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OSCILLATION CRITERION FOR GENERALIZED EULER DIFFERENCE EQUATIONS

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Abstract. Using a modification of the adapted Riccati transformation, we prove an oscillation criterion for generalizations of linear and half-linear Euler difference equations. Our main result complements a large number of previously known oscillation criteria about several similar generalizations of Euler difference equations. The major part of this paper is formed by the proof of the main theorem. To illustrate the fact that the presented criterion is new even for linear equations with periodic coefficients, we finish this paper with the corresponding corollary together with concrete examples of simple equations whose oscillatory properties do not follow from previously known criteria.

1. Introduction

1.1. Treated equations. In this paper, we study the oscillation of linear and half-linear difference equations which generalize the famous Euler equation. At first, we recall that the half-linear difference equation is an equation in the form

$$\Delta\big(c_k\Phi(\Delta x_k)\big) + d_k\Phi(x_{k+1}) = 0, \tag{1.1}$$

where $\Phi(x) := |x|^{p-1}\operatorname{sgn} x$ for an arbitrarily given $p > 1$ and $c_k > 0$, $d_k \in \mathbb{R}$ for all considered $k \in \mathbb{N}$. It is seen that (1.1) is a linear equation if and only if $p = 2$. In particular, it suffices to formulate results for $p > 1$ (linear equations are covered by the half-linear ones).

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In this paragraph, we mention the needed basic definitions from the oscillation theory of (1.1). We say that an interval $(l, l+1]$ (where $l \in \mathbb{N}$ is sufficiently large) contains the generalized zero of a solution $\{x_k\}$ of (1.1) if $x_l \neq 0$ and if $x_l x_{l+1} \leq 0$. Equation (1.1) is called disconjugate on $\{l, l+1, \ldots, l+j\}$ (where $j \in \mathbb{N}$) if any solution of (1.1) has at most one generalized zero on $(l, l+j+1]$ and any solution $\{x_k\}$ fulfilling $x_l = 0$ has no generalized zero on $(l, l+j+1]$. In the other case, (1.1) is called conjugate on $\{l, l+1, \ldots, l+j\}$. Finally, (1.1) is called non-oscillatory if there exists $l \in \mathbb{N}$ such that the equation is disconjugate on $\{l, l+1, \ldots, l+j\}$ for all $j \in \mathbb{N}$. In the other case, (1.1) is called oscillatory.

From the famous Sturm separation theorem (see, e.g., [3, Theorem 3.3.6]), it follows that either any non-trivial solution of (1.1) has infinitely many generalized zeros or none on a neighborhood of infinity. Therefore, (1.1) is oscillatory if its non-trivial solution has infinitely many generalized zeros. Especially, in the oscillation theory, it suffices to consider all equations only for $k \in \mathbb{N}_a := \{l \in \mathbb{N};\, l \geq a\}$, where $a \in \mathbb{N}$ is a sufficiently large integer. Note that we consider $a \geq p$. We will also use the standard notation concerning the conjugate numbers p, q, i.e., let $q > 1$ be such that

$$p + q = pq. \tag{1.2}$$

Then, the inverse function of $\Phi(x) = |x|^{p-1} \operatorname{sgn} x$ is $\Phi^{-1}(x) = |x|^{q-1} \operatorname{sgn} x$.

We analyze generalizations of the Euler half-linear difference equation whose coefficients are given by asymptotically periodic sequences and the generalized power function. Thus, we recall these concepts in the following two definitions.

Definition 1. We say that a sequence $\{r_k\}_{k \in \mathbb{N}_a}$ is asymptotically periodic if there exist sequences $\{r_k^1\}_{k \in \mathbb{N}_a}$, $\{r_k^2\}_{k \in \mathbb{N}_a}$ such that $r_k = r_k^1 + r_k^2$, $k \in \mathbb{N}_a$, where $\{r_k^1\}_{k \in \mathbb{N}_a}$ is periodic and $\{r_k^2\}_{k \in \mathbb{N}_a}$ satisfies $\lim_{k\to\infty} r_k^2 = 0$. The sequence $\{r_k^1\}_{k \in \mathbb{N}_a}$ is called the periodic part of the sequence $\{r_k\}_{k \in \mathbb{N}_a}$.

Definition 2. The generalized power function is defined as

$$k^{(s)} := \begin{cases} 1 & \text{for } s = 0; \\ k(k-1)\cdots(k-s+1) & \text{for } s \in \mathbb{N}; \\ \frac{1}{(k+1)(k+2)\cdots(k-s)} & \text{for } -s \in \mathbb{N}; \\ \frac{\Gamma(k+1)}{\Gamma(k-s+1)} & \text{for } s \notin \mathbb{Z}, \end{cases}$$

where $k \in \mathbb{N}$ is sufficiently large and Γ denotes the Euler gamma function

$$\Gamma(x) := \int_0^\infty \mathrm{e}^{-s} s^{x-1} \, \mathrm{d}s, \quad x > 0.$$

Now, we can explicitly mention the form of studied equations whose oscillation is analyzed. We consider the Euler type half-linear difference equations

$$\Delta\Big(\frac{r_k^{1-p}}{k}\,\Phi(\Delta x_k)\Big) + \frac{s_k}{(k+1)^{(p+1)}}\,\Phi(x_{k+1}) = 0, \quad k \in \mathbb{N}_a, \tag{1.3}$$

where $\{r_k\}_{k\in\mathbb{N}_a}$ is an asymptotically periodic sequence with the property that $\inf_{k\in\mathbb{N}_a} r_k > 0$ and $\{s_k\}_{k\in\mathbb{N}_a}$ is bounded. Considering Definition 1, one can put

$$S(r_k) := \sup_{k\in\mathbb{N}_a} r_k < \infty, \quad I(r_k) := \inf_{k\in\mathbb{N}_a} r_k > 0, \quad S(s_k) := \sup_{k\in\mathbb{N}_a} |s_k| < \infty. \tag{1.4}$$

Let $n \in \mathbb{N}$ be a period of the periodic part of $\{r_k\}_{k\in\mathbb{N}_a}$. We consider the situation when there exists $\varepsilon > 0$ such that the inequality

$$\frac{1}{n}\sum_{i=k}^{k+n-1} s_i > \varepsilon + \Big(\frac{1}{n}\sum_{i=k}^{k+n-1} r_i\Big)^{1-p} \tag{1.5}$$

is valid for all large $k \in \mathbb{N}$. Henceforward, we will consider (1.3) with coefficients given by an asymptotically periodic sequence $\{r_k\}_{k\in\mathbb{N}_a}$ satisfying $\inf_{k\in\mathbb{N}_a} r_k > 0$ and having periodic part with period n and by a bounded sequence $\{s_k\}_{k\in\mathbb{N}_a}$ such that (1.5) is true for some $\varepsilon > 0$ and all large $k \in \mathbb{N}$.

1.2. Previous results. We highlight that the strongest known oscillation criteria about the studied generalizations of Euler (linear and half-linear) difference equations are proved in [15,20,22,24,25,34,36,51,56,58] (for special cases, we refer to [13,18,35,55] as well). Some of the most relevant results are explicitly mentioned below.

THEOREM 1 [51]. *Let us consider the equation*

$$\Delta\big(k r_k^{1-p}\,\Phi(\Delta x_k)\big) + \frac{s_k}{(k+1)^{(p-1)}}\,\Phi(x_{k+1}) = 0, \quad k \in \mathbb{N}_a, \tag{1.6}$$

where $p > 2$, $\{r_k\}_{k\in\mathbb{N}_a}$ is an asymptotically periodic sequence satisfying $\inf_{k\in\mathbb{N}_a} r_k > 0$ whose periodic part has period $n \in \mathbb{N}$, and $\{s_k\}_{k\in\mathbb{N}_a}$ is a positive and bounded sequence. If there exists $\varepsilon > 0$ with the property that

$$\frac{1}{n}\sum_{i=k}^{k+n-1} s_i > \varepsilon + \Big(\frac{p-2}{p}\Big)^p\Big(\frac{1}{n}\sum_{i=k}^{k+n-1} r_i\Big)^{1-p}$$

holds for all large $k \in \mathbb{N}$, then (1.6) is oscillatory.

THEOREM 2 [34,36]. *Let us consider the equation*

$$\Delta\big(r_k^{1-p}\,\Phi(\Delta x_k)\big)+\frac{s_k}{k^p}\,\Phi(x_{k+1})=0,\quad k\in\mathbb{N}, \tag{1.7}$$

where $\{r_k\}_{k\in\mathbb{N}}, \{s_k\}_{k\in\mathbb{N}}$ *are positive sequences. Let* $n\in\mathbb{N}$ *be such that*

$$\sum_{i=kn+1}^{(k+1)n} r_i=\sum_{i=1}^{n} r_i,\qquad \sum_{i=kn+1}^{(k+1)n} s_i=\sum_{i=1}^{n} s_i,\quad k\in\mathbb{N}.$$

Equation (1.7) *is oscillatory if and only if*

$$\frac{1}{n}\sum_{i=1}^{n} s_i>\left(\frac{p-1}{p}\right)^p\left(\frac{1}{n}\sum_{i=1}^{n} r_i\right)^{1-p}.$$

THEOREM 3 [22]. *Let us consider the equation*

$$\Delta\left(\frac{r_k^{1-p}}{k}\,\Phi(\Delta x_k)\right)+\frac{s_k}{(k+1)^{p+1}}\,\Phi(x_{k+1})=0,\quad k\in\mathbb{N}, \tag{1.8}$$

where $\{r_k\}_{k\in\mathbb{N}}$, $\{s_k\}_{k\in\mathbb{N}}$ *satisfy*

$$0<\inf_{k\in\mathbb{N}} r_k\le\sup_{k\in\mathbb{N}} r_k<\infty,\quad 0<\inf_{k\in\mathbb{N}} s_k\le\sup_{k\in\mathbb{N}} s_k<\infty.$$

- *If there exist* $N,n\in\mathbb{N}$ *such that*

$$1>\sup_{k\in\mathbb{N}_N}\left(\frac{1}{n}\sum_{i=k}^{k+n-1} r_i\right)^{1-p}\left(\frac{1}{n}\sum_{i=k}^{k+n-1} s_i\right)^{-1},$$

then (1.8) *is oscillatory.*

- *If there exist* $N,n\in\mathbb{N}$ *such that*

$$1<\inf_{k\in\mathbb{N}_N}\left(\frac{1}{n}\sum_{i=k}^{k+n-1} r_i\right)^{1-p}\left(\frac{1}{n}\sum_{i=k}^{k+n-1} s_i\right)^{-1},$$

then (1.8) *is non-oscillatory.*

We use the theorems above to describe the goal of our paper. We study equations whose second coefficients can change sign (note that the coefficients of the analyzed equations in Theorems 1–3 have to be positive). We obtain an oscillation criterion similar to Theorem 1 or Theorem 2 which are proved for different generalizations of the Euler half-linear difference equation. The presented criterion improves in a certain sense the oscillatory part of Theorem 3 whose non-oscillatory part shows that our result cannot be substantially generalized. We point out that, for (1.3), all previously applied processes cannot be used if $\{s_k\}_{k\in\mathbb{N}_a}$ can change its sign. This fact documents the novelty of our approach.

Fundamental results about the oscillation of linear and half-linear difference equations are mentioned, e.g., in [3,7]. For other relevant oscillation criteria, we refer to [1,2,8,11,28,30,38,39,41,43–46,52–54]. The oscillation of the perturbed Euler type difference equations is studied in [29,59] (see also [33]). The oscillation of the Euler type dynamic equations on time scales is analyzed in [14,26,27,57] (see also [31,32]).

The oscillation of linear and half-linear difference equations is primarily motivated by its continuous counterpart, i.e., by oscillation criteria for the corresponding differential equations. The most relevant results about generalizations of the corresponding Euler linear and half-linear differential equations are proved in [6,12,16,23,48,49] (see also [5,17,47,50]). The oscillation of perturbations of the Euler type differential equations is studied, e.g., in [4,9,10,21,40,42].

1.3. Riccati transformation. To prove the main result, we use a modification of the Riccati transformation. From (1.1), applying the basic Riccati transformation

$$w_k = c_k\Phi\Big(\frac{\Delta x_k}{x_k}\Big),$$

we obtain the so-called Riccati equation

$$\Delta w_k + d_k + w_k\Big(1 - \frac{c_k}{\Phi[\Phi^{-1}(c_k) + \Phi^{-1}(w_k)]}\Big) = 0.$$

Considering [3, Lemma 3.2.6, (I_8)] for $w_k + c_k > 0$, one can express the Riccati equation associated to (1.1) in the form

$$\Delta w_k + d_k + \frac{(p-1)\,|w_k|^q\,|\alpha_k|^{p-2}}{\Phi[\Phi^{-1}(c_k) + \Phi^{-1}(w_k)]} = 0,$$

where α_k is between $\Phi^{-1}(c_k)$ and $\Phi^{-1}(c_k) + \Phi^{-1}(w_k)$ for all considered $k \in \mathbb{N}$, i.e., $k \in \mathbb{N}_a$.

We recall that we analyze (1.1) for

$$c_k = \frac{r_k^{1-p}}{k}, \quad d_k = \frac{s_k}{(k+1)^{(p+1)}}, \quad k \in \mathbb{N}_a,$$

i.e., we consider the Riccati equation in the form

$$\Delta w_k + \frac{s_k}{(k+1)^{(p+1)}} + \frac{(p-1)|w_k|^q|\alpha_k|^{p-2}}{\Phi[\Phi^{-1}(k^{-1}r_k^{1-p}) + \Phi^{-1}(w_k)]} = 0, \tag{1.9}$$

where α_k is between $\Phi^{-1}(k^{-1}r_k^{1-p})$ and $\Phi^{-1}(k^{-1}r_k^{1-p})+\Phi^{-1}(w_k)$.

The non-oscillation of (1.3) is connected with the solvability of (1.9) as follows.

THEOREM 4 [3, Theorem 3.3.4]. *Equation (1.3) is non-oscillatory if and only if there exists a solution $\{w_k\}_{k\in\mathbb{N}_b}$ of (1.9) satisfying $w_k+k^{-1}r_k^{1-p}>0$, $k\in\mathbb{N}_b$.*

Finally, we mention the announced modification of the half-linear Riccati transformation for $k\in\mathbb{N}_a$. Putting

$$\zeta_k=-k^{(p)}w_k, \tag{1.10}$$

from (1.9), we obtain

$$\begin{aligned}\Delta\zeta_k&=-pk^{(p-1)}w_k-(k+1)^{(p)}\Delta w_k\\&=pk^{(p-1)}\frac{\zeta_k}{k^{(p)}}+(k+1)^{(p)}\frac{s_k}{(k+1)^{(p+1)}}\\&+(k+1)^{(p)}\frac{(p-1)|\zeta_k|^q(k^{(p)})^{-q}|\alpha_k|^{p-2}}{\Phi\left[\Phi^{-1}(k^{-1}r_k^{1-p})+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p)}}\right)\right]}.\end{aligned}$$

From Definition 2, it follows

$$\frac{k^{(p-1)}}{k^{(p)}}=\frac{(k+1)^{(p)}}{(k+1)^{(p+1)}}=\frac{1}{k-p+1},\qquad\frac{(k+1)^{(p+1)}}{k^{(p)}}=k+1$$

for all considered large $k\in\mathbb{N}$. Thus, for $k\in\mathbb{N}_a$, the modified Riccati equation associated to (1.3) takes the form

$$\begin{aligned}\Delta\zeta_k&=\frac{1}{k-p+1}\left[p\zeta_k+s_k+(k+1)^{(p+1)}\frac{(p-1)|\zeta_k|^q(k^{(p)})^{-q}|\alpha_k|^{p-2}}{\Phi\left[\Phi^{-1}(k^{-1}r_k^{1-p})+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p)}}\right)\right]}\right]\\&=\frac{1}{k-p+1}\left[p\zeta_k+s_k+\frac{(k+1)(k^{(p)})^{1-q}(p-1)|\zeta_k|^q|\alpha_k|^{p-2}}{\Phi\left[\Phi^{-1}\left(k^{-1}r_k^{1-p}\right)+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p)}}\right)\right]}\right],\end{aligned}$$

i.e.,

$$\Delta\zeta_k=\frac{1}{k-p+1}\left[p\zeta_k+s_k+\frac{(k+1)(p-1)|\zeta_k|^q|\alpha_k|^{p-2}}{(k^{(p)})^{q-1}\Phi\left[\Phi^{-1}(k^{-1}r_k^{1-p})+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p)}}\right)\right]}\right], \tag{1.11}$$

where α_k is between $\Phi^{-1}\left(k^{-1}r_k^{1-p}\right)$ and $\Phi^{-1}\left(k^{-1}r_k^{1-p}\right)+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p)}}\right)$.

2. Oscillation criterion with consequences

At first, we prove two auxiliary results.

LEMMA 1. *If* (1.3) *is non-oscillatory, then there exists a negative solution* $\{\zeta_k\}_{k\in\mathbb{N}_b}$ *of* (1.11) *such that*

$$\lim_{k\to\infty} \frac{\zeta_k}{k^{p-1}} = 0. \tag{2.1}$$

PROOF. From Theorem 4, it follows that the non-oscillation of (1.3) implies the existence of a solution $\{w_k\}_{k\in\mathbb{N}_b}$ of (1.9) for which $w_k + k^{-1}r_k^{1-p} > 0$, $k \in \mathbb{N}_b$, where $b \in \mathbb{N}_a$. Using (1.10), we obtain the corresponding solution $\{\zeta_k\}_{k\in\mathbb{N}_b} \equiv \{-k^{(p)}w_k\}_{k\in\mathbb{N}_b}$ of (1.11). It suffices to prove that this solution $\{\zeta_k\}_{k\in\mathbb{N}_b}$ is negative and satisfies (2.1).

From known results (see, e.g., [7, Theorems 8.2.8 and 8.2.10]), it follows that

$$\lim_{k\to\infty} w_k = 0. \tag{2.2}$$

Actually, we show that the solution $\{w_k\}_{k\in\mathbb{N}_b}$ of (1.9) satisfies

$$\lim_{k\to\infty} kw_k = 0. \tag{2.3}$$

We consider the sequences $\{w_{nk}\}$, $\{w_{nk+1}\}$, ..., $\{w_{nk+n-1}\}$ for large $k \in \mathbb{N}$. In fact, we choose $j \in \{0, 1, \dots, n-1\}$ arbitrarily and consider the sequence $\{w_{nk+j}\}$ for large $k \in \mathbb{N}$.

From (1.5) and the positivity of $\{r_k\}_{k\in\mathbb{N}_a}$, we have

$$\sum_{i=nk+j}^{n(k+1)+j-1} s_i > \varepsilon$$

for all large $k \in \mathbb{N}$ and, from Definition 2, we have

$$\lim_{m\to\infty} \frac{(m+i)^{(p+1)}}{m^{(p+1)}} = 1, \quad i \in \{1, 2, \dots, n-1\}.$$

Hence, (1.4) yields

$$\sum_{i=nk+j}^{n(k+1)+j-1} \frac{s_i}{(i+1)^{(p+1)}} > 0 \tag{2.4}$$

for all large $k \in \mathbb{N}$. At the same time, $w_k + k^{-1}r_k^{1-p} > 0$ implies

$$\Phi\big[\Phi^{-1}\big(k^{-1}r_k^{1-p}\big) + \Phi^{-1}(w_k)\big] > 0 \tag{2.5}$$

for all $k \in \mathbb{N}_b$. Considering (1.9) together with (2.4) and (2.5), for all large $k \in \mathbb{N}$, we have

$$w_{n(k+1)+j} - w_{nk+j} = - \sum_{i=nk+j}^{n(k+1)+j-1} \frac{s_i}{(i+1)^{(p+1)}} - \sum_{i=nk+j}^{n(k+1)+j-1} \frac{(p-1)|w_i|^q|\alpha_i|^{p-2}}{\Phi\big[\Phi^{-1}(i^{-1}r_i^{1-p}) + \Phi^{-1}(w_i)\big]} < 0, \tag{2.6}$$

i.e., the sequence $\{w_{nk+j}\}$ is decreasing for considered large $k \in \mathbb{N}$. In addition, the inequality $w_k + k^{-1}r_k^{1-p} > 0$ and (1.4) guarantee the positivity of the sequence $\{w_{nk+j}\}$. Indeed, if w_{nk_0+j} is negative for some large $k_0 \in \mathbb{N}$, then

$$\liminf_{k\to\infty}(w_k + k^{-1}r_k^{1-p}) = \liminf_{k\to\infty} w_k < w_{nk_0+j} < 0,$$

which is a contradiction (see also (2.2)). We have proved that $\{w_{nk+j}\}$ is decreasing and positive. In particular, the sequence $\{\zeta_k\}_{k\in\mathbb{N}_b}$ is negative.

Our aim is to prove

$$\lim_{k\to\infty} (nk+j)w_{nk+j} = 0 \tag{2.7}$$

which implies (2.3) (consider that $j \in \{0, 1, \dots, n-1\}$ is arbitrarily given). From (2.4) and (2.6), we have

$$(n(k+1)+j)w_{n(k+1)+j} - (nk+j)w_{nk+j} < nw_{n(k+1)+j} - (nk+j) \sum_{i=nk+j}^{n(k+1)+j-1} \frac{(p-1)|w_i|^q|\alpha_i|^{p-2}}{\Phi\big[\Phi^{-1}(i^{-1}r_i^{1-p}) + \Phi^{-1}(w_i)\big]}. \tag{2.8}$$

Let us assume that the inequality

$$(nk+j)w_{nk+j} > \vartheta \tag{2.9}$$

holds for some large $k \in \mathbb{N}$ and some $\vartheta > 0$. We put

$$\Omega_k := w_{nk+j}\left(\frac{r_{nk+j}^{1-p}}{\vartheta} + 1\right) \tag{2.10}$$

for large $k \in \mathbb{N}$. Taking into account (1.2), (1.4), (2.5), (2.9), and (2.10), we have

$$(nk+j) \sum_{i=nk+j}^{n(k+1)+j-1} \frac{(p-1)|w_i|^q|\alpha_i|^{p-2}}{\Phi\big[\Phi^{-1}(i^{-1}r_i^{1-p}) + \Phi^{-1}(w_i)\big]}$$

$$\geq (nk+j)\frac{(p-1)|w_{nk+j}|^q|\alpha_{nk+j}|^{p-2}}{\Phi\big[\Phi^{-1}\big((nk+j)^{-1}r_{nk+j}^{1-p}\big)+\Phi^{-1}(w_{nk+j})\big]}$$

$$= (nk+j)\frac{(p-1)|w_{nk+j}|^q|\alpha_{nk+j}|^{p-1}}{\Phi\big[\Phi^{-1}\big((nk+j)^{-1}r_{nk+j}^{1-p}\big)+\Phi^{-1}(w_{nk+j})\big]\,|\alpha_{nk+j}|}$$

$$\geq (nk+j)\frac{(p-1)|w_{nk+j}|^q\big(\Phi^{-1}((nk+j)^{-1}r_{nk+j}^{1-p})\big)^{p-1}}{\Phi\big[\Phi^{-1}\big((nk+j)^{-1}r_{nk+j}^{1-p}\big)+\Phi^{-1}(w_{nk+j})\big]\,|\alpha_{nk+j}|}$$

$$= \frac{(p-1)|w_{nk+j}|^q r_{nk+j}^{1-p}}{\Phi\big[\Phi^{-1}\big((nk+j)^{-1}r_{nk+j}^{1-p}\big)+\Phi^{-1}(w_{nk+j})\big]\,|\alpha_{nk+j}|}$$

$$> \frac{(p-1)|w_{nk+j}|^q(S(r_k))^{1-p}}{\Phi\big[\Phi^{-1}\big(w_{nk+j}\vartheta^{-1}r_{nk+j}^{1-p}\big)+\Phi^{-1}(w_{nk+j})\big]}$$

$$\times \frac{1}{\big|\Phi^{-1}\big(w_{nk+j}\vartheta^{-1}r_{nk+j}^{1-p}\big)+\Phi^{-1}(w_{nk+j})\big|}$$

$$> \frac{(p-1)|w_{nk+j}|^q(S(r_k))^{1-p}}{\Phi\big[\Phi^{-1}(\Omega_k)+\Phi^{-1}(\Omega_k)\big]\,\big|\Phi^{-1}(\Omega_k)+\Phi^{-1}(\Omega_k)\big|}$$

$$= \frac{(p-1)|w_{nk+j}|^q(S(r_k))^{1-p}}{2^p\,\Omega_k\,\Phi^{-1}(\Omega_k)} = \frac{(p-1)w_{nk+j}^q(S(r_k))^{1-p}}{2^p\,w_{nk+j}^q\big(\frac{r_{nk+j}^{1-p}}{\vartheta}+1\big)^q}$$

$$\geq \frac{(p-1)(S(r_k))^{1-p}}{2^p\big(\frac{(I(r_k))^{1-p}}{\vartheta}+1\big)^q},$$

i.e., if (2.9) is valid, then

$$(nk+j)\sum_{i=nk+j}^{n(k+1)+j-1}\frac{(p-1)|w_i|^q|\alpha_i|^{p-2}}{\Phi\big[\Phi^{-1}(i^{-1}r_i^{1-p})+\Phi^{-1}(w_i)\big]} > \delta(\vartheta), \tag{2.11}$$

where

$$\delta(\vartheta) := \frac{p-1}{2^p(S(r_k))^{p-1}\big(\frac{(I(r_k))^{1-p}}{\vartheta}+1\big)^q} > 0$$

is constant for an arbitrarily given number $\vartheta > 0$.

Considering (2.8) and (2.11) (see also (2.2)), if (2.9) is valid for large $k \in \mathbb{N}$ and for some $\vartheta > 0$, then

$$(n(k+1)+j)\,w_{n(k+1)+j} - (nk+j)w_{nk+j} < -\frac{\delta(\vartheta)}{2} < 0.$$

We point out that $\{w_{nk+j}\}$ is positive and $\vartheta > 0$ is arbitrary in (2.9). Therefore, for large $k \in \mathbb{N}$, the obtained implication

$$(nk+j)w_{nk+j} > \eta \ \Rightarrow \ (n(k+1)+j)w_{n(k+1)+j} - (nk+j)w_{nk+j} < -\frac{\delta(\eta)}{2}$$

proves that (2.9) cannot be valid for infinitely many large $k \in \mathbb{N}$ for any $\vartheta = 2\eta > 0$ (consider (2.8) and the positivity of $\{(nk+j)w_{nk+j}\}$). We have proved (2.7) which implies (2.3).

Now, we use the well-known asymptotic equivalence of $k^{(\gamma)}$ and k^γ for $\gamma \geq 1$, i.e.,

$$\lim_{k\to\infty} \frac{k^\gamma}{k^{(\gamma)}} = 1, \quad \gamma \geq 1. \tag{2.12}$$

Note that (2.12) can be easily derived from Definition 2 (we can also refer to [37]). Finally, from (1.10), (2.3), and (2.12), we obtain

$$\lim_{k\to\infty} \frac{\zeta_k}{k^{p-1}} = \lim_{k\to\infty} \frac{-k^{(p)}w_k}{k^{p-1}} = \lim_{k\to\infty} -kw_k = 0,$$

i.e., (2.1) is satisfied as well. □

Lemma 2. *If (1.3) is non-oscillatory, then there exists a negative and bounded solution $\{\zeta_k\}_{k\in\mathbb{N}_b}$ of (1.11) such that*

$$\lim_{k\to\infty} \frac{\Phi^{-1}(k^{-1}r_k^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_k}{k^{(p)}}\big)}{\Phi^{-1}(k^{-1}r_k^{1-p})} = 1. \tag{2.13}$$

Proof. Lemma 1 guarantees the existence of a negative solution $\{\zeta_k\}_{k\in\mathbb{N}_b}$ of (1.11) satisfying (2.1). From (1.4), (2.1), and (2.12), we have (2.13). Therefore, it suffices to show that $\{\zeta_k\}_{k\in\mathbb{N}_b}$ is bounded from below.

It holds (see (1.4) and (1.11))

$$\Delta\zeta_k \geq \frac{1}{k-p+1}\left[p\zeta_k - S(s_k) + \frac{(k+1)(p-1)|\zeta_k|^q \alpha_k^{p-2}}{(k^{(p)})^{q-1}\Phi\big(\Phi^{-1}(k^{-1}r_k^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_k}{k^{(p)}}\big)\big)}\right] \tag{2.14}$$

for $k \in \mathbb{N}_b$, where

$$0 < \Phi^{-1}(k^{-1}r_k^{1-p}) \leq \alpha_k \leq \Phi^{-1}(k^{-1}r_k^{1-p}) + \Phi^{-1}\Big(-\frac{\zeta_k}{k^{(p)}}\Big), \quad k \in \mathbb{N}_b. \tag{2.15}$$

Considering (2.13) and (2.15), we obtain

$$\frac{(k^{-1}r_k^{1-p})^{2-q}}{\sqrt[3]{2}} < \alpha_k^{p-2} < \sqrt[3]{2}\big(k^{-1}r_k^{1-p}\big)^{2-q} \tag{2.16}$$

for all large $k \in \mathbb{N}$. Similarly, considering (2.13), we obtain

$$\frac{k^{-1} r_k^{1-p}}{\sqrt[3]{2}} < \Phi\Big(\Phi^{-1}(k^{-1} r_k^{1-p}) + \Phi^{-1}\Big(-\frac{\zeta_k}{k^{(p)}}\Big)\Big) < \sqrt[3]{2}\, k^{-1} r_k^{1-p} \tag{2.17}$$

for all large $k \in \mathbb{N}$. We also use

$$\lim_{k\to\infty} \frac{(k+1)k^{q-1}}{(k^{(p)})^{q-1}} = 1 \tag{2.18}$$

which follows from (1.2) and (2.12). Hence,

$$\frac{k^{1-q}}{\sqrt[3]{2}} < \frac{k+1}{(k^{(p)})^{q-1}} < \sqrt[3]{2}\, k^{1-q} \tag{2.19}$$

for all sufficiently large $k \in \mathbb{N}$. Thus, we have (see (1.4), (2.13), (2.15), and (2.18))

$$\begin{aligned} &\lim_{k\to\infty} \left(\frac{(k+1)\alpha_k^{p-2}}{(k^{(p)})^{q-1} \Phi\big(\Phi^{-1}(k^{-1} r_k^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_k}{k^{(p)}}\big)\big)} - r_k \right) \\ &= \lim_{k\to\infty} \left(\frac{(k+1)\big(\Phi^{-1}(k^{-1} r_k^{1-p})\big)^{p-2}}{(k^{(p)})^{q-1} k^{-1} r_k^{1-p}} - r_k \right) \\ &= \lim_{k\to\infty} \left(r_k \frac{(k+1)k^{q-1}}{(k^{(p)})^{q-1}} - r_k \right) = 0. \end{aligned} \tag{2.20}$$

Similarly as in (2.20), we obtain (see (1.4), (2.14), (2.16), (2.17), and (2.19))

$$\begin{aligned} \Delta\zeta_k &> \frac{1}{k-p+1}\left[p\zeta_k - S(s_k) + \frac{(k+1)(p-1)|\zeta_k|^q (k^{-1} r_k^{1-p})^{2-q}}{\sqrt[3]{4}(k^{(p)})^{q-1} k^{-1} r_k^{1-p}} \right] \\ &= \frac{1}{k-p+1}\left[p\zeta_k - S(s_k) + \frac{(k+1)(p-1)|\zeta_k|^q (k^{-1} r_k^{1-p})^{1-q}}{\sqrt[3]{4}(k^{(p)})^{q-1}} \right] \\ &> \frac{1}{k-p+1}\left[p\zeta_k - S(s_k) + \frac{(p-1) r_k |\zeta_k|^q k^{q-1}}{\sqrt[3]{8}\, k^{q-1}} \right] \\ &\geq \frac{1}{k-p+1}\left[p\zeta_k - S(s_k) + \frac{(p-1) I(r_k) |\zeta_k|^q}{2} \right] \end{aligned} \tag{2.21}$$

for large $k \in \mathbb{N}$. Let

$$\zeta_k \leq -\left(\frac{2(p + S(s_k))}{(p-1) I(r_k)} \right)^{\frac{1}{q-1}} - 1$$

for some large $k \in \mathbb{N}$. Then,

$$\frac{(p-1)I(r_k)\,|\zeta_k|^q}{2} = \frac{(p-1)I(r_k)|\zeta_k| \cdot |\zeta_k|^{q-1}}{2}$$
$$\geq p|\zeta_k| + S(s_k)|\zeta_k| \geq p|\zeta_k| + S(s_k)$$

which yields

$$\Delta\zeta_k > \frac{1}{k-p+1}\left[p\zeta_k - S(s_k) + \frac{(p-1)I(r_k)|\zeta_k|^q}{2}\right] \geq 0.$$

Analogously as in (2.21), we have

$$|\Delta\zeta_k| < \frac{1}{k-p+1}\left[p|\zeta_k| + S(s_k) + 2(p-1)S(r_k)|\zeta_k|^q\right]$$

for large $k \in \mathbb{N}$. Let

$$0 > \zeta_k > -\left(\frac{2(p+S(s_k))}{(p-1)I(r_k)}\right)^{\frac{1}{q-1}} - 1$$

for some large $k \in \mathbb{N}$. If we denote

$$C := \left(\frac{2(p+S(s_k))}{(p-1)I(r_k)}\right)^{\frac{1}{q-1}} + 1,$$

then

$$|\Delta\zeta_k| < \frac{1}{k-p+1}\left[pC + S(s_k) + 2(p-1)S(r_k)C^q\right].$$

We recapitulate the estimations mentioned above. For large $k \in \mathbb{N}$, we have proved that $\Delta\zeta_k$ is positive if $\zeta_k \leq -C$, and $|\Delta\zeta_k| < D/(k-p+1)$ for some $D > 0$ if $\zeta_k > -C$. Thus, the considered solution $\{\zeta_k\}_{k\in\mathbb{N}_b}$ has to be bounded from below. □

The main result of this paper reads as follows.

THEOREM 5. *Let us consider* (1.3), *where* $\{r_k\}_{k\in\mathbb{N}_a} \subset (0,\infty)$ *is an asymptotically periodic sequence with period* $n \in \mathbb{N}$ *of its periodic part satisfying* $\liminf_{k\to\infty} r_k > 0$ *and* $\{s_k\}_{k\in\mathbb{N}_a}$ *is a bounded sequence. If* (1.5) *is valid for some* $\varepsilon > 0$ *and for all large* $k \in \mathbb{N}$, *then* (1.3) *is oscillatory.*

PROOF. On contrary, let us consider that (1.5) is true for some $\varepsilon > 0$ and for all $k \in \mathbb{N}_a$ and that (1.3) is non-oscillatory. We remark that we assume the validity of (1.5) for all $k \in \mathbb{N}_a$ for the sake of simplicity. From Lemma 2, we know that there exists a negative and bounded solution $\{\zeta_k\}_{k\in\mathbb{N}_b}$ of (1.11) satisfying (2.13), where $b \in \mathbb{N}_a$. Especially, there exists $M > 0$ with the property that

$$\zeta_k \in (-M, 0), \quad k \in \mathbb{N}_b. \tag{2.22}$$

Similarly as in (2.21), we can again use (2.16), (2.17), and (2.19). Thus, we have (consider also (1.11) together with (1.4) and (2.22))

$$\begin{aligned} \Delta\zeta_k &< \frac{1}{k-p+1}\left[0 + S(s_k) + \frac{\sqrt[3]{4}(k+1)(p-1)M^q(k^{-1}r_k^{1-p})^{2-q}}{(k^{(p)})^{q-1}k^{-1}r_k^{1-p}}\right] \\ &= \frac{1}{k-p+1}\left[S(s_k) + \frac{\sqrt[3]{4}(k+1)(p-1)M^q(k^{-1}r_k^{1-p})^{1-q}}{(k^{(p)})^{q-1}}\right] \\ &< \frac{1}{k-p+1}\left[S(s_k) + \frac{\sqrt[3]{8}(p-1)r_k M^q k^{q-1}}{k^{q-1}}\right] \\ &\le \frac{1}{k-p+1}\left[S(s_k) + 2(p-1)S(r_k)M^q\right] \end{aligned} \tag{2.23}$$

and, at the same time,

$$\Delta\zeta_k > \frac{1}{k-p+1}\left[p\zeta_k + s_k + 0\right] > \frac{1}{k-p+1}\left[-pM - S(s_k)\right] \tag{2.24}$$

for all large $k \in \mathbb{N}$. Combining (2.23) and (2.24), we obtain

$$|\Delta\zeta_k| < \frac{S(s_k) + 2(p-1)S(r_k)M^q + pM}{k-p+1}$$

for all large $k \in \mathbb{N}$, i.e., there exists $N > 0$ for which

$$|\Delta\zeta_k| \le \frac{N}{k-p+1}, \quad k \in \mathbb{N}_b.$$

Consequently,

$$|\zeta_{k+i} - \zeta_{k+j}| \le \frac{(n-1)N}{k-p+1}, \quad i, j \in \{0, 1, \dots, n-1\}, \ k \in \mathbb{N}_b. \tag{2.25}$$

For $k \in \mathbb{N}_b$, we put

$$\xi_k := \frac{1}{n}\sum_{i=k}^{k+n-1} \zeta_i, \tag{2.26}$$

$$A_k := p\left(\frac{p}{n}\sum_{i=k}^{k+n-1} r_i\right)^{-\frac{1}{q}}, \tag{2.27}$$

$$B_k := -\xi_k \left(\frac{p}{n} \sum_{i=k}^{k+n-1} r_i \right)^{\frac{1}{q}}, \tag{2.28}$$

$$U_k := p \frac{k-p+n}{k-p+1} \xi_k + \frac{A_k^p}{p} + \frac{B_k^q}{q}, \tag{2.29}$$

$$V_k := \frac{1}{n} \sum_{i=k}^{k+n-1} s_i - \frac{A_k^p}{p} - \frac{\varepsilon}{4}, \tag{2.30}$$

and

$$W_k := \frac{1}{n} \sum_{i=k}^{k+n-1} \frac{(i+1)(p-1)|\zeta_i|^q |\alpha_i|^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} - \frac{B_k^q}{q}. \tag{2.31}$$

Evidently (see (2.22) and (2.26)),

$$\xi_k \in (-M, 0), \quad k \in \mathbb{N}_b. \tag{2.32}$$

Using (2.22), (2.25), (2.26), and (2.32), we obtain $L > 0$ such that

$$\big| |\zeta_i|^q - |\xi_k|^q \big| \le \frac{L}{k-p+1}, \quad i \in \{k, k+1, \dots, k+n-1\}, \ k \in \mathbb{N}_b. \tag{2.33}$$

For all large $k \in \mathbb{N}$, considering (1.11) together with (1.5), (2.22), and (2.26), we have

$$\Delta \xi_k = \frac{1}{n} \sum_{i=k}^{k+n-1} \Delta \zeta_i$$

$$= \frac{1}{n} \sum_{i=k}^{k+n-1} \frac{1}{i-p+1} \left[p\zeta_i + s_i + \frac{(i+1)(p-1)|\zeta_i|^q |\alpha_i|^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} \right]$$

$$\ge \frac{p}{k-p+1} \xi_k + \frac{1}{k-p+n}$$

$$\times \left[\frac{1}{n} \sum_{i=k}^{k+n-1} s_i - \frac{\varepsilon}{4} + \frac{1}{n} \sum_{i=k}^{k+n-1} \frac{(i+1)(p-1)|\zeta_i|^q |\alpha_i|^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} \right]$$

$$= \frac{1}{k-p+n} \left[p \frac{k-p+n}{k-p+1} \xi_k + \frac{A_k^p}{p} + \frac{B_k^q}{q} + \frac{1}{n} \sum_{i=k}^{k+n-1} s_i - \frac{A_k^p}{p} - \frac{\varepsilon}{4} \right.$$

$$\left. + \frac{1}{n} \sum_{i=k}^{k+n-1} \frac{(i+1)(p-1)|\zeta_i|^q |\alpha_i|^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} - \frac{B_k^q}{q} \right].$$

Hence, we obtain (see (2.29), (2.30), and (2.31))

$$\Delta \xi_k \geq \frac{U_k + V_k + W_k}{k - p + n} \tag{2.34}$$

for all large $k \in \mathbb{N}$.

The Young inequality gives

$$p\xi_k + \frac{A_k^p}{p} + \frac{B_k^q}{q} = \frac{A_k^p}{p} + \frac{B_k^q}{q} - A_k B_k \geq 0, \quad k \in \mathbb{N}_b. \tag{2.35}$$

Considering

$$\lim_{k\to\infty} \frac{k-p+n}{k-p+1} = 1$$

together with (2.32) and (2.35), we have

$$U_k > -\frac{\varepsilon}{4} \tag{2.36}$$

for all large $k \in \mathbb{N}$.

Taking into account (1.5), we also have (see (2.27) and (2.30))

$$\begin{aligned} V_k &= \frac{1}{n}\sum_{i=k}^{k+n-1} s_i - \frac{A_k^p}{p} - \frac{\varepsilon}{4} = \frac{1}{n}\sum_{i=k}^{k+n-1} s_i - \frac{p^p}{p}\left(\frac{p}{n}\sum_{i=k}^{k+n-1} r_i\right)^{1-p} - \frac{\varepsilon}{4} \\ &= \frac{1}{n}\sum_{i=k}^{k+n-1} s_i - \left(\frac{1}{n}\sum_{i=k}^{k+n-1} r_i\right)^{1-p} - \frac{\varepsilon}{4} > \frac{3\varepsilon}{4}, \end{aligned}$$

i.e.,

$$V_k > \frac{3\varepsilon}{4} \quad \text{for all large } k \in \mathbb{N}. \tag{2.37}$$

It holds (see (2.20) and (2.22))

$$\lim_{k\to\infty}\left(|\zeta_k|^q \frac{(k+1)\alpha_k^{p-2}}{(k^{(p)})^{q-1}\Phi\big(\Phi^{-1}(k^{-1}r_k^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_k}{k^{(p)}}\big)\big)} - |\zeta_k|^q r_k\right) = 0$$

and (see (1.2), (2.15), (2.28), (2.31), and (2.32))

$$W_k = \frac{1}{n}\sum_{i=k}^{k+n-1} \frac{(i+1)(p-1)|\zeta_i|^q|\alpha_i|^{p-2}}{(i^{(p)})^{q-1}\Phi\big(\Phi^{-1}(i^{-1}r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} - \frac{B_k^q}{q}$$

$$= \frac{1}{n} \sum_{i=k}^{k+n-1} \frac{(i+1)(p-1)|\zeta_i|^q |\alpha_i|^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} - \frac{|\xi_k|^q}{q} \cdot \frac{p}{n} \sum_{i=k}^{k+n-1} r_i$$

$$= \frac{p-1}{n} \sum_{i=k}^{k+n-1} \left(|\zeta_i|^q \frac{(i+1)|\alpha_i|^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} - |\xi_k|^q r_i \right)$$

$$= \frac{p-1}{n} \sum_{i=k}^{k+n-1} \left(|\zeta_i|^q \frac{(i+1)\alpha_i^{p-2}}{(i^{(p)})^{q-1} \Phi\big(\Phi^{-1}(i^{-1} r_i^{1-p}) + \Phi^{-1}\big(-\frac{\zeta_i}{i^{(p)}}\big)\big)} - |\zeta_i|^q r_i \right)$$

$$+ \frac{p-1}{n} \sum_{i=k}^{k+n-1} \left(|\zeta_i|^q r_i - |\xi_k|^q r_i \right)$$

for all $k \in \mathbb{N}_b$. Thus (see (1.4) and (2.33)),

$$\limsup_{k \to \infty} |W_k| \le \limsup_{k \to \infty} \frac{p-1}{n} \sum_{i=k}^{k+n-1} \big| |\zeta_i|^q r_i - |\xi_k|^q r_i \big| \tag{2.38}$$

$$\le S(r_k) \frac{p-1}{n} \limsup_{k \to \infty} \sum_{i=k}^{k+n-1} \frac{L}{k-p+1} = 0.$$

In particular,

$$W_k > -\frac{\varepsilon}{4} \tag{2.39}$$

for all large $k \in \mathbb{N}$.

Finally, (2.34), (2.36), (2.37), and (2.39) imply

$$\Delta \xi_k > \frac{-\frac{\varepsilon}{4} + \frac{3\varepsilon}{4} - \frac{\varepsilon}{4}}{k-p+n} = \frac{\varepsilon}{4\,(k-p+n)} \tag{2.40}$$

for all large $k \in \mathbb{N}$. Because of

$$\sum_{i=l}^{\infty} \frac{1}{i-p+n} = \infty$$

for any large $l \in \mathbb{N}$, from (2.40), we obtain

$$\lim_{k \to \infty} \xi_k = \infty,$$

i.e., $\{\xi_k\}_{k \in \mathbb{N}_b}$ has to be positive for large $k \in \mathbb{N}$. This contradiction (see (2.32)) completes the proof. □

REMARK 1. Based on Theorems 2 and 3 (see also [24] together with constructions in [19]), we conjecture that Theorem 5 is not true for some n-periodic sequence $\{r_k\}_{k\in\mathbb{N}_a}$ and a bounded positive sequence $\{s_k\}_{k\in\mathbb{N}_a}$ satisfying

$$\lim_{k\to\infty} \frac{1}{n} \sum_{i=k}^{k+n-1} s_i = \left(\frac{1}{n} \sum_{i=a}^{a+n-1} r_i \right)^{1-p}.$$

Nevertheless, it remains an open problem.

Let a sequence $\{h_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ satisfy

$$\lim_{k\to\infty} \frac{h_k}{(k+1)^{(p+1)}} = 1.$$

For such a sequence, it is easy to reformulate Theorem 5 for equations in the form

$$\Delta\left(\frac{r_k^{1-p}}{k} \, \Phi(\Delta x_k) \right) + \frac{s_k}{h_k} \, \Phi(x_{k+1}) = 0, \quad k \in \mathbb{N}.$$

A basic reformulation is embodied into the following theorem, whose statement does not contain the generalized power function.

THEOREM 6. *Let us consider the equation*

$$\Delta\left(\frac{r_k^{1-p}}{k} \, \Phi(\Delta x_k) \right) + \frac{s_k}{k^{p+1}} \, \Phi(x_{k+1}) = 0, \quad k \in \mathbb{N}, \tag{2.41}$$

where $\{r_k\}_{k\in\mathbb{N}}$ is an asymptotically periodic sequence satisfying $\inf_{k\in\mathbb{N}} r_k > 0$ whose periodic part has period $n \in \mathbb{N}$ and $\{s_k\}_{k\in\mathbb{N}}$ is a bounded sequence. If there exists $\varepsilon > 0$ with the property that (1.5) is valid for all large $k \in \mathbb{N}$, then (2.41) is oscillatory.

PROOF. It suffices to consider Theorem 5 together with (2.12) from the proof of Lemma 1. $\square$

To highlight the novelty of our research in the linear case, we formulate the corresponding oscillation criterion for linear equations.

COROLLARY 1. *Let a sequence $\{h_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ satisfy*

$$\lim_{k\to\infty} \frac{h_k}{k^3} = 1 \tag{2.42}$$

and let us consider the equation

$$\Delta\left(\frac{1}{k\, r_k} \, \Delta x_k \right) + \frac{s_k}{h_k} \, x_{k+1} = 0, \quad k \in \mathbb{N}, \tag{2.43}$$

where $\{r_k\}_{k\in\mathbb{N}}$ *is an asymptotically periodic sequence with period* $n \in \mathbb{N}$ *of its periodic part and satisfying* $\inf_{k\in\mathbb{N}} r_k > 0$ *and* $\{s_k\}_{k\in\mathbb{N}}$ *is a bounded sequence. If there exists* $\varepsilon > 0$ *with the property that*

$$\frac{1}{n}\sum_{i=k}^{k+n-1} s_i > \varepsilon + \left(\frac{1}{n}\sum_{i=k}^{k+n-1} r_i\right)^{-1}$$

is valid for all large $k \in \mathbb{N}$, *then* (2.43) *is oscillatory.*

PROOF. The corollary follows from Theorem 6 for $p = 2$. □

Corollary 1 is new even for linear equations with periodic coefficients (see Corollary 2 below). We also mention examples of simple equations whose oscillation does not follow from any previously known result.

COROLLARY 2. *Let a sequence* $\{h_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ *satisfy* (2.42) *and let us consider* (2.43), *where* $\{r_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ *and* $\{s_k\}_{k\in\mathbb{N}}$ *are* n*-periodic sequences. If*

$$\left(\frac{1}{n}\sum_{i=1}^{n} r_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} s_i\right) > 1,$$

then (2.43) *is oscillatory.*

PROOF. This is a special case of Corollary 1. □

EXAMPLE 1. Let $n_1, n_2 \in \mathbb{N}$ and $\Lambda > 1$ and let us consider the equations

$$\Delta\left(\left[k\left(1+\frac{\sin\frac{k\pi}{n_1}}{2}\right)\right]^{-1}\Delta x_k\right) + \frac{\Lambda\left(1+2\sin\frac{k\pi}{n_2}\right)}{k^3}\, x_{k+1} = 0, \quad k\in\mathbb{N},$$

$$\Delta\left(\left[k\left(1+\frac{\sin\frac{k\pi}{n_1}}{2}\right)\right]^{-1}\Delta x_k\right) + \frac{\Lambda\left(1+2\sin\frac{k\pi}{n_2}\right)}{k(k+4)(k+8)}\, x_{k+1} = 0, \quad k\in\mathbb{N},$$

$$\Delta\left(\left[k\left(1+\frac{\sin\frac{k\pi}{n_1}}{2}\right)\right]^{-1}\Delta x_k\right) + \frac{\Lambda\left(1+2\sin\frac{k\pi}{n_2}\right)}{\left(k+\sqrt{k}\right)^{(3)}}\, x_{k+1} = 0, \quad k\in\mathbb{N},$$

$$\Delta\left(\left[k\left(1+\frac{\sin\frac{k\pi}{n_1}}{2}\right)\right]^{-1}\Delta x_k\right) + \frac{\Lambda\left(1+2\sin\frac{k\pi}{n_2}\right)}{k^3+k^2-k+1}\, x_{k+1} = 0, \quad k\in\mathbb{N},$$

$$\Delta\left(\left[k\left(1+\frac{\sin\frac{k\pi}{n_1}}{2}\right)\right]^{-1}\Delta x_k\right) + \frac{\Lambda\left(1+2\sin\frac{k\pi}{n_2}\right)}{k^3+\sqrt[3]{k^8+1}\arctan(k+2)}\, x_{k+1} = 0, \ k\in\mathbb{N},$$

$$\Delta\left(\left[k\left(1+\frac{\sin\frac{k\pi}{n_1}}{2}\right)\right]^{-1}\Delta x_k\right) + \frac{\Lambda\left(1+2\sin\frac{k\pi}{n_2}\right)}{k^3+k^2\sin k}\, x_{k+1} = 0, \quad k\in\mathbb{N}.$$

We use Corollary 2 for $n = 2n_1n_2$. Since

$$\left(\frac{1}{n}\sum_{i=1}^{n} r_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} s_i\right) = \Lambda$$

for each one of the equations above, we obtain its oscillation.

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