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ON THE STRUCTURE OF THE IWASAWA MODULE FOR $\mathbb{Z}_2$-EXTENSIONS OF CERTAIN REAL BIQUADRATIC FIELDS

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Abstract. For an infinite family of real biquadratic fields k we give the structure of the Iwasawa module $X = X(k_\infty)$ of the $\mathbb{Z}_2$-extension of k. For these fields, we obtain that $\lambda = \mu = 0$ and $\nu = 2$. where λ, μ and ν are the Iwasawa invariants of the cyclotomic $\mathbb{Z}_2$-extension of k.

1. Introduction

Let ℓ be a prime number and k a number field. A Galois extension k_∞/k is called a $\mathbb{Z}_\ell$-extension if the topological group $\mathrm{Gal}(k_\infty/k)$ is isomorphic to the additive group $\mathbb{Z}_\ell$ of ℓ-adic integers. Except for the trivial subgroup, all the closed subgroups of $\mathbb{Z}_\ell$ have finite index. Such a closed subgroup is of the form $\ell^n\mathbb{Z}_\ell$ for some positive integer n and the corresponding quotient group is cyclic of order ℓ^n. Thus, if k_∞/k is a $\mathbb{Z}_\ell$-extension, there is a unique field k_n of degree ℓ^n over k for all n, which called the n^{th} layer of k_∞/k. These k_n and k_∞, are the only fields between k and k_∞.

Every number field k, has at least one $\mathbb{Z}_\ell$-extension, namely the cyclotomic $\mathbb{Z}_\ell$-extension. It is obtained by the compositum $k_\infty = k\mathbb{Q}_\infty$, where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_\ell$-extension of the field of rational numbers $\mathbb{Q}$.

For each positive integer n, let $a_n = 2\cos(\frac{2\pi}{2^{n+2}})$ and $\mathbb{Q}_n = \mathbb{Q}(a_n)$, then $\mathbb{Q}_n \subset \mathbb{Q}_{n+1}$ by $a_{n+1} = \sqrt{2 + a_n}$. The extension $\mathbb{Q}_n$ is cyclic of degree 2^n over $\mathbb{Q}$. This mean that $\mathbb{Q}_\infty = \bigcup_{n=0}^{\infty} \mathbb{Q}_n$ is the unique $\mathbb{Z}_2$-extension of $\mathbb{Q}$. Specifically, the first layer $\mathbb{Q}_1$ of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$ is the real quadratic field $\mathbb{Q}(\sqrt{2})$. Accordingly if $\sqrt{2} \notin k$, the first layer k_1 of the cyclotomic $\mathbb{Z}_2$-extension of a number field k is $k_1 = k(\sqrt{2})$.

Key words and phrases: Iwasawa theory, $\mathbb{Z}_2$-extension, real biquadratic field, 2-class group, class field theory, unit.

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Let $A(k_n)$ be the ℓ-Sylow subgroup of ideal class group of n^{th} layer k_n, and $X(k_\infty) = \varprojlim A(k_n)$ be the inverse limit with respect the norm map. For all sufficiently large n, the order $\# A(k_n)$ is described as,

$$\# A(k_n) = \ell^{\lambda n + \mu p^n + \nu}$$

by the Iwasawa invariants λ, μ and ν. The inverse limit $X(k_\infty) = \varprojlim A(k_n)$ is called the Iwasawa module for k_∞/k. Greenberg conjectured claims [8] that λ and μ both vanish for any prime number ℓ and any totally real number field k. When k is abelian over the field of rational numbers $\mathbb{Q}$, and k_∞ is the cyclotomic $\mathbb{Z}_\ell$-extension of k, Ferrero and Washington [2] proved that $\mu = 0$.

In the previous years, many authors work on Greenberg's conjecture for totally real fields. For example, Ozaki and Taya [17] proved the existence of infinitely many real quadratic fields k, with $\lambda = \mu = 0$ in various situations. Y. Mizusawa [16] discusses some cases of real quadratic fields, for which Greenberg's conjecture hold. On the other hand when $k = \mathbb{Q}(\sqrt{p})$ is real quadratic field with prime number p, T. Fukuda and K. Kamotsue [4,5] have given some sufficient conditions for the conjecture to be true, mainly in terms of units of the n^{th} layer k_n of the cyclotomic $\mathbb{Z}_2$-extension for some n. Comparing with previous papers, the main novelty of this article is to construct an infinite family of real biquadratic fields k, such that the Iwasawa module $X(k_\infty)$, is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then we prove that the Iwasawa λ and μ-invariants of k_∞/k vanish, which confirms a conjecture of Greenberg's.

The aim of this article is to prove the following theorem:

THEOREM 1. *Let p, q and s be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \text{and} \quad s \equiv 7 \pmod 8,$$

and let k be one of the biquadratic fields

$$\mathbb{Q}\big(\sqrt{qs}, \sqrt{2pq}\big), \quad \mathbb{Q}\big(\sqrt{qs}, \sqrt{pq}\big) \quad \text{or} \quad \mathbb{Q}\big(\sqrt{2qs}, \sqrt{pq}\big).$$

Assume that the condition

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

is satisfied. Then the Iwasawa module $X(k_\infty)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Consequently $\lambda = \mu = 0$ and $\nu = 2$.

2. Preliminary results

During this paper, we fix the following notations.

k	number field.
U_k	the unit group of k.
O_k	the ring of integers of k.
k_1	the first layer of the cyclotomic $\mathbb{Z}_2$-extension of a number field k.
A_1	the 2-Sylow subgroup of the ideal class group k_1.
A	the 2-Sylow subgroup of the ideal class group k.
$N_{K/k}$	the relative norm of K/k.
$r(K/k)$	the number of primes of k ramified in K.
m	positive integer.
ε_m	the fundamental unit of $\mathbb{Q}(\sqrt{m})$.
$h(k)$	the class number of k.
$h(m)$	the class number for the quadratic number field $\mathbb{Q}(\sqrt{m})$.
$(\frac{*,*}{*})_m$	the m^{th} power residue symbol.
$(\frac{*,*}{*})$	the norm residue symbol.
$\#$	the order of a finite group.

In this section, we are collecting some results that will be useful in the sequel. The following result gives the rank of 2-Sylow subgroup of ideal class group of a number field K, such that K contains a number field k with odd class number, and the extension K/k is quadratic. Recall that the 2-rank of the ideal class group of k, meant to be the dimension of $A(K)/2A(K)$ as a $\mathbb{F}_2$-vector space.

LEMMA 1 [6]. *Let K/k be a quadratic extension of number fields. Assume that the class number of k is odd, then the rank of the 2-Sylow subgroup of the ideal class group of K, is equal to $r(K/k) - e - 1$ where $2^e = [U_k : U_k \cap N_{K/k}(K^*)]$.*

Let k be a number field and d a square-free integer satisfying $\sqrt{d} \notin k$. The determination of the integer e return to search units of k that are norms in the extension $k(\sqrt{d})/k$. A unit ε of k is norm in $k(\sqrt{d})/k$ if and only if the value of the norm residue symbol $(\frac{\varepsilon,d}{\mathcal{P}})$ equals 1, for each prime ideal $\mathcal{P}$ of k that ramifies in $k(\sqrt{d})$. For instance, when all units of k are norms in the extension $k(\sqrt{d})/k$ we have $e = 0$. Note that the definition of norm residue symbol can be extended to any extension of the form $k(\sqrt[m]{d})/k$ where m is a positive integer and k contains the m^{th} roots of unity.

Let $K/\mathbb{Q}$ be a real biquadratic field. The field K has the three real quadratic subextensions $F_i/\mathbb{Q}$ $(i = 1, 2, 3)$. Let ε_i be the fundamental unit of F_i $(i = 1, 2, 3)$, and $A(K)$, $A(F_i)$ the 2-Sylow subgroup of ideal class group of K, F_i, respectively. Put the group index $Q_K = [U_K : \langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3\rangle]$. Then, we have $Q_K = 1, 2$ or 4. S. Kuroda [14] proved the following equation:

$$\#A(K) = \frac{1}{4}Q_K.\#A(F_1).\#A(F_2).\#A(F_3).$$

This is often called Kuroda's class number formula. Furthermore, a system of the fundamental units of K is one of the following types (cf. [13, p. 72, Satz 1]):

(1) $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$,
(2) $\{\sqrt{\varepsilon_1}, \varepsilon_2, \varepsilon_3\}$, $(N_{F_1/\mathbb{Q}}(\varepsilon_1) = 1)$,
(3) $\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \varepsilon_3\}$ $(N_{F_1/\mathbb{Q}}(\varepsilon_1) = N_{F_2/\mathbb{Q}}(\varepsilon_2) = 1)$,
(4) $\{\sqrt{\varepsilon_1\varepsilon_2}, \varepsilon_2, \varepsilon_3\}$, $(N_{F_1/\mathbb{Q}}(\varepsilon_1) = N_{F_2/\mathbb{Q}}(\varepsilon_2) = 1)$,
(5) $\{\sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_3}, \varepsilon_2\}$, $(N_{F_1/\mathbb{Q}}(\varepsilon_1) = N_{F_2/\mathbb{Q}}(\varepsilon_2) = N_{\mathbb{Q}(F_3/\mathbb{Q}}(\varepsilon_3) = 1)$,
(6) $\{\sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{\varepsilon_1\varepsilon_3}\}$ $(N_{F_1/\mathbb{Q}}(\varepsilon_1) = N_{F_2/\mathbb{Q}}(\varepsilon_2) = N_{\mathbb{Q}(F_3/\mathbb{Q}}(\varepsilon_3) = 1)$,
(7) $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$, $(N_{F_1/\mathbb{Q}}(\varepsilon_1) = N_{F_2/\mathbb{Q}}(\varepsilon_2) = N_{\mathbb{Q}(F_3/\mathbb{Q}}(\varepsilon_3) = \pm 1)$.

LEMMA 2 [11]. *If* $N_{\mathbb{Q}(\sqrt{m})/\mathbb{Q}}(\varepsilon_m) = -1$, *then all odd prime factors of* m *are congruent to* 1 *modulo* 4.

The following result plays a crucial role in the proofs of our results.

LEMMA 3 [15]. *Let* F *be a real quadratic number field with fundamental unit* ε *and discriminant* D. *Suppose that* $N_{F/\mathbb{Q}}(\varepsilon) = 1$. *Then, there exists a positive square free integer* m *dividing* D *such that* $m\varepsilon$ *is a square in* F.

REMARK 1. As in the proof of Lemma 3, the integer m is norm in the extension $F/\mathbb{Q}$.

PROPOSITION 1 [1]. *Let* p, q *and* r *be distinct prime numbers with*

$$p \equiv -q \equiv -s \equiv 1 \pmod 4$$

and let $k = \mathbb{Q}\big(\sqrt{qs}, \sqrt{pq}\big)$. *Then the rank of* 2*-Sylow subgroup of the ideal class group of* k *equal to* 2, *if and only if the condition*

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1$$

is satisfied,

Add to the above proposition the following theorem which plays an important role in the proof of our main theorem.

THEOREM 2 [3]. *Let* k_∞/k *be any* $\mathbb{Z}_p$*-extension such that any prime of* k_∞ *which is ramified in* k_∞/k *is totally ramified.*

(1) *If* $\operatorname{rank}(A_1) = \operatorname{rank}(A)$, *then* $\operatorname{rank} A(k_n) = \operatorname{rank}(A)$ *for all* $n \geq 1$.
(2) *If* $\#A_1 = \#A$, *then* $\#A(k_n) = \#A$ *for all* $n \geq 1$.

Let us close this preliminary reminder by recalling the following known result that we shall use through our computations.

THEOREM 3 [9]. *Let k be a number field containing the m-th roots of unity and K be a finite extension of K. Let $\alpha \in k^*$, and $\beta \in K^*$. For an ideal prime $\mathcal{P}$ of k we have*

$$\prod_{\overline{\mathcal{P}}} \Big(\frac{\beta, \alpha}{\overline{\mathcal{P}}}\Big)_m = \Big(\frac{N_{K/k}(\beta), \alpha}{\mathcal{P}}\Big)_m,$$

where the product is taken over all the prime ideals of K above $\mathcal{P}$.

3. Rank of Iwasawa module of the cyclotomic $\mathbb{Z}_2$-extensions of certain real biquadratic fields

PROPOSITION 2. *Let q and s be prime numbers such that $q \equiv s \equiv -1 \pmod 4$. Then we have*

$$\sqrt{q\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs}) \quad \text{or} \quad \sqrt{s\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs}).$$

Consequently $\varepsilon_{qs} = qu^2$ or $\varepsilon_{qs} = sv^2$ where u and v are two elements in $\mathbb{Q}(\sqrt{qs})$.

PROOF. The discriminant of $\mathbb{Q}(\sqrt{qs})$ is equal to qs. By Lemma 2 we have $N_{\mathbb{Q}(\sqrt{qs})/\mathbb{Q}}(\varepsilon_{qs}) = 1$. Lemma 3 gives that there exists an integer $m \mid qs$ such $\sqrt{m\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs})$. Since ε_{qs} is the fundamental unit of $\mathbb{Q}(\sqrt{qs})$ then m must be contained in $\{q, s\}$. Either way, we can conclude that

$$\sqrt{q\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs}) \quad \text{or} \quad \sqrt{s\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs}).$$

Therefore $\varepsilon_{qs} = qu^2$ or $\varepsilon_{qs} = sv^2$ where u and v are two elements in $\mathbb{Q}(\sqrt{qs})$ as desired. □

PROPOSITION 3. *Let q and s be prime numbers such that $q \equiv 3 \pmod 8$ and $s \equiv 7 \pmod 8$. Then,*

$$\sqrt{s\varepsilon_{2qs}} \in \mathbb{Q}(\sqrt{2qs}).$$

Consequently, $\varepsilon_{2qs} = sa^2$ where a is an element in $\mathbb{Q}(\sqrt{2qs})$.

PROOF. The discriminant of $\mathbb{Q}(\sqrt{2qs})$ is equal to $8qs$, and

$$N_{\mathbb{Q}(\sqrt{2qs})/\mathbb{Q}}(\varepsilon_{2qs}) = 1$$

(see Lemma 2). By Lemma 3 and Remark 1 there exists an integer $m \mid 2qs$ such that m is a norm in the extension $\mathbb{Q}(\sqrt{2qs})/\mathbb{Q}$ and $\sqrt{m\varepsilon_{2qs}} \in \mathbb{Q}(\sqrt{2qs})$. By the facts $\left(\frac{2}{q}\right) = -1$, 2 and q are not norms in the extension $\mathbb{Q}(\sqrt{2qs})/\mathbb{Q}$, hence we deduce

$$\sqrt{s\varepsilon_{2qs}} \in \mathbb{Q}\big(\sqrt{2qs}\big).$$

Therefore $\varepsilon_{2qs} = sa^2$ where a is an element in $\mathbb{Q}(\sqrt{2qs})$. This establishes the proposition. □

PROPOSITION 4. *Let p and q be prime numbers such that $p \equiv 5 \pmod 8$ and $q \equiv 3 \pmod 8$. Then,*

$$\sqrt{p\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq}) \quad \textit{or} \quad \sqrt{q\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq}).$$

PROOF. The discriminant of $\mathbb{Q}(\sqrt{pq})$ is equal to $4pq$, and

$$N_{\mathbb{Q}(\sqrt{pq})/\mathbb{Q}}(\varepsilon_{pq}) = 1.$$

By Lemma 3 there exists an integer $m \mid 2pq$ such that m is a norm in the extension $\mathbb{Q}(\sqrt{pq})/\mathbb{Q}$ (see Remark 1) and $\sqrt{m\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq})$. Since ε_{pq} is the fundamental unit of $\mathbb{Q}(\sqrt{pq})$ then m must be contained in $\{2, p, q, 2p, 2q, 2pq\}$. On the other hand we have $p \equiv 5 \pmod 8$ and $q \equiv 3 \pmod 8$, which means: $\left(\frac{2}{p}\right) = \left(\frac{2}{q}\right) = -1$. Then 2, $2p$, $2q$ and $2pq$ are not norms in the extension $\mathbb{Q}(\sqrt{pq})/\mathbb{Q}$. Therefore

$$\sqrt{p\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq}) \quad \text{or} \quad \sqrt{q\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq}).$$

This shows the statement. □

LEMMA 4. *Let q and s be distinct prime numbers with*

$$q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8$$

and let L be the biquadratic field $L = \mathbb{Q}\big(\sqrt{qs}, \sqrt{2}\big)$. Then, $\big\{\sqrt{\varepsilon_{2qs}\varepsilon_{qs}}, \varepsilon_{qs}, \varepsilon_2\big\}$ is a fundamental system of units of biquadratic field L. Therefore the Hasse unit index Q_L is equal to 2.

PROOF. By Proposition 2 we have

$$\sqrt{q\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs}) \quad \text{or} \quad \sqrt{s\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs}).$$

Proposition 3 gives that $\sqrt{s\varepsilon_{2qs}} \in \mathbb{Q}(\sqrt{2qs})$. Therefore,

$$\sqrt{\varepsilon_{qs}\varepsilon_{2qs}} \in L.$$

Since $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\varepsilon_2) = -1$, ε_2 is not a square root of an element of L. It follows that $\{\sqrt{\varepsilon_{2qs}\varepsilon_{qs}}, \varepsilon_{qs}, \varepsilon_2\}$ is a fundamental system of units of biquadratic field L, which gives that the Hasse unit index Q_L is equal to 2. □

LEMMA 5. *Let q and s be distinct prime numbers with*

$$q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8.$$

Then the class number of $L = \mathbb{Q}\big(\sqrt{qs}, \sqrt{2}\big)$ is odd.

PROOF. Assume that q and s satisfy the conditions in Lemma 5. By Lemma 4 the Hasse unit index for the biquadratic number field L is equal to 2. On the other hand, the class number formula gives that

$$h(L) = \frac{2h(2qs)h(qs)h(2)}{4}.$$

We have $h(2) = 1$ and $h(qs)$ is odd [19]. Moreover since $q \equiv 3 \pmod 8$, from [10] we have $h(2qs) \equiv 2 \pmod 4$. This allows us to conclude that the class number of biquadratic number field $L = \mathbb{Q}(\sqrt{qs}, \sqrt{2})$ is odd. □

PROPOSITION 5. *Let p, q and r be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8$$

and

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

Let F be the biquadratic field $F = \mathbb{Q}\big(\sqrt{2qs}, \sqrt{pq}\big)$. Then, the Hasse unit index Q_F is equal to 2.

PROOF. By Proposition 3 we have $\sqrt{s\varepsilon_{2qs}} \in \mathbb{Q}(\sqrt{2qs})$, and Proposition 4 gives that $\sqrt{p\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq})$ or $\sqrt{q\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq})$. On the other hand, the discriminant of $\mathbb{Q}(\sqrt{2ps})$ is equal to $8ps$, and $N_{\mathbb{Q}(\sqrt{2ps})/\mathbb{Q}}(\varepsilon_{2ps}) = 1$. By Lemma 3 there exists an integer $m \mid 2ps$ such that m is a norm in the extension $\mathbb{Q}(\sqrt{2ps})/\mathbb{Q}$ and $\sqrt{m\varepsilon_{2ps}} \in \mathbb{Q}(\sqrt{2ps})$. On account of the fact that ε_{2ps} is the fundamental unit of $\mathbb{Q}(\sqrt{2ps})$, m must be contained in $\{2, p, s, 2p, 2s\}$. By the facts $p \equiv 5 \pmod 8$, we have 2, $2p$ and $2s$ are not norms in the extension $\mathbb{Q}(\sqrt{2ps})/\mathbb{Q}$, hence we deduce $\sqrt{p\varepsilon_{2ps}} \in \mathbb{Q}(\sqrt{pq})$ or $\sqrt{s\varepsilon_{2ps}} \in \mathbb{Q}(\sqrt{2ps})$. Therefore, $\sqrt{\varepsilon_{pq}\varepsilon_{2ps}}$, $\sqrt{\varepsilon_{2ps}\varepsilon_{2qs}}$ or $\sqrt{\varepsilon_{pq}\varepsilon_{2ps}}$ is in the biquadratic field $F = \mathbb{Q}(\sqrt{2qs}, \sqrt{pq})$. It follows that, a system of the fundamental units of F is one of the types $\{\sqrt{\varepsilon_{pq}\varepsilon_{2qs}}, \varepsilon_{2qs}, \varepsilon_{2ps}\}$, $\{\sqrt{\varepsilon_{2ps}\varepsilon_{2qs}}, \varepsilon_{pq}, \varepsilon_{2ps}\}$ or $\{\sqrt{\varepsilon_{pq}\varepsilon_{2ps}}, \varepsilon_{2qs}, \varepsilon_{2ps}\}$. (See a system of the fundamental units of biquadratic fields at the beginning of page 4). Either way, we can conclude that the Hasse unit index Q_F is equal to 2. □

PROPOSITION 6. *Let p, q and r be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8$$

and

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

Let k be the biquadratic field $k = \mathbb{Q}\big(\sqrt{pq}, \sqrt{qs}\big)$. Then, the Hasse unit index Q_k is equal to 4.

PROOF. By Proposition 2, $\sqrt{q\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{pq})$ or $\sqrt{s\varepsilon_{qs}} \in \mathbb{Q}(\sqrt{qs})$. On the other hand, Proposition 4 gives that $\sqrt{p\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq})$ or $\sqrt{q\varepsilon_{pq}} \in \mathbb{Q}(\sqrt{pq})$. Then

$$\sqrt{\varepsilon_{pq}\varepsilon_{qs}}, \quad \sqrt{\varepsilon_{pq}\varepsilon_{ps}} \quad \text{and} \quad \sqrt{\varepsilon_{ps}\varepsilon_{qs}},$$

are in the biquadratic field $k = \mathbb{Q}(\sqrt{pq}, \sqrt{qs})$. This allows us to conclude that a fundamental system of units of the biquadratic number field k, is $\{\sqrt{\varepsilon_{pq}\varepsilon_{qs}}, \sqrt{\varepsilon_{pq}\varepsilon_{ps}}, \sqrt{\varepsilon_{ps}\varepsilon_{qs}}\}$. Therefore, the Hasse unit index Q_k for the biquadratic number field k is equal to 4. Thus, we have proved the desired result. □

In order to prove Theorem 1, we use the following proposition.

PROPOSITION 7. *Let p, q and s be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8,$$

and $k = \mathbb{Q}(\sqrt{pq}, \sqrt{qs})$. Assume that the condition

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

is satisfied. Then the rank of 2-Sylow subgroup of the ideal class group of $k_1 = k(\sqrt{2}) = \mathbb{Q}(\sqrt{pq}, \sqrt{qs}, \sqrt{2})$ is equal to 2.

PROOF. We see that $k_1 = L(\sqrt{pq})$. From Lemma 5 the class number L is odd, moreover the number of primes of L which are ramified in k_1 is equal to 3. Consequently for Lemma 1 the rank of 2-Sylow subgroup of the ideal class group of k_1 is equal to $r(k_1/L) - e - 1$ such that $r(k_1/L) = 3$ and $2^e = [U_L : U_L \cap N(k_1^{\times})]$. Then to prove that the rank of 2-Sylow subgroup of the ideal class group of k_1 is equal to 2, it suffices to show that all units of L are norms in the extension k_1/L. Since $\left(\frac{qs}{p}\right) = 1$, the rational prime p splits in $\mathbb{Q}(\sqrt{qs})$. Then $pO_{\mathbb{Q}(\sqrt{qs})} = \mathcal{P}_1\mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two prime ideals of $\mathbb{Q}(\sqrt{qs})$ lying above p. By assumption $\left(\frac{2}{p}\right) = -1$, we have $\mathcal{P}_1 O_L = \mathfrak{B}_1$ and $\mathcal{P}_2 O_L = \mathfrak{B}_2$, where $\mathfrak{B}_1$, $\mathfrak{B}_2$ the two prime ideals of L lying respectively above $\mathcal{P}_1$ and $\mathcal{P}_2$. So from the properties of the norm residue symbol we get

$$\left(\frac{\varepsilon_{qs}, pq}{\mathfrak{B}_1}\right) = \left(\frac{N_{L/\mathbb{Q}(\sqrt{qs})}(\varepsilon_{qs}), pq}{\mathcal{P}_1}\right) = \left(\frac{\varepsilon_{qs}^2, pq}{\mathcal{P}_1}\right) = 1,$$

$$\left(\frac{\sqrt{\varepsilon_{qs}}, pq}{\mathfrak{B}_1}\right) = \left(\frac{N_{L/\mathbb{Q}(\sqrt{qs})}(\sqrt{\varepsilon_{qs}}), pq}{\mathcal{P}_1}\right) = \left(\frac{\pm\varepsilon_{qs}, p}{\mathcal{P}_1}\right)$$
$$= \left(\frac{\pm qu^2}{p}\right) = \left(\frac{\pm q}{p}\right) = 1,$$

$$\left(\frac{\varepsilon_2, pq}{\mathfrak{B}_1}\right) = \left(\frac{N_{L/\mathbb{Q}(\sqrt{qs})}(\varepsilon_2), pq}{\mathcal{P}_1}\right) = \left(\frac{-1, p}{\mathcal{P}_1}\right) = \left(\frac{-1}{p}\right) = 1,$$

$$\left(\frac{-1, pq}{\mathfrak{B}_1}\right) = \left(\frac{N_{L/\mathbb{Q}(\sqrt{qs})}(-1), pq}{\mathcal{P}_1}\right) = \left(\frac{1, pq}{\mathcal{P}_1}\right) = 1.$$

On the other hand by Proposition 3, $\varepsilon_{2qs} = sa^2$. By the facts $\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1$, and $\left(\frac{2}{p}\right) = -1$, if we denote by $\mathcal{P}'$ the prime ideal of $\mathbb{Q}(\sqrt{2qs})$ lying above the prime p, we have $pO_{\mathbb{Q}(\sqrt{2qs})} = \mathcal{P}'$ and $\mathcal{P}'O_L = \mathfrak{B}_1\mathfrak{B}_2$, then

$$\left(\frac{\sqrt{\varepsilon_{2qs}}, pq}{\mathfrak{B}_1}\right) = \left(\frac{N_{L/\mathbb{Q}(\sqrt{2qs})}(\sqrt{\varepsilon_{2qs}}), pq}{\mathcal{P}'}\right) = \left(\frac{\varepsilon_{2qs}, p}{\mathcal{P}'}\right)$$

$$= \left(\frac{sa^2, p}{\mathcal{P}'}\right) = \left(\frac{s, p}{\mathcal{P}'}\right) = \left(\frac{s}{p}\right) = 1.$$

By the condition $q \equiv 3 \pmod 8$ and $s \equiv 7 \pmod 8$ we can see that rational prime 2, remain inert in $\mathbb{Q}(\sqrt{qs})$. Hence $2O_{\mathbb{Q}(\sqrt{qs})} = \mathcal{S}$ and $\mathcal{S}O_L = \mathfrak{R}^2$. For all unit u in L we have

$$\left(\frac{u, qs}{\mathfrak{R}}\right) = \left(\frac{N_{L/\mathbb{Q}(\sqrt{qs})}(u), pq}{\mathcal{S}}\right) = 1.$$

Consequently all units of L are norms in the extension k_1/L. This allows us to conclude that $e = 0$ and complete the proof of the proposition. □

THEOREM 4. *Let p, q and s be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8$$

and let k be one of the following biquadratic fields

$$\mathbb{Q}\left(\sqrt{qs}, \sqrt{2pq}\right), \quad \mathbb{Q}\left(\sqrt{qs}, \sqrt{pq}\right) \quad \textit{or} \quad \mathbb{Q}\left(\sqrt{2qs}, \sqrt{pq}\right).$$

Assume that the condition

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

is satisfied. Then the rank of the Iwasawa module $X(k_\infty)$ is equal to 2.

PROOF. The extension k_1/k is ramified this means that the extension k_∞/k is totally ramified. In Proposition 1 the rank of the 2-Sylow subgroup of the ideal class group of k is equal to 2. From Proposition 7 the rank of the 2-Sylow subgroup of the ideal class group of k_1 is equal to 2. Therefore we obtain

$$\mathrm{rank}(A) = \mathrm{rank}(A_1) = 2.$$

By using Theorem 2, the rank of $X(k_\infty)$ is equal to 2. □

4. Proof of main Theorem 1

Before giving the proof of the main theorem, we are going to give some preliminary results.

It is known ([7, Ch. 5, Theorem 4.5]) that there exist exactly three infinite families of non-abelian finite 2-groups G of which the largest abelian factor groups G^{ab} are isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Namely, the generalized quaternion groups Q_m, dihedral groups D_m and the semidihedral groups S_m, of order exactly 2^m, with $m\geq 3$ for the first two families and $m\geq 4$ for the last. A representation by generators and relations of these three families are given by

$$Q_m=\langle x,y \mid x^{2^{m-2}}=y^2,\ y^4=1,\ y^{-1}xy=x^{-1}\rangle \ \text{ with } m\geq 3,$$
$$D_m=\langle x,y \mid x^{2^{m-1}}=y^2=1,\ y^{-1}xy=x^{-1}\rangle \ \text{ with } m\geq 3,$$
$$S_m=\langle x,y \mid x^{2^{m-1}}=y^2,\ y^{-1}xy=x^{2^{m-2}-1}\rangle \ \text{ with } m\geq 4.$$

In this section we will use the following known properties of these groups G (see, for instance, [12, pp. 272–273] and [7, Ch. 5]). The commutator subgroup $[G,G]$ of G is always cyclic: $[G,G]=\langle x^2\rangle$. These groups G possess exactly three sub-groups of index 2. Namely, $H_1=\langle x\rangle$, $H_2=\langle x^2,y\rangle$, $H_3=\langle x^2,xy\rangle$. When G is not the quaternion group of order 8, only one of the three maximal sub-groups of G is cyclic. When $m\geq 4$, the other two maximal sub-groups of G are not abelian and their maximal abelian factor groups, are again isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Of course, when G is the quaternion group of order 8, its three maximal subgroups are cyclic and when G is the dihedral group of order 8, its three subgroups are abelian.

Now let k be a number field whose 2-class group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ $\times\,\mathbb{Z}/2\mathbb{Z}$. By Taussky [18], the Hilbert 2-class field tower of k terminates in at most two steps. Denote by $L(k)$ the Hilbert 2-class field of k and by $L^2(k)$ that of $L(k)$. Let H_i $(i=1,2,3)$ be the subgroups of $G=\mathrm{Gal}(L^2(k)/k)$ associated to the above notations. There are just three quadratic subextensions F_i/k $(i=1,2,3)$ such that $H_i=\mathrm{Gal}(L^2(k)/F_i)$ and the 2-Sylow subgroup of the ideal class group $A(F_i)\simeq H_i^{ab}$. If $G\simeq Q_8$, or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, all the three 2-class groups $A(F_i)(i=1,2,3)$ are cyclic. If $G\simeq Q_{2^m}(m\geq 4)$ or D_{2^m}, SD_{2^m}, then $A(F_1)$ is cyclic and $A(F_2)\simeq A(F_3)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

Lemma 6. *Let p, q and s be distinct prime numbers with*

$$p\equiv 5 \pmod 8,\quad q\equiv 3 \pmod 8 \quad \textit{and} \quad s\equiv 7 \pmod 8$$

and

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

Let F be the biquadratic field $F = \mathbb{Q}\big(\sqrt{2qs}, \sqrt{pq}\big)$. Then, the 2-class numbers of F is equal to 4.

PROOF. By Proposition 5 the Hasse unit index for biquadratic number field F is equal to 2. The class number formula gives that:

$$h(F) = \frac{h(2qs)h(2ps)h(pq)}{2}.$$

Moreover, since $\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = -\left(\frac{2}{p}\right) = 1$, from [10] we have $h(2qs) \equiv h(pq) \equiv h(2ps) \equiv 2 \pmod 4$. This allows us to conclude that the 2-class number of biquadratic number field F is equal to 4. $\square$

LEMMA 7. *Let p, q and s be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8$$

and

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

Let k be the biquadratic field $k = \mathbb{Q}\big(\sqrt{pq}, \sqrt{qs}\big)$. Then, the 2-Sylow subgroup of the ideal class group of k is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

PROOF. By Proposition 6 the Hasse unit index for biquadratic number field k is equal to 4. On the other hand, the class number formula gives that:

$$h(k) = h(pq)h(qs)h(ps),$$

we have $h(qs)$ is odd [19]. Moreover since $\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = -\left(\frac{2}{p}\right) = 1$. From [10] we have $h(pq) \equiv h(ps) \equiv 2 \pmod 4$. This allows us to conclude that the 2-class number of biquadratic number field k is equal to 4. By Proposition 1 the rank of the 2-ideal class group of k is equal to 2. From the above results, we have the 2-Sylow subgroup of the ideal class group of k is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, we have proved the desired result. $\square$

LEMMA 8. *Let p, q and r be distinct prime numbers with*

$$p \equiv 5 \pmod 8, \quad q \equiv 3 \pmod 8 \quad \textit{and} \quad s \equiv 7 \pmod 8$$

and

$$\left(\frac{p}{q}\right) = \left(\frac{p}{s}\right) = 1.$$

Let F be the biquadratic field $F=\mathbb{Q}\big(\sqrt{2qs},\sqrt{pq}\big)$. Then, the 2-Sylow subgroup of the ideal class group of F is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

PROOF. By Proposition 5 the 2-class numbers of F is equal to 4, we can see that the Hilbert 2-class field of F is the field $L(F)=\mathbb{Q}\big(\sqrt{p},\sqrt{q},\sqrt{s},\sqrt{2}\big)$. The three quadratic unramified subextensions of $L(F)/F$ are: $k_1=\mathbb{Q}\big(\sqrt{qs},\sqrt{pq},\sqrt{2}\big)$, $F_1=\mathbb{Q}\big(\sqrt{q},\sqrt{p},\sqrt{2s}\big)$ and $F_2=\mathbb{Q}\big(\sqrt{2q},\sqrt{2p},\sqrt{s}\big)$. Then the 2-Sylow subgroup of the ideal class group of F is not cyclic. This allows us to conclude that $A(F)\simeq \mathrm{Gal}(L(F)/F)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. $\square$

THEOREM 5. *Let p, q and r be distinct prime numbers with*

$$p\equiv 5 \pmod 8,\quad q\equiv 3 \pmod 8\quad \textit{and}\quad s\equiv 7 \pmod 8,$$

Assume that the condition

$$\left(\frac{p}{q}\right)=\left(\frac{p}{s}\right)=1$$

is satisfied. Then, the 2-Sylow subgroup of the ideal class group of $k_1=\mathbb{Q}\big(\sqrt{pq},\sqrt{qs},\sqrt{2}\big)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

PROOF. By Lemma 8 the 2-Sylow subgroup of the ideal class group of biquadratic field $F=\mathbb{Q}\big(\sqrt{2qs},\sqrt{pq}\big)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. The three quadratic unramified subextensions of F are: $k_1=\mathbb{Q}\big(\sqrt{qs},\sqrt{pq},\sqrt{2}\big)$, $F_1=\mathbb{Q}\big(\sqrt{q},\sqrt{p},\sqrt{2s}\big)$ and $F_2=\mathbb{Q}\big(\sqrt{2q},\sqrt{2p},\sqrt{s}\big)$. By Proposition 7 we have, the rank of the 2-ideal class group of k_1 is equal to 2. This means that the 2-Sylow subgroup of the ideal class group of k_1 is not cyclic. So we conclude that the 2-Sylow subgroup of the ideal class group of k_1 is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, (see the properties of 2-group G such that G^{ab} is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ at the beginning of this section). Thus, we have proved the desired result. $\square$

Now, we are ready to prove Theorem 1.

The extension k_1/k is ramified, i.e. k_∞/k is totally ramified. It was noted in Lemma 7 that the 2-Sylow subgroup of the ideal class group of k is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, and from Theorem 5 we have the 2-Sylow subgroup of the ideal class group of k_1 is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ therefore we obtain

$$A\simeq A_1\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

By applying Theorem 2 we get $X(k_\infty)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. This means that $\lambda=\mu=0$ and $\nu=2$. Finally, the three biquadratic fields

$$\mathbb{Q}\big(\sqrt{qr},\sqrt{pq}\big),\quad \mathbb{Q}\big(\sqrt{2qr},\sqrt{pq}\big)\quad\text{and}\quad \mathbb{Q}\big(\sqrt{qr},\sqrt{2pq}\big),$$

have the same cyclotomic $\mathbb{Z}_2$-extension k_∞, so the Iwasawa invariants are also the same.

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