytic set*, and sets that are complements of analytic sets are called *coanalytic*. As usual, we denote by Σ_1^1 all the analytic sets and Π_1^1 all coanalytic sets in a given Polish space. By a classical Souslin's theorem, we know that $\Sigma_1^1 \cap \Pi_1^1$ is exactly the Borel σ-algebra. Given a standard Borel space X, a set $A \subseteq X$ is said Σ_1^1-*hard* (resp. Π_1^1-*hard*) if for every Polish space Y, any analytic (resp. coanalytic) $B \subseteq Y$ can be written as $B = f^{-1}(A)$ for some Borel map $f : Y \longrightarrow X$. If moreover, A is analytic (resp. coanalytic), then it is called a *complete analytic set* (resp. *complete coanalytic set*). Since there exist analytic sets (as well as coanalytic sets) that are not Borel, it follows that Σ_1^1-hard and Π_1^1-hard sets are always not Borel.

By a classical theorem of Banach, every separable Banach space is isometrically isomorphic to a subspace of $C(2^\omega)$. Thus, denoting by $\mathcal{F}(C(2^\omega))$ the set of all closed subsets of $C(2^\omega)$, we consider by $\mathcal{SB}$ to be the subset of $\mathcal{F}(C(2^\omega))$ consisting of all linear subspaces of $C(2^\omega)$ endowed with the relative *Effros–Borel* structure. We refer the reader to the book [12] for all notions and notations regarding descriptive set theory.

The following result can be found in [4].

PROPOSITION 4. *$\mathcal{SB}$ is a Borel subset of $\mathcal{F}(C(2^\omega))$ equipped with the Effros–Borel structure.*

Recently, Godefroy and Saint-Raymond [10] proved that for any admissible topology τ, $\mathcal{SB}$ is a G_δ subset of $\mathcal{F}(C(2^\omega))$ equipped with τ, and thus it is a Polish space for the topology induced by τ. For the definition of admissible topologies, see Section 2 of [10].

We denote by $\omega = \{0, 1, \dots\}$ the first infinite ordinal. If we consider ω with the discrete topology, then ω^ω becomes a Polish space. Let $\omega^{<\omega}$ be the set of all finite sequences in ω. If $s = (s(0), \dots, s(n-1)) \in \omega^{<\omega}$, we denote its length n by $|s|$. In particular, the empty sequence $\emptyset$ has length 0. For every $i \in \{0, \dots, n\}$, let us denote by $s|_i = (s(0), \dots, s(i-1))$ and $s|_0 = \emptyset$. For $s, t \in \omega^{<\omega}$, we say that $s \le t$ if and only if $|s| \le |t|$ and $s_i = t_i$ for every $i \in \{0, \dots, |s|-1\}$; in such a case we say that t is an *extension* of s. For $s = (s(0), \dots, s(n-1))$ and $t = (t(0), \dots, t(k-1))$, the *concatenation* $s^\frown t$ is defined by

$$s^\frown t = (s(0), \dots, s(n-1), t(0), \dots, t(k-1)).$$

A subset $T \subseteq \omega^{<\omega}$ is said to be a *tree* on ω if $t|_i \in T$ for every $i \in \{0, \dots, |t|-1\}$, whenever $t \in T$. Let us denote by $\mathcal{T}$ the set of all trees on ω. A subset I of a tree T is called a *segment* if it is completely ordered and if $s, t \in I$ with $s \le t$, then every $s \le \ell \le t \Rightarrow \ell \in I$. Two segments I_1, I_2 are called completely incomparable if neither $s \le t$ nor $t \le s$ holds for every $s \in I_1$ and $t \in I_2$. Since $\mathcal{T}$ can be seen as a closed subset of $2^{\omega^{<\omega}}$ (observe each element as its characteristic function) and the last one is a Polish space, it follows that $\mathcal{T}$ is Polish as well. For $\theta \in \mathcal{T}$, a *branch* through θ is an $\varepsilon \in \omega^\omega$ such that for all $n \in \omega$,

$$\varepsilon|_n = (\varepsilon(0), \dots, \varepsilon(n-1)) \in \theta.$$

We denote by

$$[\theta] = \{\varepsilon \in \omega^\omega : \varepsilon \text{ is a branch through } \theta\},$$

usually called the *body* of θ.

A tree θ is *well-founded* iff $[\theta] = \emptyset$; i.e., θ has no branches. Otherwise, we will call θ *ill-founded*. We will denote by $\mathcal{WF}$ (resp. $\mathcal{IF}$) the set of well-founded trees (resp. ill-founded trees) on ω. The following is fundamental for our purpose.

THEOREM 5 [12, Theorem 27.1]. *The set $\mathcal{IF}$ is Σ^1_1-complete.*

For a tree $\theta \in \mathcal{T}$, one can consider the derivate procedure; i.e., let $\theta' = \{s \in \theta : \exists t \in \theta, s < t\}$, and by standard transfinite induction one defines

$$\theta^0 = \theta, \quad \theta^{\alpha+1} = (\theta^\alpha)', \quad \theta^\alpha = \bigcap_{\beta<\alpha} \theta^\beta \text{ if } \alpha \text{ is a limit ordinal.}$$

If there exists $\alpha < \omega_1$ (the first uncountable ordinal) such that $\theta^\alpha = \emptyset$, we define $o(\theta) = \min\{\alpha < \omega_1 : \theta^\alpha = \emptyset\}$. If there is no such countable α such that $\theta^\alpha = \emptyset$, then we set $o(\theta) = \omega_1$.

The following is a particular version of the Kunen–Martin theorem.

THEOREM 6. *A tree $\theta \in \mathcal{T}$ is well-founded if and only if $o(\theta) < \omega_1$.*

For $\theta \in \mathcal{T}$ and $k \in \omega$, let $\theta(k) = \{(n_1, \dots, n_l) \in \omega^{<\omega} : (k, n_1, \dots, n_l) \in \theta\}$, and $\theta_k = \{s \in \theta : (k) \leq s\}$. The next proposition follows by an easy transfinite induction procedure.

PROPOSITION 7. *Let $\theta \in \mathcal{WF}$ with $o(\theta) > 1$. Then*

$$o(\theta) = \sup_{k\in\omega}[o(\theta(k))] + 1;$$

In particular, $o(\theta(k)) < o(\theta)$ for all $k \in \omega$.

Before concluding this section, we recall the following theorem which is a consequence of the Kuratowski–Ryll–Nardzewski selection theorem (see [12, Theorem 12.13]).

THEOREM 8. *There exists a sequence of Borel functions $d_n \colon \mathcal{SB} \longrightarrow C(2^\omega)$ such that for every $X \in \mathcal{SB}$ the sequence $\{d_n(X) : n \in \mathbb{N}\}$ is dense in X.*

2.1. Lavrentiev and Bourgain indices. Let us recall two special indices (see [5]).

DEFINITION 9. Let K be a compact metric space and let A, B be two disjoint G_δ subsets of K. Let $R(A, B)$ consists of all transfinite (strictly) increasing sequences $(G_\alpha)_{\alpha\leq\beta}$ ($\beta < \omega_1$) of open subsets of K such that

1. $G_{\alpha+1} \setminus G_\alpha$ is disjoint with either A or B, for all $\alpha < \beta$;
2. $G_\gamma = \bigcup_{\alpha<\gamma} G_\alpha$ if $\gamma \leq \beta$ is a limit ordinal;
3. $G_0 = \emptyset$ and $G_\beta = K$.

Then define

$$L(A, B) = \min\{\beta < \omega_1 : \text{there exists } (G_\alpha)_{\alpha\leq\beta} \text{ in } R(A, B)\}$$

DEFINITION 10. For each $\alpha < \omega_1$, let

$$G_\alpha = \{(A, B) \in 2^K \times 2^K : A, B \text{ are disjoint } G_\delta \text{ sets and } L(A, B) \leq \alpha\}.$$

If the cardinality of K is the continuum, then G_α's are strictly increasing [5, p. 240].

In particular, if f is a first Baire function in a classical sense, and $a, b \in \mathbb{R}$ with $a < b$, then $\{x \in K : f(x) \leq a\}$ and $\{x \in K : f(x) \geq b\}$ are disjoint G_δ subsets of K. Then one can use the notation

$$L(f, a, b) = L(\{x \in K : f(x) \leq a\}, \{x \in K : f(x) \geq b\}).$$

DEFINITION 11. For each ordinal α, by induction one defines a new ordinal $[\alpha]$:
$[0] = 0$;
$[\beta] = \sup_{\alpha<\beta}[\beta]\omega_0 + 1$.

DEFINITION 12. Let X be a separable Banach space and $\delta > 0$. Let

$$T(X,\delta) = \bigcup_{k=1}^{\infty} \Big\{ (x_1,\dots,x_k) \in X^k : \|x_i\| \le 1 \ (1 \le i \le k)$$

$$\text{and } \Big\| \sum_{i=1}^{k} \lambda_i x_i \Big\| \ge \delta \sum_{i=1}^{k} |\lambda_i| \text{ for all } (\lambda_i)_{i=1}^k \in \mathbb{R}^k \Big\}.$$

Of course, $T(X,\delta)$ is a norm closed tree on X. The one can define the Bourgain ℓ_1-index as

$$o(X,\delta) = o(T(X,\delta)) + 1.$$

The main result in [5] can be state as follows.

THEOREM 13. *Assume X is a separable Banach space which does not contain ℓ_1. Denote by $K = (B_{X^*}, weak^*)$. Then, for every $x^{**} \in X^{**}$, with $\|x^{**}\| \le 1$, and for every reals a, b, δ with $0 < \delta < \frac{1}{2}(b-a)$ one has*

$$L(x^{**}|_K, a, b) \le [o(X,\delta)].$$

DEFINITION 14. Let $(A_n, B_n)_{n\in\mathbb{N}}$ be a sequence of pairs of subsets of K. We let

$$T(A_n,B_n;n) = \bigcup_{k=1}^{\infty} \Big\{ (n_1,\dots,n_k) \in \mathbb{N}^k : n_1 < \dots < n_k$$

$$\text{and for each } (\varepsilon_1,\dots,\varepsilon_k) \in \{-1,1\}^k \text{ holds } \bigcap_{\ell=1}^{k} \varepsilon_\ell A_{n_\ell} \ne \emptyset \Big\}.$$

which is of course a tree on $\mathbb{N}$, we use the notation $1 \cdot A_m = A_m$ and $(-1) \cdot A_m = B_m$. Take

$$o(A_n,B_n;n) := o(T(A_n,B_n;n)) + 1.$$

The following has been proved in [5, Theorem 25].

THEOREM 15. *Assume $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on K, which is pointwise relatively compact in the space of first Baire functions. Then, whenever f is a pointwise cluster point of $(f_n)_{n\in\mathbb{N}}$ and $a < c < d < b$ are reals, it holds*

$$L(f,a,b) \le [o(\{f_n < c\},\{f_n > d\};n)] < \omega_1.$$

2.2. Pointwise stabilized sequences. Let K be a compact metric space.

Definition 16. We say that a sequence $(f_n)_n \subseteq C(K)$ pointwise stabilizes if $f_m(s) = f_n(s)$ for n, m big enough and for each $s \in K$.

Let us recall the following well known result.

Proposition 17. *If the set of all extreme points of the dual ball of a Banach space is a norm separable set, then the dual itself is separable.*

Using the lifting property of extreme points, it follows

Corollary 18. *Let X be a Banach space, Y a closed subspace of X and $i\colon Y \longrightarrow X$ the canonical injection. Let E be the set of all extreme points in the dual ball of X and suppose that $i^*(E)$ is a separable subset of Y^*. Then Y^* itself is separable.*

We also recall the following fact:

Proposition 19. *If $R : X \longrightarrow Y$ is a bounded linear operator and $\overline{R(X)}^{w^*}$ is separable, then also $R^*(Y^*)$ is separable.*

From the last two propositions, we deduce the following.

Proposition 20. *If $(f_n)_n$ is a sequence in $C(K)$ which pointwise stabilizes, then $X = \overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$ has separable dual.*

Proof. Denote by $i\colon X \longrightarrow C(K)$ the canonical injection. By Corollary 18, it is enough to show that $i^*(K)$ is a separable set in X^* (identifying elements of $k \in K$ as Dirac measures δ_k, then as functionals on $C(K)$). For each $m \in \mathbb{N}$ define the compact set

$$K_m = \{s \in K : f_n(s) = f_m(s) \text{ for } n \geq m\},$$

and the restriction map $\rho_m : C(K) \longrightarrow C(K_m)$. It is clear that $\rho_m i(X)$ is finite dimensional and thus it has separable dual. By Proposition 19 we have $i^*\rho_m^*(K_m) = i^*(K_m)$ is a separable set. To finish the proof, it is enough to note that $K = \bigcup_m K_m$. $\square$

3. Some geometry of Banach spaces

Let us start with some geometrical lemmas useful to study families of Banach spaces. The proof of the first lemma is omitted since it follows directly from the Hahn–Banach separation theorem. Using a standard notation, for a subset A of a Banach space X, we denote by $\operatorname{co}(A)$ the convex hull of A. Moreover, unless specifically said, all closed subspaces are considered infinite dimensional.

LEMMA 21. *Let X be a Banach space and $A, B \subseteq X$ be so that $\overline{A}^{w^*} \cap \overline{B}^{w^*} \neq \emptyset$, where the weak* closures are taken in X^{**}. Then*

$$\|\cdot\|\text{- dist}(\mathrm{co}(A), \mathrm{co}(B)) = 0,$$

where $\mathrm{co}(A)$ *denotes the convex hull of A.*

LEMMA 22. *Let X be a Banach space and Y, Z be two closed subspaces of X with canonical embeddings $i\colon Y \longrightarrow X$ and $j\colon Z \longrightarrow X$ respectively. If $i^{**}(y^{**}) = j^{**}(z^{**})$, for some $y^{**} \in B_1(Y)$ and $z^{**} \in Z^{**}$, then for every sequence $(\varepsilon_n)_n$ of positive numbers, there exist $(y_n)_n \subseteq Y$ and $(z_n)_n \subseteq Z$ such that $w^*\text{- }\lim_{n\to\infty} y_n = y^{**}$ and $\|y_n - z_n\| < \varepsilon_n$ for each $n \in \mathbb{N}$.*

Consequently, $w^*\text{- }\lim_{n\to\infty} z_n = z^{**}$, and thus $z^{**} \in B_1(Z)$.

PROOF. We can assume that $\|y^{**}\| = \|z^{**}\| = 1$. We know there exists $(x_n)_n \subseteq Y$ such that $w^*\text{- }\lim_{n\to\infty} x_n = y^{**}$. Let $A_m = \{x_n : n \geq m\}$. Then, by assumption $\overline{A_m}^{w^*} \cap \overline{B_Z}^{w^*} \neq \emptyset$ for each $m \in \mathbb{N}$. The previous Lemma tells us $\|\cdot\|$-dist$(\mathrm{co}(A_m), B_Z) = 0$. Therefore, there exist $y_m \in \mathrm{co}(A_m)$ and $z_m \in B_Z$ such that $\|y_m - z_m\| < \varepsilon_m$ for each $m \in \mathbb{N}$. Of course, it follows $w^*\text{- }\lim_{n\to\infty} y_n = y^{**}$. $\square$

PROPOSITION 23. *Let $x^{**} \in X^{**}$ be such that dist$(x^{**}, X) > \varepsilon > 0$ and $x^{**} = w^*\text{- }\lim_{n\to\infty} x_n$, for some sequence $(x_n)_n \subseteq X$. Then $(x_n)_n$ admits a subsequence $(y_n)_n$ such that*

$$\max_{1\leq i\leq n} |a_i| \leq \frac{4}{\varepsilon}\left\| \sum_{i=1}^{n} a_i y_i \right\| \tag{3.1}$$

whenever $(a_i)_{i=1}^n \subseteq \mathbb{R}$.

PROOF. By induction, one can construct a subsequence $(y_n)_n$ of $(x_n)_n$, and a sequence $(y_n^*)_n$ in the unit sphere of X^* such that

(i) $x^{**}(y_n^*) > \varepsilon$;

(ii) $y_q^*(y_p) = 0$ if $p < q$;

(iii) $|y_q^*(y_p) - x^{**}(y_q^*)| < \varepsilon 7^{-p}$ if $q \leq p$.

Indeed, let us pick $x_1 \in X$ such that $\|x^{**} - x_1\| > \varepsilon$. Since dist$(x^{**}, X) > \varepsilon$, it follows that $\|x^{**}\| > \varepsilon$. Let us fix $y_1^* \in S_{X^*}$ such that $y_1^*(x^{**}) := x^{**}(y_1^*) > \varepsilon$. By assumption, let us consider $n_1 \in \omega$ such that

$$|y_1^*(x_{n_1} - x^{**})| < \varepsilon 7^{-1}.$$

Therefore $y_1 = x_{n_1}$ and y_1^* satisfy (i), (ii), and (iii).

Let us assume that $y_1, \ldots, y_k$ (with $y_i = x_{n_i}$) and $y_1^*, \ldots, y_k^*$ have been found. Take $\widetilde{y}^* \in \{y_1, \ldots y_k\}^\perp \cap S_{X^*}$ such that $\widetilde{y}^*(x^{**}) > \varepsilon$. By assumption, let us consider $n_{k+1} > n_k$ such that

$$|\widetilde{y}^*(x_{n_{k+1}}) - \widetilde{y}^*(x^{**})| < \frac{\varepsilon}{7^{k+1}}$$

and

$$|y_i^*(x_{n_{k+1}}) - y_i^*(x^{**})| < \frac{\varepsilon}{7^{k+1}} \quad i \in \{1, \ldots, k\}.$$

It is easy to check that $y_1, \ldots, y_k, x_{n_{k+1}}$ and $y_1^*, \ldots, y_k^*, \widetilde{y}^*$ satisfy (i), (ii), and (iii).

Let us fix $(a_i)_{i=1}^n \subseteq \mathbb{R}$ and suppose that $|a_k| = \max_{1\le i\le n} |a_i|$. We have

$$\begin{aligned}
&\|a_1y_1 + a_2(y_2 - y_1) + \cdots + a_n(y_n - y_{n-1})\| \\
&\ge |a_k||y_k^*(y_k)| - \sum_{s=k+1}^{n} |a_s| \cdot |y_k^*(y_s) - y_k^*(y_{s-1})| \\
&\ge |a_k| \cdot [|x^{**}(y_k^*)| - |x^{**}(y_k^*) - y_k^*(y_k)|] \\
&- \sum_{s=k+1}^{n} |a_s| \cdot [|y_k^*(y_s) - x^{**}(y_k^*)| + |y_k^*(y_{s-1}) - x^{**}(y_k^*)|] \\
&\ge \Big(\varepsilon - \frac{\varepsilon}{7^k}\Big)|a_k| - \sum_{s=k+1}^{n} |a_s|\Big(\frac{\varepsilon}{7^s} + \frac{\varepsilon}{7^{s-1}}\Big) \ge \frac{\varepsilon}{2}|a_k| = \frac{\varepsilon}{2}\max_{1\le i\le n}|a_i|.
\end{aligned}$$

By applying this inequality for $b_k = a_k + \cdots + a_n$, we obtain

$$\begin{aligned}
&\max_{1\le i\le n}|a_i| \le 2\max_{1\le i\le n}|b_i| \\
&\le \frac{4}{\varepsilon}\big\| b_1y_1 + b_2(y_2 - y_1) + \cdots + b_n(y_n - y_{n-1})\big\| = \frac{4}{\varepsilon}\Big\|\sum_{i=1}^{n} a_iy_i\Big\|. \quad \square
\end{aligned}$$

LEMMA 24. *Let X be a Banach space and Y, Z be two closed subspaces with canonical embeddings $i\colon Y \longrightarrow X$ and $j\colon Z \longrightarrow X$ respectively. If $x^{**} \in [i^{**}(B_1(Y)) \cap j^{**}(B_1(Z))] \setminus X$, then Y contains an infinite dimensional subspace W which is isomorphic to a subspace of Z and such that $x^{**} \in k^{**}(B_1(W))$, where $k\colon W \longrightarrow X$ is the natural injection.*

PROOF. Assume that $\operatorname{dist}(x^{**}, X) > \varepsilon > 0$. By Lemma 22, there exist $(y_n)_n \subseteq Y$ and $(z_n)_n \subseteq Z$ such that $w^*\text{-}\lim_{n\to\infty} y_n = x^{**}$ and $\|y_n - z_n\| < \varepsilon 2^{-n-3}$ for each $n \in \mathbb{N}$. By the previous proposition, one can pick $(y_{n_k})_k$ a subsequence of $(y_n)_n$ such that (3.1) holds. This implies that $W = \overline{\operatorname{span}}\{y_{n_k} : k \in \mathbb{N}\}$ is an infinite dimensional subspace of Y and, as $\|y_{n_k} - z_{n_k}\| < \frac{\varepsilon}{8}2^{-k}$, it follows that W and $\overline{\operatorname{span}}\{z_{n_k} : k \in \mathbb{N}\}$ are isomorphic and $x^{**} \in k^{**}(B_1(W))$, as wished. $\square$

Proposition 25. *Let X be a separable Banach space, Y be a closed subspace of X and $i\colon Y \longrightarrow X$ be the natural embedding. If $y^{**} \in B_1(Y)$, then*

$$y^{**} \in B_1^1(Y) \quad \textit{if and only if} \quad i^{**}(y^{**}) \in B_1^1(X).$$

Proof. One implication is of course trivial. If $x^{**} = i^{**}(y^{**}) \in B_1^1(X)$, then X has a closed subspace Z with no ℓ_1-subspace, such that $x^{**} \in j^{**}(B_1(Z))$, where $j\colon Z \longrightarrow X$ is the natural injection. Since $y^{**} \in B_1(Y)$, by previous lemma there exists a closed subspace W of Y which is isomorphic to a subspace of Z such that $x^{**} \in k^{**}(B_1(W))$, where $k\colon W \longrightarrow X$ is the injection. Therefore W has no ℓ_1-subspace and hence $y^{**} \in B_1^1(Y)$. $\square$

Analogously, one has

Proposition 26. *Let X be a separable Banach space, Y be a closed subspace of X and $i\colon Y \longrightarrow X$ be the natural embedding. If $y^{**} \in B_1(Y)$, then*

$$y^{**} \in B_1^0(Y) \quad \textit{if and only if} \quad i^{**}(y^{**}) \in B_1^0(X).$$

4. Bourgain's example and more

It is well known that if F is a nonempty closed subset of $[0,1]$, then $[0,1] \setminus F$ is the union of countably many disjoint intervals; i.e.,

$$[0,1] \setminus F = [0,a[\cup \Big(\bigcup_n]c_n, d_n[\Big) \cup]b,1],$$

where $a = \inf F$, $b = \sup F$ and $c_n, d_n \in F$ for each n.

Assume now f is a real valued function on F, and let consider the following extension

$$E_F f(t) := \begin{cases} f(t), & \text{for } t \in F, \\ \frac{t}{a} f(a), & \text{for } t \in [0,a[\\ \frac{1-t}{1-b} f(b), & \text{for } t \in]b,1] \\ \frac{d_n - t}{d_n - c_n} f(c_n) + \frac{t - c_n}{d_n - c_n} f(d_n), & \text{for } t \in]c_n, d_n[\,. \end{cases}$$

We also denote by $R_F f$ the restriction of f on F. The following properties are easy to verify.

PROPOSITION 27. *Let* K, L *be non empty closed subsets of* $[0,1]$ *such that* $K \subseteq L$. *Then* $E_K f = E_L R_L E_K f$.

PROPOSITION 28. *If* F *is a non empty closed subset of* $[0,1]$, *then* E_F *induces an isometric embedding of* $C(F)$ *into* $C([0,1])$.

For all $i \in \mathbb{N}$ and $j = 1, \dots, 2^i$ we denote by $I_{i,j} = \left](j-1)2^{-i}, j2^{-i}\right[$.

PROPOSITION 29. *There are sequences* $(K_i)_i$ *and* $(L_i)_i$ *of nonempty closed subsets of* $[0,1]$ *and a sequence* $E_i : C(K_i) \longrightarrow C([0,1])$ *of isometrically embeddings, satisfying*:

(1) $|K_i| = \mathfrak{c}$ (*where* $|\cdot|$ *denotes the cardinality of a set and* $\mathfrak{c}$ *the continuum*);

(2) $K_i \subseteq L_i$ *and* $\{j2^{-i} : 0 \leq j \leq 2^i\} \subseteq L_i$;

(3) L_i *is meager and the sequence is increasing*;

(4) $E_i f$ *extends* f

(5) $E_i f$ *is* 2^{-i}*-periodic*;

(6) $E_i f = E_{L_i} R_{L_i} E_i f$;

(7) $E_{i_2} f(t) = 0$ *if* $t \in L_{i_1}$ *where* $i_1 < i_2$.

PROOF. In the construction, we proceed by induction on i. Let K_1 be any closed and meager subset of $]0,1[$ such that $|K_1| = c$ and take $L_1 = K_1 \cup \{0,1\}$. Let E_1 be induced by E_{K_1}.

Suppose L_i are obtained. It is clear that the set $L = \bigcup_{j=1}^{2^{i+1}} [(L_i \cap \overline{I_{i+1,j}}) - (j-1)2^{-(i+1)}]$ is a meager subset of $\overline{I_{i+1,1}}$. Consider any closed and meager subset K_{i+1} of $I_{i+1,1}$ such that $|K_{i+1}| = c$ and $L \cap K_{i+1} = \emptyset$. Take $M = L \cup K_{i+1} \cup \{0, 2^{-(i+1)}\}$ and $L_{i+1} = \bigcup_{j=1}^{2^{i+1}} [(M + (j-1)2^{-(i+1)}]$, which are still meager. It remains to introduce E_{i+1}. Let $f \in C(K_{i+1})$. The function $\hat{f} \in C(M)$ will be obtained by taking $\hat{f}(t) = f(t)$ for $t \in K_{i+1}$ and $\hat{f}(t) = 0$ otherwise. If $\hat{\hat{f}} \in C(L_{i+1})$ is the $2^{-(i+1)}$ periodic extension of $\hat{f}$ to L_{i+1} , then take $E_{i+1} f = E_{L_{i+1}} \hat{\hat{f}}$. It is easy to check that all the conditions are satisfied with this construction. □

4.1. Setting on a sequence of Banach spaces. For each $i \in \mathbb{N}$, assume that X_i is a subspace of $C(K_i)$ and $(e_i^k)_k$ a sequence in X_i satisfying:

(1) $\|e_i^k\| = 1$;

(2) $X_i = \overline{\operatorname{span}}\{e_i^k : k\}$;

(3) $\| \sum_{k \in \mathbb{N}} a_k e_0^k \| \geq \delta_0 \max\{|a_k| : k \in \mathbb{N}\}$;

(4) For each i, the operator $T_i \colon X_{i+1} \longrightarrow X_i$ which maps e_{i+1}^k to e_i^k exists and it is moreover strictly singular (there is no infinite dimensional subspace Y of X_{i+1} such that the restriction $T_i|_Y$ is an isomorphism).

4.2. The construction of the space. For each $i, k \in \mathbb{N}$, let $\varepsilon_i = 2^{-i}$, and define

$$f_i^k = E_i e_i^k;\ g_i^k = \sum_{j \le i} \varepsilon_j f_j^k;\ h_i^k = \sum_{j > i} \varepsilon_j f_j^k;\ f^k = \sum_i \varepsilon_i f_i^k.$$

Let us introduce the spaces

$$Y_i = \overline{\operatorname{span}}\{g_i^k : k \in \mathbb{N}\};\quad Z_i = \overline{\operatorname{span}}\{h_i^k : k \in \mathbb{N}\};$$

Finally, let

$$X := \overline{\operatorname{span}}\{f^k : k \in \mathbb{N}\}.$$

PROPOSITION 30. *For each $i \in \mathbb{N}$ and $k \in \mathbb{N}$ we have*
a. $f_{i_1}^k = E_{L_i} R_{L_i} f_{i_1}^k$ *if* $i_1 \le i$;
b. $R_{L_i} f_{i_1}^k = 0$ *if* $i_1 > i$;
c. $g_i^k = E_{L_i} R_{L_i} g_i^k$;
d. $R_{L_i} h_i^k = 0$.
For each $i \in \mathbb{N}$, the following operators exist
e. $\alpha_i \colon X \longrightarrow Y_i$ *which maps* f^k *to* g_i^k;
f. $\beta_i \colon X \longrightarrow Z_i$ *which maps* f^k *to* h_i^k;
g. $\pi_i \colon X \longrightarrow X_i$ *which maps* f^k *to* e_i^k;
Moreover, $\|\alpha_i\| \le 1$, $\|\beta_i\| \le 2$, $\|\pi_i\| \le 2\varepsilon_i^{-1}$. *Finally, for each $i \in \mathbb{N}$*

$$T_i \pi_{i+1} = \pi_i. \tag{4.1}$$

PROOF. a. follows from Proposition 29(6) and from Proposition 27. b. follows from Proposition 29(7). c. and d. follow directly from the previous a. and b.

For each $i \in \mathbb{N}$, let (a_k) be any sequence of reals with are zero except finitely many. Then,

$$\Big\|\sum_k a_k f^k\Big\| \ge \Big\|\sum_k a_k R_{L_i}(g_i^k + h_i^k)\Big\|_{C(L_i)}$$
$$= \Big\|\sum_k a_k R_{L_i} g_i^k\Big\|_{C(L_i)} = \Big\|\sum_k a_k R_{L_i} g_i^k\Big\| = \Big\|\sum_k a_k g_i^k\Big\|.$$

Thus the operator α_i exists and $\|\alpha_i\| \le 1$.

Let $I \colon X \longrightarrow C([0,1])$ be the canonical injection. Then for the operator β_i it is enough to consider $I - \alpha_i$.

Finally, observe $\pi_0 = \varepsilon_0^{-1} R_{K_0} \alpha_0$ and $\pi_i = \varepsilon_i^{-1} R_{K_i}(\alpha_i - \alpha_{i-1})$. The last equality is obvious. $\square$

Now, let us study some geometrical property of X.

Proposition 31. *If $(x_n)_n$ is weakly null in X, then*

$$\lim_{i\to+\infty}\sup_{n\in\mathbb{N}}\|\beta_i(x_n)\|=0.$$

Proof. Fix $\varepsilon>0$. By Baire's classical theorem there exist $i\in\mathbb{N}$, $j\in\{1,\dots,2^i\}$ and some $n_0\in\mathbb{N}$ such that

$$|x_n(t)|\le\varepsilon/2,\quad \text{if } t\in\overline{I_{i,j}} \text{ and } n>n_0. \tag{4.2}$$

Take $i_1\ge i$ so that $\|\beta_{i_2}(x_n)\|<\varepsilon$ for $n=1,\dots,n_0$ and $i_2\ge i_1$; assume, by contradiction, that $\|\beta_{i_2}(x_n)\|>\varepsilon$ for some $n\ge n_0$ and $i_2\ge i_1$. Let $j_2\in\{1,\dots,2^{i_2}\}$ be such that $I_{i_2,j_2}\subseteq I_{i,j}$. Since $\beta_{i_2}(x_n)$ is $2^{-(i_2+1)}$-periodic and L_{i_2} is meager, we obtain that $|\beta_{i_2}(x_n)(t)|>\varepsilon$ for some $t\in I_{i_2,j_2}\setminus L_{i_2}$.

Now, $\varepsilon/2\ge|x_n(t)|\ge|\beta_{i_2}(x_n)(t)|-|\alpha_{i_2}(x_n)(t)|$ and thus $|\alpha_{i_2}(x_n)(t)|>\varepsilon/2$.

Since $\alpha_{i_2}(x_n)=E_{L_{i_2}}R_{L_{i_2}}\alpha_{i_2}(x_n)$, then

$$\alpha_{i_2}(x_n)(t)=\lambda\alpha_{i_2}(x_n)(c)+(1-\lambda)\alpha_{i_2}(x_n)(d),$$

where $t\in]c,d[$, $]c,d[\cap L_{i_2}=\emptyset$, $c,d\in L_{i_2}$ and $\lambda=\frac{d-t}{d-c}$.

Since $\{k2^{-i_2};\ 0\le k\le 2^{i_2}\}\subseteq L_{i_2}$, we have that $c,d\in\overline{I_{i_2,j_2}}$. Of course, some $t_2\in\{c,d\}$ is such that $|\alpha_{i_2}(x_n)(t_2)|>\varepsilon/2$ and hence also $|(x_n)(t_2)|>\varepsilon/2$ (remember that $\beta_{i_2}(x_n)\equiv 0$ on L_{i_2}). By (4.2), this yields the required contradiction. $\square$

Using a standard argument, we deduce from the previous proposition the following.

Corollary 32. *If (x_n) is a weak* converging sequence in X then*

$$\lim_{i\to+\infty}\sup_{n\in\mathbb{N}}\|\beta_i(x_n)\|=0.$$

Proposition 33. *If the Banach spaces X_i are weakly sequentially complete, then X is weakly sequentially complete.*

Proof. Assume $(x_n)_n\subseteq X$ is a sequence converging to $x^{**}\in X^{**}$. Fix $\varepsilon>0$ and by the previous corollary there exists $i\in\mathbb{N}$ such that $\sup_n\|\beta_i(x_n)\|<\varepsilon/4$. Since $\beta_i^{**}(x^{**})=w^*\text{-}\lim_n\beta_i(x_n)$ in Z_i^{**}, we also have $\|\beta_i^{**}(x^{**})\|\le\varepsilon/4$.

On the other hand, $\pi_i^{**}(x^{**})=w^*\text{-}\lim_n\pi_i(x_n)$ in X_i^{**} and since X_i is weakly sequentially complete, by Mazur's lemma, we have $\pi_i^{**}(x^{**})\in\overline{\mathrm{co}}\{\pi_i(x_n):n\in\mathbb{N}\}$. Let $M=\max\{\|T_j\|:0\le j\le i\}$ and let $(\lambda_n)_n$ positive reals such that $\sum_n\lambda_n=1$ and

$$\left\|\pi_i^{**}(x^{**})-\sum_n\lambda_n\pi_i(x_n)\right\|<\frac{\varepsilon}{4M^i}.$$

Since $x^{**} = \alpha_i^{**}(x^{**}) + \beta_i^{**}(x^{**})$, we also have

$$\begin{aligned}\Big\| x^{**} - \sum_n \lambda_n x_n \Big\| &\le \Big\| \alpha_i^{**}(x^{**}) - \sum_n \lambda_n \alpha_i(x_n) \| + \Big\| \beta_i^{**}(x^{**}) - \sum_n \lambda_n \beta_i(x_n) \Big\| \\ &\le \sum_{j\le i} \varepsilon_j \Big\| E_j^{**}\pi_j^{**}(x^{**}) - \sum_n \lambda_n E_j \pi_j(x_n) \Big\| + \frac{\varepsilon}{2} \\ &= \sum_{j\le i} \varepsilon_j \Big\| \pi_j^{**}(x^{**}) - \sum_n \lambda_n \pi_j(x_n) \Big\| + \frac{\varepsilon}{2}.\end{aligned}$$

If $j < i$, then $\pi_j = T_j \cdots T_{i-1}\pi_i$ and thus

$$\begin{aligned}\Big\| \pi_j^{**}(x^{**}) - \sum_n \lambda_n \pi_j(x_n) \Big\| &\le \|T_j\| \cdots \|T_{i-1}\| \Big\| \pi_i^{**}(x^{**}) - \sum_n \lambda_n \pi_j(x_n) \Big\| \\ &\le M^{i-j} \frac{\varepsilon}{4M^i} \le \frac{\varepsilon}{4}.\end{aligned}$$

Therefore,

$$\Big\| x^{**} - \sum_n \lambda_n x_n \Big\| < \sum_{j<i} \varepsilon_j \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that $x^{**} \in X$. $\square$

PROPOSITION 34. *The space X is hereditarily ℓ_1.*

PROOF. Let Y be an infinite dimensional subspace of X and let $(y_n)_n$ be a basic sequence in Y. Therefore, there exists $c > 0$ such that

$$\Big\| \sum_{i=1}^n a_i y_i \Big\| \ge c \cdot \max\{|a_i| : 1 \le i \le n\},$$

whenever $(a_i)_i$ is a finite set of reals. Let $(z_n)_n \subseteq \operatorname{span}\{f^k : k \in \mathbb{N}\}$ so that $\|z_n - y_n\| < c5^{-n}$ for each $n \in \mathbb{N}$. By 4.1(3), for each finite set of reals $(a_i)_i$,

$$\begin{aligned}\|a_1 f^1 + \cdots + a_n f^n\| &\ge \|a_1 g_0^1 + \cdots a_n g_0^n\| \\ = \varepsilon_0 \|a_1 e_0^1 + \cdots a_n e_0^n\| &\ge \varepsilon_0 \delta_0 \cdot \max\{|a_i| : 1 \le i \le n\}.\end{aligned}$$

By a standard procedure (see for example [8, p. 46]), it is possible to pick two increasing sequences $(p_r)_r, (q_r)_r$ of integers and a block subsequence $(x_r)_r$ of $(f^k)_k$ such that

(1) $p_r < q_r < p_{r+1}$;
(2) $\|x_r - (z_{p_r} - z_{q_r})\| < \frac{c}{5^r}$.

Therefore,

$$\begin{aligned}
&\Big\|\sum_{r=1}^{\infty} a_r x_r - \sum_{r=1}^{\infty} a_r (y_{p_r} - y_{q_r})\Big\| \\
&\le \sum_{r=1}^{\infty} |a_r| \|x_r - (z_{p_r} - z_{q_r})\| + \sum_{r=1}^{\infty} |a_r| \|y_{p_r} - z_{p_r}\| + \sum_{r=1}^{\infty} |a_r| \|y_{q_r} - z_{q_r}\| \\
&\le \max\{|a_r| : r \in \mathbb{N}\} \\
&\times \Big[\sum_{r=1}^{\infty} \|x_r - (z_{p_r} - z_{q_r})\| + \sum_{r=1}^{\infty} \|y_{p_r} - z_{p_r}\| + \sum_{r=1}^{\infty} \|y_{q_r} - z_{q_r}\|\Big] \\
&\le \max\{|a_r| : r \in \mathbb{N}\} \cdot c \cdot \Big[3 \sum_{r=1}^{\infty} \frac{1}{5^r}\Big] \\
&= \frac{3}{4} c \max\{|a_r| : r \in \mathbb{N}\} \le \frac{3}{4} \Big\|\sum_r a_r (y_{p_r} - y_{q_r})\Big\|,
\end{aligned}$$

and hence

$$\frac{1}{4}\Big\|\sum_r a_r (y_{p_r} - y_{q_r})\Big\| \le \Big\|\sum_r a_r x_r\Big\| \le \frac{7}{4}\Big\|\sum_r a_r (y_{p_r} - y_{q_r})\Big\|.$$

Actually this means that $(y_{p_r} - y_{q_r})_r$ and $(x_r)_r$ are equivalent.

To show that ℓ_1 embeds in $\overline{\operatorname{span}}\{x_r : r \in \mathbb{N}\}$, it is enough to show (see [18]) for each $i \in \mathbb{N}$ there exists $\xi_i \in \overline{\operatorname{span}}\{x_r : r \in \mathbb{N}\}$ of unit norm such that

$$\inf \xi_i(I_{i,j}) \le \frac{1}{4} \quad \text{and} \quad \sup \xi_i(I_{i,j}) \ge \frac{1}{2} \quad \text{for each } 1 \le j \le 2^i.$$

Let $M = \max\{\|T_0\|, \dots, \|T_i\|\}$. By 4.1(3) it follows that $(\pi_0(x_r))_r$ are linearly independent in X_0 as well as $(\pi_{i+1}(x_r))_r$ are linearly independent in X_{i+1}. Since $\overline{\pi_{i+1}(\overline{\operatorname{span}}\{x_r : r \in \mathbb{N}\})}$ is an infinite dimensional subspace of X_{i+1} and the operator $T_i \colon X_{i+1} \longrightarrow X_i$ is strictly singular, there must exists $\xi \in \overline{\operatorname{span}}\{x_r : r \in \mathbb{N}\}$ so that $\|\xi\| = 1$ and

$$\|\pi_i(\xi)\| = \|T_i \pi_{i+1}(\xi)\| \le \frac{\varepsilon_{i+1}}{16 M^i} \|\pi_{i+1}(\xi)\| \le \frac{1}{8 M^i}.$$

Thus,

$$\|\pi_j(\xi)\| \le \|T_j\| \cdots \|T_{i-1}\| \|\pi_i(\xi)\| \le \frac{1}{8} \quad \text{if } j \le i - 1$$

and therefore,

$$\|\alpha_i(\xi)\| \leq \sum_{j\leq i} \varepsilon_j \|\pi_j(\xi)\| \leq \frac{1}{4}.$$

Since $\xi = \alpha_i(\xi) + \beta_i(\xi)$, ti follows that $\|\beta_i(\xi)\| \geq \frac{3}{4}$. Recall that $\beta_i(\xi)$ is 2^{-i}-periodic, then $\sup \beta_i(\xi)(I_{i,j}) \geq \frac{3}{4}$, so that $\sup \xi(I_{i,j}) \geq \frac{1}{2}$ for all $j = 1, \dots, 2^i$. On the other hand, $R_{L_i}\beta_i(\xi) = 0$, this implies that $|\xi(t)| \leq \frac{1}{4}$ for all $t \in L_i$. Finally, to complete the proof it is enough to recall $\{j2^{-i} : 0 \leq j \leq 2^i\} \subseteq L_i$. □

4.3. Bourgain's example. Let us recall the following James space.

PROPOSITION 35. *Let* $1 < p < \infty$ *and let* c_{00} *endowed by the following norm*

$$\|x\|_p = \sup_r \sup_{n_1 < \cdots n_r} \left[\sum_{s=1}^{r} \left| \sum_{k=n_{s-1}+1}^{n_s} x_k \right|^p \right]^{1/p}.$$

Let J_p *be the completion. Then the following hold*:

a. J_p *is hereditarily* ℓ_p;

b. J_p *is not reflexive. More precisely, the sequence* $(e_k)_k$ *of the unit vectors is a monotone basis which weak* converges in* J_p^{**} *with no weak limit point.*

The following proposition is well known.

PROPOSITION 36. *Let* $1 \leq p < q < \infty$. *Then the spaces* ℓ_p *and* ℓ_q *are totally incomparable*; *i.e.*, *there is no infinite dimensional which is isomorphic to a subspace of* ℓ_p *as well as a subspace of* ℓ_q.

THEOREM 37. *There exists a Banach space* $\mathcal{X}_B$ *which is hereditarily* ℓ_1 *but not weakly sequentially complete.*

PROOF. To realize $\mathcal{X}_B$, we construct a sequence of Banach spaces X_i with the properties 4.1. Let $X_i = J_{\frac{i+2}{i+1}}$ viewed as a subspace of $C(K_i)$ and let $(e_i^k)_k$ be the unit vectors of $J_{\frac{i+2}{i+1}}$. To show that the operators T_i exist, it is enough to observe that $\|x\|_p \geq \|x\|_q$ whenever $1 < p < q < \infty$. Since X_i is hereditarily $\ell_{\frac{i+2}{i+1}}$ we get that T_i is strictly singular by Proposition 36. Let $\mathcal{X}_B$ be the Banach space obtained via 4.2. For each $i \in \mathbb{N}$, $(e_i^k)_k$ is a weakly Cauchy sequence in X_i and then $(f_i^k)_k$ is a weakly Cauchy sequence in $C([0,1])$. Therefore, $(f^k)_k$ is a weakly Cauchy sequence in $\mathcal{X}_B$. But $(f^k)_k$ does not weakly converge since $(\pi_0(f^k))_k$ is not weakly convergent in X_0. Finally, by Proposition 34 it follows that $\mathcal{X}_B$ is ℓ_1-hereditarily. □

For what we have said above, the following now is easy to get.

Corollary 38. *For the space above the following phenomenon happens*:

$$B_1^0(\mathcal{X}_B) = B_1^1(\mathcal{X}_B) = \mathcal{X}_B \quad \textit{and} \quad B_1(\mathcal{X}_B) \neq \mathcal{X}_B.$$

4.4. Another example: a variant.

Theorem 39. *There exists a Banach space $\mathcal{Y}$ which is hereditarily ℓ_1 and weakly sequentially complete, but not a Schur space.*

Proof. As before, we realize the space $\mathcal{Y}$ by a sequence of Banach spaces Y_i with the properties 4.1. For each $i \in \mathbb{N}$ let $Y_i = \ell_{\frac{i+2}{i+1}}$, which can be viewed as a subspace of $C(K_i)$. For $(e_i^k)_k$ we simply take the unit vectors of $\ell_{\frac{i+2}{i+1}}$. For the same reason as before, the operator T_i is well defined and strictly singular by Proposition 36. Let $\mathcal{Y}$ be the Banach space obtained via 4.2. For each $i \in \mathbb{N}$, $w\text{-}\lim_{k\to\infty} e_i^k = 0$ in Y_i and thus $w\text{-}\lim_{k\to\infty} f_i^k = 0$ in $C([0,1])$. Therefore also $w\text{-}\lim_{k\to\infty} f^k = 0$ in $C([0,1])$. On the other hand

$$\|f^k\| \geq \|\alpha_0 f^k\| = \varepsilon_0 \|f_0^k\| = \varepsilon_0.$$

This means that $\mathcal{Y}$ is not a Schur space. Of course $\mathcal{Y}$ is hereditarily ℓ_1 by Proposition 34. Finally, since each Y_i is weakly sequentially complete we also get $\mathcal{Y}$ weakly sequentially complete by Proposition 33. $\square$

4.5. On the construction of particular first Baire functionals. Firstly, let us recall that if K is a compact metric space and $X = C(K)$ then every 1-first Baire function φ on K extends to some $\hat{\varphi} \in B_1(X)$ by taking

$$\hat{\varphi}(\mu) = \int_K \varphi \, d\mu, \quad \mu \in C(K)^*.$$

This extension is unique by Choquet's theorem.

Before to present the next example, we need to recall a type of convergence for a sequence.

Definition 40. Let C be a set and $(f_n)_n$ be a sequence of real valued functions defined on C. We say that $(f_n)_n$ converges strictly on some $A \subseteq C$ if

$$\sum_n |f_{n+1}(x) - f_n(x)| < +\infty \quad \text{for each } x \in A.$$

Of course, strict convergence implies pointwise convergence. Let us also recall the following easy fact.

Lemma 41. *Suppose $(f_n)_n$ strictly converges in some $A \subseteq C$ and $(n_k)_k$ is a strictly increasing sequence of integers. If for each $k \in \mathbb{N}$, $g_k \in \operatorname{co}\{f_n : n_k \leq n < n_{k+1}\}$, then also $(g_k)_k$ is strictly convergent on A.*

PROOF. It is enough to note that, for $x \in A$

$$\begin{aligned}|g_{k+1}(x) - g_k(x)| &\leq \sup\{|f_q(x) - f_p(x)| : n_k \leq p < n_{k+1} \leq q < n_{k+2}\} \\ &\leq \sum_{n=n_k}^{n_{k+2}-1} |f_{n+1}(x) - f_n(x)|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_k |g_{k+1}(x) - g_k(x)| \\ &\leq \sum_k \Bigg[\sum_{n=n_k}^{n_{k+1}-1} |f_{n+1}(x) - f_n(x)| + \sum_{n=n_{k+1}}^{n_{k+2}-1} |f_{n+1}(x) - f_n(x)| \Bigg] \\ &\leq 2 \sum_n |f_{n+1}(x) - f_n(x)| < \infty. \quad \square\end{aligned}$$

Let us denote by $\operatorname{ext}(B_{X^*})$ the set of all extreme points of the unit dual ball endowed by the weak* topology.

DEFINITION 42. Another subspace (not closed) of $B_1(X)$ is defined as

$$\begin{aligned}B_1^{sc}(X) = \big\{ x^{**} \in X^{**} : x^{**} &\text{ is pointwise limit of a sequence } (x_n)_n \text{ of } X \\ &\text{such that } (x_n)_n \text{ converges strictly on } \operatorname{ext}(B_{X^*}) \big\}.\end{aligned}$$

The following will be really useful.

THEOREM 43. *The following inclusion* $B_1^{sc}(X) \subseteq B_1^0(X)$ *holds.*

PROOF. Let $x^{**} \in B_1^{sc}(X)$, of course we can assume $x^{**} \notin X$. Let $(x_n)_n \subseteq X$ pointwise convergent to x^{**} which is strictly convergent on $\operatorname{ext}(B_{X^*})$, and also bounded by $\|x^{**}\|$. For each $r \in \mathbb{N}$, let

$$K_r = \left\{ x^* \in B_{X^*} : \sum_n |x^*(x_{n+1}) - x^*(x_n)| \leq r \right\},$$

then $(K_r)_r$ is an increasing sequence of weak* compact sets such that $\operatorname{ext}(B_{X^*}) \subseteq \bigcup_r K_r$. Let $\rho_r : X \longrightarrow C(K_r)$ be the restriction map, identifying elements of K_r with Dirac measures, we have $K_r = \rho_r^*(K_r)$. By Proposition 23 (see the proof), there is a subsequence $(y_n)_n$ of $(x_n)_n$ and some $\varepsilon > 0$ such that, if $e_1 = y_1$ and $e_n = y_n - y_{n-1}$ we have

$$\frac{\varepsilon}{2} \max\{|a_n| : n \in \mathbb{N}\} \leq \left\| \sum_n a_n e_n \right\|$$

for all finite sets $(a_n)_n$ of reals.

Let $Y = \overline{\text{span}}\{y_n : n \in \mathbb{N}\} = \overline{\text{span}}\{e_n : n \in \mathbb{N}\}$. By the above inequality, it is clear that the operator $s\colon Y \longrightarrow c_0$ such that

$$s(\sum_n a_n e_n) = (a_n)_n$$

is well defined. Let us show that Y^* is separable.

Let $i\colon Y \longrightarrow X$ be the canonical injection, then $i^*(\text{ext}(B_{X^*})) \subseteq \bigcup_r i^*(K_r)$. By Corollary 18 it is enough to show the separability of $i^*(K_r)$ for each $r \in \mathbb{N}$. Since for $x^* \in K_r$ we have that $\sum_n |x^*(e_n)| \leq r$ then there exists an operator $\ell : c_0 \longrightarrow C(K_r)$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "s"'] & X \arrow[r, "\rho_r"] & C(K_r) \\ & c_0 \arrow[ur, "\ell"'] & \end{tikzcd}$$

commutes (just define ℓ of the n-basis element of c_0 to take value $\rho_r(e_n)$). In particular, we have

$$i^*(K_r) = i^*\rho_r^*(K_r) = s^*\ell^*(K_r).$$

Since $\ell^*(K_r)$ is a subset of ℓ_1, we have that $s^*\ell^*(K_r)$ is a separable subset of Y^*, as wished. $\square$

4.6. An alterative example. In this subsection we use notations and results given in Sections 2.1 and 2.2, to make this section more readable we suggest to have a look before.

Let $K = 2^\omega$ be the Cantor set and consider the space $X = C(K)$. In this section we will show that another phenomenon as Corollary 38 occurs for X.

Let us denote by $(K_{r,s})_{r\in\mathbb{N},\ 1\leq s\leq 2^r}$ the system of Cantor intervals. If f is a function on $K_{r,1}$ we can define a function $\overrightarrow{f}$ on $K_{r,s}$ defined by

$$\overrightarrow{f}(t) = f(t - (s-1)2^{-r}), \quad t \in K_{r,s}.$$

For each $r \in \mathbb{N}$ let $\varepsilon_r = 5^{-r}$; by induction let us construct A_r, B_r disjoints G_δ sets in $K_{r,1}$, a function $\varphi_r \in B_1(K_{r,1})$, a sequence $(f_{r,n})_n \subseteq C(K_{r,1})$ and a subspace X_r of $C(K)$ such that the following happens:

1. $\varphi_r \equiv \begin{cases} 1, & \text{on } A_r, \\ -1, & \text{on } B_r, \end{cases}$ with $\|\varphi_r\|_{C(K)} = 1$;

2. The sequence $(f_{r,n})_n$ is a stabilizing sequence pointwise bounded by 1 which pointwise converges to φ_r;

3. $X_r = \overline{\text{span}}\{\sum_{s=1}^r \varepsilon_s \overrightarrow{f}_{s,n} : n \in \mathbb{N}\}$;

4. $L(A_r, B_r) > [o(X_{r-1}, \frac{\varepsilon_r}{8})]$.

To make this construction possible, it is enough to observe that disjoint G_δ sets can be separated by a set which is both G_δ and F_σ, then one can consider the characteristic functions as limits of a pointwise stabilizing sequence of continuous functions (see [11]). By Proposition 20, X_r^* is separable and thus X_r does not contain ℓ_1. By [5, p. 245] it follows that $o(X_r, \delta) < \omega_1$ for all $\delta > 0$. Moreover, for what we said in Definition 10, the classes G_α are strictly increasing.

For each $r \in \mathbb{N}$, let $\delta_r = 2^{-r}$ and ϕ_r be a first Baire function on K given by

$$\psi_r(t) = \begin{cases} \overrightarrow{\varphi}_r(t), & \text{if } t \in K_{r,2^r-1} \\ 0, & \text{otherwise.} \end{cases}$$

Finally, define

$$\Phi = \sum_r \varepsilon_r \overrightarrow{\varphi}_r \quad \text{and} \quad \Psi = \sum_r \delta_r \psi_r,$$

which are obviously in $B_1(X)$. Here the main result of this section (recall the notation at the begin of Section 4.5).

THEOREM 44. *The following hold*:
1. $\hat{\Psi} \in B_1^{sc}(X)$ *and* $\hat{\Phi} + \hat{\Psi} \in B_1^0(X)$;
2. $\hat{\Phi} \notin B_1^1(X)$.

PROOF. For each $r, n \in \mathbb{N}$ let us define

$$g_{r,n}(t) = \begin{cases} \overrightarrow{f}_{r,n}(t), & \text{if } t \in K_{r,2^r-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence $g_n = \sum_r \delta_r g_{r,n}$ in $C(K)$ which is bounded by 1 and pointwise stabilizing with limit Ψ. By Proposition 20, $\overline{\text{span}}\{g_n,\ n \in \mathbb{N}\}$ has separable dual and $\hat{\Psi} \in B_1^{sc}(X)$.

It is clear that Φ is pointwise limit of the sequence $(f_n)_n$ in $C(K)$, where $f_n = \sum_r \varepsilon_r \overrightarrow{f}_{r,n}$. Thus $\Phi + \Psi = \lim_n (f_n + g_n)$ pointwise. Let us show that $\overline{\text{span}}\{g_n,\ n \in \mathbb{N}\}$ and $\overline{\text{span}}\{f_n + g_n,\ n \in \mathbb{N}\}$ are isomorphic, which implies that $\hat{\Phi} + \hat{\Psi} \in B_1^0(X)$. Indeed,

$$\begin{aligned} \left\| \sum_n a_n f_n \right\| &\le \sum_r \varepsilon_r \left\| \sum_n a_n \overrightarrow{f}_{r,n} \right\| = \sum_r \varepsilon_r \left\| \sum_n a_n f_{r,n} \right\| \\ &= \sum_r \varepsilon_r \left\| \sum_n a_n g_{r,n} \right\| \le \frac{4}{5} \sum_r \delta_r \left\| \sum_n a_n \delta_r g_{r,n} \right\| \end{aligned}$$

$$\leq \frac{4}{5}\sum_r \delta_r \Big\| \sum_n a_n g_n \Big\| = \frac{4}{5}\Big\| \sum_n a_n g_n \Big\|,$$

and so

$$\frac{1}{5}\Big\| \sum_n a_n g_n \Big\| \leq \Big\| \sum_n a_n g_n + f_n \Big\| \leq \frac{9}{5}\Big\| \sum_n a_n g_n \Big\|.$$

Let us show that if $(h_m)_m$ is a uniformly bounded sequence in $C(K)$ with pointwise limit Φ, then $\overline{\mathrm{span}}\{h_m,\ m \in \mathbb{N}\}$ contains ℓ_1. Without loss of generality, we can assume that $h_m \in \mathrm{co}\{f_n : n \geq m\}$. Thus, for each $r \in \mathbb{N}$, we obtain a sequence $(h_{r,m})_m$ in $C(K_{r,1})$ such that for each m

i) $h_{r,m}$ is convex combination of $f_{r,n}$ $(n \geq m)$;

ii) $h_m = \sum_r \varepsilon_r \overrightarrow{h}_{r,m}$.

Let us prove that for each $r \in \mathbb{N}$ there is $h \in \overline{\mathrm{span}}\{h_m,\ m \in \mathbb{N}\}$ such that

(4.3)
$$\|h\| \leq \frac{3}{2}, \quad \inf h(K_{r,s}) < -\frac{1}{8}, \quad \sup h(K_{r,s}) > \frac{1}{8}, \quad \text{for all } s = 1, \dots 2^r;$$

this will complete the proof by [17].

Since $\lim_m h_{r,m} = \lim_n f_{r,n} = \varphi_r$ pointwise in $K_{r,1}$ by Theorem 15 it follows that

$$L(A_r, B_r) \leq L(\varphi_r, -1, 1) \leq [o(C_m, D_m; m)]$$

where $C_m = \{t \in K_{r,1} : h_{r,m}(t) < -\frac{1}{2}\}$ and $D_m = \{t \in K_{r,1} : h_{r,m}(t) > \frac{1}{2}\}$. Therefore,

$$o\Big(X_{r-1}, \frac{\varepsilon_r}{8}\Big) < o(C_m, D_m; m).$$

From i) we have that if $x_m = \sum_{s=1}^{r-1} \varepsilon_s \overrightarrow{h}_{s,m}$, then $(x_m)_m$ is a sequence in the unit ball of X_{r-1}. Let us define the following tree on $\mathbb{N}$:

$$T = \bigcup_{k=1}^{\infty} \Big\{ (m_1, \dots, m_k) \in \mathbb{N}^k : m_1 < \dots < m_k$$
$$\text{and } (x_{m_1}, \dots, x_{m_k}) \in T\Big(X_{r-1}, \frac{\varepsilon_r}{8}\Big) \Big\}.$$

Obviously $o(T) \leq o[T(X_{r-1}, \frac{\varepsilon_r}{8})] < o[T(C_m, D_m; m)]$. This means there are $m_1 < \dots < m_k$ such that $(m_1, \dots, m_k) \in T(C_m, D_m; m)$ and $\| \sum_{i=1}^k \lambda_i x_{m_i} \| < \frac{\varepsilon_r}{8}$ for some $(\lambda_i)_{i=1}^k \in \mathbb{R}^k$ with $\sum_{i=1}^k |\lambda_i| = 1$. Consider $(\nu_i)_{i=1}^k \in \{\pm 1\}^k$ so that $\sum_{i=1}^k \nu_i \lambda_i = 1$. Since $(m_1, \dots, m_k) \in T(C_m, D_m; m)$ both sets $\bigcap_{i=1}^k \nu_i C_{m_i}$ and $\bigcap_{i=1}^k \nu_i D_{m_i}$ are nonempty (we are just using the notation $1 \cdot C_m = C_m$

and $(-1)\cdot C_m = D_m$). Hence, if $u = \sum_i \lambda_i h_{r,m_i}$, then $\inf u(K_{r,1}) < -\frac{1}{2}$ and $\sup u(K_{r,1}) > \frac{1}{2}$, as well as $\inf \overrightarrow{u}(K_{r,s}) < -\frac{1}{2}$ and $\sup \overrightarrow{u}(K_{r,s}) > \frac{1}{2}$, for all $s = 1, \ldots 2^r$. Let us also observe that

$$h = \varepsilon_r^{-1} \sum_i \lambda_i h_{m_i} = \varepsilon_r^{-1} \sum_i \lambda_i x_{m_i} + \overrightarrow{u} + \varepsilon_r^{-1} \sum_i \lambda_i \sum_{s>r} \varepsilon_s \overrightarrow{h}_{s,m_i},$$

so that $\|h - \overrightarrow{u}\| \le \varepsilon_r^{-1}\big(\frac{\varepsilon_r}{8} + \sum_{s>r} \varepsilon_s\big) = \frac{3}{8}$. Since $\|\overrightarrow{u}\| \le 1$ we have that h satisfies (4.3). □

As consequence, we get

Corollary 45. *For $X = C(2^\omega)$, then $B_1^1(X)$ and $B_1^0(X)$ are not vector spaces, in particular it follows*

$$X \subsetneqq B_1^0(X), B_1^1(X) \subsetneqq B_1(X).$$

5. Families of Banach spaces

5.1. An auxiliary space. Let us denote by $(u_n)_n$ the standard Schauder basis of $C(2^\omega)$. We denote by $c_{00}(T)$ the space of finitely supported functions from $T = \omega^{<\omega}$ to $\mathbb{R}$ and by $\chi_s \colon T \longrightarrow \{0,1\}$ the characteristic function of $\{s\}$ for every $s \in T$. Thus $c_{00}(T) = \operatorname{span}\{\chi_s : s \in T\}$.

An *admissible choice of intervals* is a finite set $\{I_j : 0 \le j \le k\}$ of intervals of T such that every branch of T meets at most one of these intervals.

We define the following norm on $c_{00}(T)$:

$$\|y\|_2 = \sup \left[\sum_{j=0}^{k} \left\| \sum_{s\in I_j} y(s)\ u_{|s|} \right\|_{C(2^\omega)}^2 \right]^{1/2}$$

where the supremum is taken over $k \in \omega$ and over all admissible choices of intervals $\{I_j : 0 \le j \le k\}$.

We let $U_2(T)$ to be the completion of $c_{00}(T)$ under the norm $\|\cdot\|_2$. In the sequel, for $A \subseteq \omega^{<\omega}$, we denote by $U_2(A)$ the closed subspace of $U_2(T)$ generated by $\{\chi_s : s \in A\}$. We recall some useful lemmas to understand the structure of those spaces introduced above. The next three lemmas are well known (see [4, Lemmas 1.3, 1.4, 1.5] and also [13]).

Lemma 46. *The sequence $\{\chi_s : s \in T\}$ determines a basis for $U_2(T)$. For any $A \subseteq T$, $\{\chi_s : s \in A\}$ determines a basis for $U_2(A)$.*

Lemma 47. *Let b be a branch of T. Then*

(i) *The space $U_2(b)$ is isomorphic to $C(2^\omega)$.*

(ii) *If $\theta \in \mathcal{T}$ and if b is a branch of θ, then $U_2(b)$ is complemented in $U_2(\theta)$.*

Lemma 48. *Let $(A_i)_{i\in\omega}$ be a sequence of subsets of T such that every branch of T meets at most one of these subsets. Then the spaces*

$$U_2\Big(\bigcup_{i\in\omega} A_i\Big) \quad \textit{and} \quad \Big(\bigoplus_{i\in\omega} U_2(A_i)\Big)_{\ell_2}$$

are isometric.

5.2. Different sequences of first Baire classes. Coming back to our main ingredients, the following will be essential for the main result of the paper.

Theorem 49. *Let $\theta \in \mathcal{T}$.*
(i) *If θ is ill-founded, then $B_1^1(U_2(\theta)) \subsetneqq B_1(U_2(\theta))$;*
(ii) *If θ is well-founded, then $U_2(\theta) = B_1^1(U_2(\theta)) = B_1(U_2(\theta))$.*

Proof. (i) If θ is ill founded, we pick b a branch of θ. By Lemma 47, $U_2(\theta)$ contains a complemented copy of $U_2(b) \simeq C(2^\omega)$. By Corollary 45 and Proposition 25, it follows that $B_1^1(U_2(\theta)) \subsetneqq B_1(U_2(\theta))$.

(ii) For $\theta \in \mathcal{T}$, $s \in T$ and $i \in \omega$, we define

$$s^\frown\theta = \{s^\frown t : t \in \theta\}.$$

Since $U_2(\theta) = U_2(\emptyset^\frown\theta)$, to prove the theorem it is enough to show the following

Claim. *If θ is well-founded, then for any $s \in T$, $U_2(s^\frown\theta)$ is reflexive.*

We will show the Claim using transfinite induction on $o(\theta)$.

We assume that for every tree $\tau \in \mathcal{T}$ such that $o(\tau) < \alpha < \omega_1$, $U_2(s^\frown\tau)$ is reflexive for any $s \in T$.

Let $\theta \in \mathcal{T}$ such that $o(\theta) = \alpha$, and for $s \in T$ let $N_s = \{i \in \omega : s^\frown(i) \in \theta\}$. We let $A_i = s^\frown(i)^\frown\theta(i)$ for $i \in N_s$, so that $\bigcup_{i\in N_s} A_i = (s^\frown\theta) \setminus \{s\}$ and every branch of T meets at most one of the A_i's. If $i \in N_s$, by Proposition 7 we get $o(\theta(i)) < \alpha$, thus $U_2(A_i)$ is reflexive by the induction hypothesis. By Lemma 48, we have

$$U_2((s^\frown\theta) \setminus \{s\}) = U_2\Big(\bigcup_{i\in N_s} A_i\Big) = \Big(\bigoplus_{i\in N_s} U_2(A_i)\Big)_{\ell_2},$$

and thus $U_2((s^\frown\theta) \setminus \{s\})$ is reflexive.

Since $\{\chi_{s_j} : j \in \omega,\ \ s_j \in s^\frown\theta\}$ is a basis of $U_2(s^\frown\theta)$ with the first elements χ_s and the other element generate $U_2((s^\frown\theta) \setminus \{s\})$. Then, we have that $U_2(s^\frown\theta) \cong \mathbb{R} \times U_2((s^\frown\theta) \setminus \{s\})$. Therefore $U_2(s^\frown\theta)$ is reflexive. Since reflexive, it easily follows $B_1^1(U_2(\theta)) = B_1(U_2(\theta))$ (which are both equal to $U_2(\theta)$!). $\square$

Similarly, one gets

THEOREM 50. *Let $\theta \in \mathcal{T}$.*
(i) *If θ is ill-founded, then $U_2(\theta) \subsetneqq B_1^0(U_2(\theta)) \subsetneqq B_1(U_2(\theta))$;*
(ii) *If θ is well-founded, then $U_2(\theta) = B_1^0(U_2(\theta)) = B_1(U_2(\theta))$.*

Another main ingredient for our purpose is the following.

LEMMA 51 [4, Lemma 2.4]. *The map $\varphi\colon \mathcal{T} \longrightarrow \mathcal{SE}$ defined by*

$$\varphi(\theta) = U_2(\theta)$$

is Borel.

We are ready to state the main theorem of this note.

THEOREM 52. *The family of all separable Banach spaces X such that $B_1^1(X) \subsetneqq B_1(X)$ is Σ_1^1-hard. In particular it cannot be Borel in $\mathcal{SB}$. Similarly $\{X \in \mathcal{SB}\colon B_0^1(X) \subsetneqq B_1(X)\}$ is Σ_1^1-hard.*

PROOF. Let us denote by $\mathcal{F}$ such a family. If $\mathcal{F}$ was bot Σ_1^1-hard, then $\mathcal{IF} \subseteq \varphi^{-1}(\mathcal{F})$, and the inclusion has to be strict. Indeed, $\mathcal{IF}$ is Σ_1^1-complete and thus Σ_1^1-hard, while $\varphi^{-1}(\mathcal{F})$ does not, by our assumption. Therefore, there must exist $\theta \in \mathcal{WF}$ such that $\varphi(\theta) \in \mathcal{F}$. This is in contrast with Theorem 49(*ii*). For the second part, just use Theorem 50 instead. $\square$

Analogously, we obtain

THEOREM 53. *The family of all separable Banach spaces X such that $X \subsetneqq B_1^1(X)$, as well as the family of all separable Banach spaces X such that $X \subsetneqq B_1^0(X)$ are Σ_1^1-hard.*

Finally, we would like to observe the following.

THEOREM 54. *The family of all separable Banach spaces X such that $B_0^1(X) = B_1^1(X)$ is Π_3^1.*

PROOF. Let us denote by $\mathcal{F}$ such a family. The equality $B_0^1(X) = B_1^1(X)$ holds if and only if for every $(x_n)_n$ which weak* converges to some $x^{**} \in B_1^1(X)$ such that $\overline{\text{span}}\{x_n : n \in \omega\} \not\supseteq \ell_1$, exists $(y_n)_n$ such that $w^*\text{-}\lim_n x_n = w^*\text{-}\lim_n y_n$ and $\overline{\text{span}}\{y_n : n \in \omega\}$ has separable dual.

Let

$$B = \Big\{(X, (x_n)_n, (y_n)_n) \in \mathcal{SB} \times C(2^\omega)^\omega \times C(2^\omega)^\omega :$$

$$(x_n)_n, (y_n)_n \subseteq X, (x_n)_n, (y_n)_n \in \mathcal{W}(X),$$

$$w^*\text{-}\lim(x_n)_n = w^*\text{-}\lim(y_n)_n,\ (x_n)_n \in [G_X^{-1}(C_{\ell_1})]^C,\ (y_n)_n \in G_X^{-1}(\mathcal{SD})\Big\}$$

is coanalytic, thus

$$C = \{ (X, (x_n)_n) \colon \exists (y_n)_n \in X^\omega \text{ such that } (X, (x_n)_n, (y_n)_n) \in B \}$$

is Σ^1_2, hence

$$\mathcal{F} = \{ X \in \mathcal{SB} \colon \ (X, (x_n)_n) \in C, \forall (x_n)_n \in X^\omega \}$$

is Π^1_3. $\square$

5.3. Final comments and open questions. In [16] Odell constructed a Banach space X with Schauder basis $(f_n)_n$ such that the closed linear span of $(f_n)_n$ does not contain ℓ_1 and for any strictly increasing sequence $(k_n)_n$ of integers, then the dual of $\overline{\operatorname{span}}\{f_{k_n} : n \in \mathbb{N}\}$ is not separable. It seems that the following question would be open.

QUESTION 55. *Is there a non-reflexive separable Banach space X not containing ℓ_1, such that for any sequence $(x_n)_n \subseteq X$ with $w^*\text{-}\lim_n x_n = x^{**} \in X^{**} \setminus X$, then the dual of $\overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$ is not separable?*

Finally, related to Theorem 54, the following would be quite natural to ask:

QUESTION 56. *Is the family of all separable Banach spaces X such that $B^1_0(X) = B^1_1(X)$ coanalytic?*

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Funding Open access funding provided by Università degli Studi di Catania within the CRUI-CARE Agreement