



ON ALGEBRAIC CONDITIONS FOR THE NON-VANISHING OF LINEAR FORMS IN JACOBI THETA-CONSTANTS

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Abstract. Elsner, Luca and Tachiya proved in [4] that the values of the Jacobi-theta constants $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are algebraically independent over \mathbb{Q} for distinct integers m, n under some conditions on τ . On the other hand, in [3] Elsner and Tachiya also proved that three values $\theta_3(m\tau), \theta_3(n\tau)$ and $\theta_3(\ell\tau)$ are algebraically dependent over \mathbb{Q} . In this article we prove the non-vanishing of linear forms in $\theta_3(m\tau), \theta_3(n\tau)$ and $\theta_3(\ell\tau)$ under various conditions on m, n, ℓ , and τ . Among other things we prove that for odd and distinct positive integers $m, n > 3$ the three numbers $\theta_3(\tau), \theta_3(m\tau)$ and $\theta_3(n\tau)$ are linearly independent over \mathbb{Q} when τ is an algebraic number of some degree greater or equal to 3. In some sense this fills the gap between the above-mentioned former results on theta constants. A theorem on the linear independence over $\mathbb{C}(\tau)$ of the functions $\theta_3(a_1\tau), \dots, \theta_3(a_m\tau)$ for distinct positive rational numbers a_1, \dots, a_m is also established.

1. Introduction

For a complex number τ from the upper complex half plane \mathbb{H} , the theta functions are defined as follows;

$$\theta_2(\tau) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2}, \quad \theta_3(\tau) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \quad \text{and} \quad \theta_4(\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2},$$

where $q = e^{i\pi\tau}$. For the sake of brevity we sometimes write θ_i instead of $\theta_i(\tau)$, $i = 2, 3, 4$.

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We define the j -function as follows;

$$j(\tau) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}, \quad \text{where } \lambda = \lambda(\tau) = \frac{\theta_2^4}{\theta_3^4},$$

which is a modular function with respect to the group $SL(2, \mathbb{Z})$.

The motivation of this article comes from the following sources: C. Elsner and Y. Tachiya [3] proved that for distinct integers ℓ, m and n , the functions $\theta_3(\ell\tau), \theta_3(m\tau)$ and $\theta_3(n\tau)$ are *algebraically* dependent over \mathbb{Q} . C. Elsner, F. Luca and Y. Tachiya [4] proved the following: let τ be any complex number with $\text{Im}(\tau) > 0$ such that $e^{i\pi\tau}$ is algebraic. Let $m, n \geq 1$ be distinct positive integers. Then the numbers $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are *algebraically* independent over \mathbb{Q} . In a subsequent paper [5] this occurs as a special case of [5, Theorem 1.1].

Naturally the following two questions arise.

QUESTION 1. *Let $m \geq 2$ and let a_1, a_2, \dots, a_m be distinct positive integers. Are the functions*

$$\theta_3(a_1\tau), \theta_3(a_2\tau), \dots, \theta_3(a_m\tau)$$

linearly independent over $\mathbb{C}(\tau)$?

By [5, Theorem 1.1] we know that for distinct positive integers m, n , and an algebraic number $e^{i\pi\tau}$ with $\tau \in \mathbb{H}$, the two numbers $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are *algebraically* independent over \mathbb{Q} . Then, for algebraic numbers α_1 and α_2 , which do not vanish simultaneously, the linear form

$$\alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau)$$

does not vanish. So it seems natural to consider linear forms in three values of the theta constant θ_3 .

QUESTION 2. *What are the values of τ and $\alpha_0, \alpha_1, \alpha_2$ such that the linear form*

$$L := \alpha_0\theta_3(\tau) + \alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau)$$

does not vanish?

In this article, we give the complete answer to the Question 1 and answer the Question 2 in the following way: We consider linear forms with integers $n > m > 1$, certain numbers $\tau \in \mathbb{H}$, and algebraic numbers $\alpha_0, \alpha_1, \alpha_2$, and give conditions on $\alpha_0, \alpha_1, \alpha_2$ such that $L \neq 0$.

We divide the remaining part of our article into three sections: In Section 2, we state our theorems, in Section 3 we collect all the tools to prove our results, and in the last section we give the proof of all the theorems from Section 2.

2. The results

2.1. The linear independence over $\mathbb{C}(\tau)$ of the functions $\theta_3(a_1\tau)$, \dots , $\theta_3(a_m\tau)$ in τ . We begin with the following result on the linear independence over $\mathbb{C}(\tau)$ of Jacobi-theta constants.

THEOREM 2.1. (i) *Let a_1, a_2, \dots, a_m be distinct positive rational numbers. Then the m functions*

$$\theta_3(a_1\tau), \theta_3(a_2\tau), \dots, \theta_3(a_m\tau)$$

in $\tau \in \mathbb{H}$ are linearly independent over $\mathbb{C}(\tau)$.

(ii) *Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be distinct positive real numbers. Then the m functions*

$$\theta_3(\alpha_1\tau), \theta_3(\alpha_2\tau), \dots, \theta_3(\alpha_m\tau)$$

are linearly independent over \mathbb{C} .

2.2. On the linear independence of values of Jacobi-theta constants for $\theta_3(\tau)$, $\theta_3(m\tau)$, $\theta_3(n\tau)$ with odd integers m, n .

THEOREM 2.2. *Let $3 \leq n < m$ be two odd integers. If one of the following conditions holds, namely*

1. τ is an algebraic number of degree ≥ 3 ,
2. $\tau \in \mathbb{H}$ such that $q = e^{i\pi\tau}$ is algebraic over \mathbb{Q} ,

then the three numbers

$$\theta_3(\tau), \theta_3(m\tau), \theta_3(n\tau)$$

are linearly independent over $\overline{\mathbb{Q}}$.

2.3. Results on linear forms $\alpha_0\theta_3(\tau) + \alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau)$ for $mn \equiv 0 \pmod{2}$ under certain restrictions on the coefficients. Let m, n be two distinct positive integers, and let $\tau \in \mathbb{H}$ satisfy the conditions in Theorem 2.2. Since the numbers $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are algebraically independent when $e^{i\pi\tau}$ is algebraic by [5, Theorem 1.1], we consider linear relations

$$\alpha_0\theta_3(\tau) + \alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau)$$

for real algebraic numbers $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_0\alpha_1\alpha_2 \neq 0$. In order to state our next result we introduce the following set. Let $s \geq 3$ be any odd integer. Set

$$M_s := \{ \pm\sqrt{u}, \pm i\sqrt{u} : u \in \mathbb{N} \wedge s \equiv 0 \pmod{u} \}.$$

THEOREM 2.3. *Let $m = 2^a s_1$ and $n = 2^b s_2$ be two distinct integers with $a, b \geq 1$ and odd integers $s_1, s_2 \geq 3$. Let $\tau \in \mathbb{H}$ such that $e^{i\pi\tau}$ is an algebraic number. Then, the inequality*

$$\alpha_0\theta_3(\tau) + \alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau) \neq 0$$

holds, if $\beta = \alpha_2\alpha_0^{-1}$ satisfies at least one of the following conditions:

$$\beta^{-1} \notin M_{s_2}, \quad \beta^4 \notin \mathbb{Q}, \quad R_{n,0}(\beta^{-4}) \neq 0,$$

where $R_{n,0}(X)$ is the polynomial defined by formula (3.5) in Lemma 3.5 in Section 3.

In the case when additionally s_1 and s_2 are coprime odd integers, we have $M_{s_1} \cap M_{s_2} = \{\pm 1, \pm i\}$. Then we obtain from Theorem 2.3 the following corollary.

COROLLARY 2.4. *Let $m = 2^a s_1$ and $n = 2^b s_2$ be two distinct integers with $a, b \geq 1$ and odd coprime integers $s_1, s_2 \geq 3$. Let $\tau \in \mathbb{H}$ such that $e^{i\pi\tau}$ is an algebraic number. Then, the inequality*

$$\alpha_0\theta_3(\tau) + \alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau) \neq 0$$

holds for all algebraic numbers $\alpha_0, \alpha_1, \alpha_2$, whenever at least one of the numbers α_0/α_1 or α_0/α_2 is not a unit in the ring of Gaussian integers $\mathbb{Z}[i]$.

For any integer $n > 1$ let

$$\psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where p runs through all primes dividing n and for $n = 1$, we define $\psi(1) = 1$.

THEOREM 2.5. *Let $m = 2^a s$ be an integer with $a \geq 1$ and an odd integer $s \geq 3$ and let $n \geq 3$ be an odd integer. Let $\tau \in \mathbb{H}$ be as in Theorem 2.2. Then, the inequality*

$$\alpha_0\theta_3(\tau) + \alpha_1\theta_3(m\tau) + \alpha_2\theta_3(n\tau) \neq 0$$

holds, if $\beta := \alpha_2\alpha_0^{-1}$ satisfies one of the following conditions:

$$\deg_{\mathbb{Q}}(\beta^4) > \psi(n), \quad S_{n,0}(n^2\beta^{-4})S_{n,d_n}(n^2\beta^{-4}) \neq 0,$$

where $S_{n,0}(X)$ and $S_{n,d_n}(X)$ are polynomials defined in (3.2) in Theorem 3.3 below.

The preceding theorems do not treat $\theta_3(\tau)$, $\theta_3(2\tau)$, and $\theta_3(3\tau)$ simultaneously. For this situation we cite a result from [3, Example 1.5]: Let $\tau \in \mathbb{H}$, and define

$$P(X, Y, Z) := 27X^8 - 18X^4Y^4 - 64X^2Y^4Z^2 + 64X^2Y^2Z^4 - 8X^2Z^6 - Z^8.$$

Then $P(X_0, Y_0, Z_0) = 0$ holds for

$$X_0 := \theta_3(3\tau), \quad Y_0 := \theta_3(2\tau), \quad Z_0 := \theta_3(\tau).$$

This shows that $\theta_3(\tau)$, $\theta_3(2\tau)$, $\theta_3(3\tau)$ are homogeneously algebraically dependent of degree 8.

PROPOSITION 2.6. *Let $m \geq 0$ be an integer. Let $\tau \in \mathbb{H}$ be as in Theorem 2.2. Then the three numbers*

$$\theta_3(2^m\tau), \quad \theta_3(2^{m+1}\tau), \quad \theta_3(2^{m+2}\tau)$$

are linearly independent over $\overline{\mathbb{Q}}$.

In Proposition 2.6 it is not possible to avoid the condition $\deg_{\mathbb{Q}}(\tau) \geq 3$. This follows for $\tau = i$ and $m = 0$ from the nontrivial relation

$$\sqrt{2 + \sqrt{2}}\theta_3(i) - 2\theta_3(2i) = 0$$

due to Ramanujan, cf. [1, p. 325]. We can also find similar linear relations for the theta-constants θ_2 and θ_4 :

$$\sqrt{2 - \sqrt{2}}\theta_2(i) - \sqrt[4]{8}\theta_2(2i) = 0, \quad \sqrt[8]{2}\theta_4(i) - \theta_4(2i) = 0.$$

Moreover, we have for $\tau = 1 + i\sqrt{3}$ (with $q = -e^{-\pi\sqrt{3}}$) the following linear identity involving values of θ_2 and θ_3 ,

$$2\theta_2(1 + i\sqrt{3}) - (1 + i)\sqrt[4]{28 - 16\sqrt{3}}\theta_3(1 + i\sqrt{3}) = 0.$$

3. Main tools towards the proof of our results

THEOREM 3.1 [9, p. 5]. *Let $\tau \in \mathbb{H}$ be an algebraic number of degree different than 2. Then $j(\tau)$ is transcendental.*

THEOREM 3.2 [2, Theorem 4]. *For any $\tau \in \mathbb{H}$, if $q = e^{i\pi\tau}$ is an algebraic number, and for integers $j, k, \ell \in \{2, 3, 4\}$ with $j \neq k$, then the three values $\theta_j(\tau)$, $\theta_k(\tau)$ and $D\theta_\ell(\tau)$ are algebraically independent over \mathbb{Q} . Here,*

$$D := \frac{1}{\pi i} \frac{d}{d\tau}$$

is a differential operator.

THEOREM 3.3 [7, Theorem 1, Corollary 4]. *For any odd integer $n \geq 3$ there exists an integer polynomial $P_n(X, Y)$ with $\deg_X P_n(X, Y) = \psi(n)$ such that*

$$P_n\left(n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0$$

holds for all $\tau \in \mathbb{H}$. Moreover, the polynomial $P_n(X, Y)$ is of the form

$$(3.1) \quad P_n(X, Y) := X^{\psi(n)} + R_1(Y)X^{\psi(n)-1} + \dots + R_{\psi(n)-1}(Y)X + R_{\psi(n)}(Y) \\ = \sum_{k=0}^{\psi(n)} R_k(Y)X^{\psi(n)-k},$$

where $R_j(Y) \in \mathbb{Z}[Y]$ for $j = 1, \dots, \psi(n)$, and

$$\deg R_k(Y) \leq k \cdot \frac{n-1}{n} \quad (1 \leq k \leq \psi(n)).$$

Let

$$d_n := \max_{1 \leq k \leq \psi(n)} \deg_Y R_k(Y).$$

Then, $P_n(X, Y)$ can be written as

$$(3.2) \quad P_n(X, Y) = \sum_{j=0}^{d_n} S_{n,j}(X)Y^j,$$

where $S_{n,j}(X) \in \mathbb{Z}[X]$ with $0 \leq j \leq d_n$ and $d_n > 0$, such that

$$(3.3) \quad S_{n,j}(0) = 0 \quad (1 \leq j \leq d_n), \quad S_{n,d_n}(X) \neq 0,$$

$$(3.4) \quad S_{n,0}(0) = P_n(0, 0) \in \mathbb{Z} \setminus \{0\}.$$

The properties (3.2) to (3.4) of $P_n(X, Y)$ follow from the proof of [4, Lemma 7]; cf. formula (9).

REMARK 3.4. From (3.1), we can observe that for any complex number α , the polynomial $P_n(X, \alpha)$ is non-zero.

The following lemmas are crucial for the proof of our results.

LEMMA 3.5 [3, Lemma 2.5]. *Let $n = 2^a s$ be an integer with $a \geq 1$ and an odd integer $s \geq 3$. Then there exists a polynomial $Q_n(X, Y)$ with integral coefficients such that*

$$Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0$$

holds for all $\tau \in \mathbb{H}$. Moreover, the polynomial $Q_n(X, Y)$ is of the form

$$(3.5) \quad Q_n(X, Y) = c_n^{2^a} Y^{2^a \psi(s)} + \sum_{j=0}^{2^a \psi(s) - 1} R_{n,j}(X) Y^j,$$

where c_n is a non-zero integer. Moreover,

$$(3.6) \quad \deg R_{n,j}(X) \leq 2^a \psi(s) - j \quad (0 \leq j < 2^a \psi(s)),$$

and

$$(3.7) \quad Q_n(0, Y) = c_n^{2^a} Y^{2^a \psi(s)},$$

$$(3.8) \quad R_{n,0}(X) = Q_n(X, 0) = 2^{4(2^a - 1)\psi(s)} X^{(2^a - 1)\psi(s)} P_s(X, 0).$$

PROOF. Apart from formula (3.8) the statements are given in [3, Lemma 2.5]. It remains to prove (3.8). We proceed by induction with respect to a and follow the lines of the proof of [3, Lemma 2.5]. As in the first part of the proof of Lemma 2.5 (corresponding to $a = \alpha = 1$) we construct the polynomials $B_{2s}(X, Y)$, $\tilde{Q}_{2s}(X, Y)$, and $Q_{2s}(X, Y)$, where

$$\tilde{Q}_{2s}(X^4, Y^4) = B_{2s}(X, Y) B_{2s}(X, iY)$$

and

$$Q_{2s}(X, Y) = \tilde{Q}_{2s}(X, 1 - Y).$$

From the proof of Lemma 2.5 we obtain

$$B_{2s}(X, 1) = 2^{2\psi(s)} P_s(X^4, 0), \quad B_{2s}(X, i) = 2^{2\psi(s)} X^{4\psi(s)}.$$

We sketch the proof of these two identities. We take the polynomials $B_{2s}(X, Y)$ from formula (19) in [3] by setting $n = 2s$:

$$B_{2s}(X, Y) = \sum_{\nu, \mu} 2^{2\nu} a_{\nu, \mu} X^{4\nu} (1 - Y^2)^{2\mu} (1 + Y^2)^{2(\psi(s) - \nu - \mu)};$$

where the variables $a_{\nu, \mu}$ are the coefficients of the polynomial

$$P_s(X, Y) = \sum_{\nu, \mu} a_{\nu, \mu} X^\nu Y^\mu$$

from Theorem 3.3. Thus, on the one hand, we obtain

$$B_{2s}(X, 1) = \sum_{\nu} 2^{2\nu} a_{\nu, 0} X^{4\nu} 2^{2(\psi(s) - \nu)} = 2^{2\psi(s)} \sum_{\nu} a_{\nu, 0} X^{4\nu} = 2^{2\psi(s)} P_s(X^4, 0),$$

on the other hand,

$$B_{2s}(X, i) = \sum_{\substack{\nu, \mu \\ \nu + \mu = \psi(s)}} 2^{2\nu} a_{\nu, \mu} X^{4\nu} 2^{2\mu} = 2^{2\psi(s)} \sum_{\substack{\nu, \mu \\ \nu + \mu = \psi(s)}} a_{\nu, \mu} X^{4\nu} = 2^{2\psi(s)} X^{4\psi(s)}.$$

The last identity follows from the fact that $a_{\nu, \mu} = 0$ for $\nu + \mu = \psi(s)$ with $\mu \geq 1$, which is a consequence of (3.1) and $\deg R_k(Y) < k$ for $k \geq 1$. For $\mu = 0$ and $\nu = \psi(s)$ we have $a_{0, \psi(s)} = R_0(Y) = 1$, cf. [7, p. 154].

This gives

$$\tilde{Q}_{2s}(X^4, 1) = B_{2s}(X, 1)B_{2s}(X, i) = 2^{4\psi(s)} X^{4\psi(s)} P_s(X^4, 0),$$

and, consequently,

$$Q_{2s}(X, 0) = \tilde{Q}_{2s}(X, 1) = 2^{4\psi(s)} X^{\psi(s)} P_s(X, 0).$$

This shows that (3.8) holds for $a = 1$. Next, let (3.8) be already proven for some fixed $a \geq 1$. For the induction step we construct the polynomials $B_{2^{a+1}s}(X, Y)$, $\tilde{Q}_{2^{a+1}s}(X, Y)$, and $Q_{2^{a+1}s}(X, Y)$, where

$$\begin{aligned} \tilde{Q}_{2^{a+1}s}(X^4, Y^4) &= B_{2^{a+1}s}(X, Y)B_{2^{a+1}s}(X, iY), \\ Q_{2^{a+1}s}(X, Y) &= \tilde{Q}_{2^{a+1}s}(X, 1 - Y). \end{aligned}$$

Since $\deg_X Q_{2^a s}(X, Y) = 2^a \psi(s)$, we obtain by applying the induction hypothesis,

$$\begin{aligned} B_{2^{a+1}s}(X, 1) &= 2^{2 \cdot 2^a \psi(s)} Q_{2^a s}(X^4) \\ &= 2^{2^{a+1} \psi(s)} \left(2^{4(2^a - 1)\psi(s)} X^{4(2^a - 1)\psi(s)} P_s(X^4, 0) \right), \\ B_{2^{a+1}s}(X, i) &= 2^{2 \cdot 2^a \psi(s)} X^{4 \cdot 2^a \psi(s)} = 2^{2^{a+1} \psi(s)} X^{4 \cdot 2^a \psi(s)}. \end{aligned}$$

Therefore, it turns out that

$$\begin{aligned} \tilde{Q}_{2^{a+1}s}(X^4, 1) &= B_{2^{a+1}s}(X, 1)B_{2^{a+1}s}(X, i) \\ &= 2^{2 \cdot 2^{a+1} \psi(s) + 4(2^a - 1)\psi(s)} X^{4 \cdot 2^a \psi(s) + 4(2^a - 1)\psi(s)} P_s(X^4, 0) \\ &= 2^{4(2^{a+1} - 1)\psi(s)} X^{4(2^{a+1} - 1)\psi(s)} P_s(X^4, 0). \end{aligned}$$

We complete the proof of the lemma by observing that

$$Q_{2^{a+1}s}(X, 0) = \tilde{Q}_{2^{a+1}s}(X, 1) = 2^{4(2^{a+1} - 1)\psi(s)} X^{(2^{a+1} - 1)\psi(s)} P_s(X, 0),$$

which is the identity in (3.8) with a replaced by $a + 1$. \square

LEMMA 3.6. *Let $s \geq 3$ be an odd integer and let $P_s(X, Y)$ be the integer polynomial from Theorem 3.3 with n replaced by s . Then we have*

$$P_s(X, 0) = \prod_{\substack{u|s \\ u \geq 1}} (X - u^2)^{w(u, s/u)},$$

where $w(a, b)$ is defined by the number of integers k with

$$0 \leq k < b \quad \text{and} \quad \gcd(a, b, k) = 1.$$

PROOF. The statement follows from the identity given in [4, Lemma 4].
□

LEMMA 3.7. *Let $\tau \in \mathbb{H}$ be as in Theorem 2.2. Then the numbers θ_3/θ_4 and θ_2/θ_3 are transcendental.*

PROOF. *Case 1:* τ is an algebraic number of degree ≥ 3 . By Theorem 3.1, the number

$$j(\tau) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

is transcendental. This implies that $\lambda = \theta_2^4/\theta_3^4$ is transcendental, and so is θ_2/θ_3 . Now by using the identity $\theta_2^4 + \theta_4^4 = \theta_3^4$, we conclude that the number θ_3/θ_4 is transcendental.

Case 2: $\tau \in \mathbb{H}$ such that $q = e^{i\pi\tau}$ is algebraic over \mathbb{Q} . Since $q = e^{i\pi\tau}$ is an algebraic number, θ_3 and θ_4 are algebraically independent as well as θ_2 and θ_3 (cf. Theorem 3.2). Therefore, the numbers θ_3/θ_4 and θ_2/θ_3 are transcendental. By Cases 1 and 2, we complete the proof of the lemma. □

LEMMA 3.8. *Let $m \geq 3$ be an integer which is either odd or an even number of the form $2^a s$, where $a \geq 1$, and $s \geq 3$ is an odd integer. Then, for any $\tau \in \mathbb{H}$ satisfying the conditions in Theorem 2.2, the number $\frac{\theta_3(m\tau)}{\theta_3(\tau)}$ is transcendental.*

PROOF. We assume that $\theta_3(m\tau)/\theta_3(\tau)$ is algebraic. By Theorem 3.3 and Lemma 3.5, there exists an integer polynomial $T_m(X, Y)$ defined by

$$T_m(X, Y) := \begin{cases} P_m(m^2 X, 16Y), & \text{if } m \equiv 1 \pmod{2}, \\ Q_m(X, Y), & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

such that

$$(3.9) \quad T_m\left(\frac{\theta_3^4(m\tau)}{\theta_3^4(\tau)}, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0.$$

Now we consider the polynomial $R_m(Y) = T_m\left(\frac{\theta_3^4(m\tau)}{\theta_3^4(\tau)}, Y\right) \in \overline{\mathbb{Q}}[Y]$ having algebraic coefficients by our assumption. The polynomial $R_m(Y)$ does not vanish identically: for odd integers m this follows from [3, Lemma 2.1], for even m this is a consequence of Lemma 3.5, cf. (3.5). Hence, by (3.9), we obtain

$$R_m\left(\frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = T_m\left(\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0.$$

This implies that θ_2/θ_3 is algebraic, which is a contradiction to Lemma 3.7. Therefore, we conclude that the number $\theta_3(m\tau)/\theta_3(\tau)$ is transcendental. \square

4. The proofs of our results

PROOF OF THEOREM 2.1. (i) Suppose that the functions $\theta_3(a_1\tau), \dots, \theta_3(a_m\tau)$ are linearly dependent over $\mathbb{C}(\tau)$. Then there exist $c_1(\tau), \dots, c_m(\tau) \in \mathbb{C}[\tau]$, not all zero and with minimal degree, such that

$$(4.1) \quad c_1(\tau)\theta_3(a_1\tau) + \dots + c_m(\tau)\theta_3(a_m\tau) = 0 \quad \text{for all } \tau \in \mathbb{H}.$$

Let M be the common denominator of the rational numbers a_1, \dots, a_m . Then $Ma_j \in \mathbb{Z}$ for every $j = 1, \dots, m$, and we notice that for all $j = 1, 2, \dots, m$,

$$\theta_3(a_j(\tau + 2M)) = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi a_j \tau n^2} e^{2M a_j \pi n^2} = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi a_j \tau n^2} = \theta_3(a_j\tau).$$

Hence, the functions $\theta_3(a_1\tau), \theta_3(a_2\tau), \dots, \theta_3(a_m\tau)$ are periodic.

Replacing τ by $\tau + 2M$ and using the periodicity, we have

$$(4.2) \quad c_1(\tau + 2M)\theta_3(a_1\tau) + \dots + c_m(\tau + 2M)\theta_3(a_m\tau) = 0 \quad \text{for all } \tau \in \mathbb{H}.$$

Thus, from (4.1) and (4.2), we obtain

$$(c_1(\tau) - c_1(\tau + 2M))\theta_3(a_1\tau) + \dots + (c_1(\tau) - c_m(\tau + 2M))\theta_3(a_m\tau) = 0$$

for all $\tau \in \mathbb{H}$. Note that the degree of the polynomial $c_j(\tau + 2M) - c_j(\tau)$ is strictly less than the degree of the polynomial $c_j(\tau)$. Therefore, by the minimality of the polynomials $c_1(\tau), \dots, c_m(\tau)$, we get $c_j(\tau + 2M) = c_j(\tau)$ for all $j = 1, 2, \dots, m$, which in turns implies that $c_1(\tau), \dots, c_m(\tau)$ are constant polynomials. Hence, in order to prove that these functions are linearly independent over $\mathbb{C}(\tau)$, it suffices to prove the linear independence over \mathbb{C} .

Therefore we can consider the identity

$$c_1\theta_3(a_1\tau) + \dots + c_m\theta_3(a_m\tau) = 0, \quad \text{for all } \tau \in \mathbb{H} \text{ and fixed } c_j \in \mathbb{C}.$$

This can be rewritten as

$$(4.3) \quad c_1 \left(1 + 2 \sum_{n=1}^{\infty} e^{i\pi\tau a_1 n^2} \right) + \dots + c_m \left(1 + 2 \sum_{n=1}^{\infty} e^{i\pi\tau a_m n^2} \right) = 0 \text{ for all } \tau \in \mathbb{H}.$$

Putting $\tau = iX$ and letting $X \rightarrow \infty$ in the above equality, we have

$$(c_1 + \dots + c_m) + 2 \lim_{X \rightarrow \infty} \left(c_1 \sum_{n=1}^{\infty} e^{-\pi X a_1 n^2} + \dots + c_m \sum_{n=1}^{\infty} e^{-\pi X a_m n^2} \right) = 0.$$

Since $\lim_{X \rightarrow \infty} \left(\sum_{n=1}^{\infty} e^{-\pi X a_j n^2} \right) = 0$ for all $j = 1, 2, \dots, m$, we have

$$c_1 + c_2 + \dots + c_m = 0.$$

Therefore (4.3) becomes

$$(4.4) \quad c_1 \sum_{n=1}^{\infty} e^{-\pi X a_1 n^2} + \dots + c_m \sum_{n=1}^{\infty} e^{-\pi X a_m n^2} = 0 \text{ for all } X > 0.$$

Without loss of generality we can assume that $a_1 < a_2 < \dots < a_m$. Multiplying the above equality by $e^{a_1 \pi X}$, we get

$$(4.5) \quad -c_1 = c_1 \sum_{n=2}^{\infty} e^{-\pi X a_1 n^2 + \pi X a_1} + \left(c_2 \sum_{n=1}^{\infty} e^{-\pi X a_2 n^2 + \pi X a_1} + \dots + c_m \sum_{n=1}^{\infty} e^{-\pi X a_m n^2 + \pi X a_1} \right)$$

Since $-\pi a_1 n^2 + \pi a_1 < 0$ for $n \geq 2$ and $-\pi a_j n^2 + \pi a_1 < 0$ for all $j = 2, 3, \dots, m$, we see that the right-hand side of (4.5) tends to zero as $X \rightarrow \infty$. Therefore, we conclude that $c_1 = 0$, and (4.4) becomes

$$c_2 \sum_{n=1}^{\infty} e^{-\pi X a_2 n^2} + \dots + c_m \sum_{n=1}^{\infty} e^{-\pi X a_m n^2} = 0 \text{ for all } X \in \mathbb{N}.$$

Now we multiply the above equality by $e^{a_2 \pi X}$ and proceed by the same process in order to get $c_2 = 0$. Hence, by continuing this process, we get $c_1 = c_2 = \dots = c_m = 0$, which gives a contradiction.

(ii) As we have seen in the above proof of the first statement in Theorem 2.2(1), the assumption that the numbers a_1, \dots, a_m are integers, is only used to reduce the arguments on the linear independence of m -theta functions $\theta_3(a_1\tau), \dots, \theta_3(a_m\tau)$ over the field \mathbb{C} . Therefore, by using the similar

approach as above, one can prove the second statement in Theorem 2.2(1). \square

PROOF OF THEOREM 2.2. Throughout this proof, we denote the degree of a polynomial $T(X, Y)$ with respect to X and Y by $\deg_X T(X, Y)$ and $\deg_Y T(X, Y)$, respectively; and by $\deg T(X, Y)$ we denote the total degree of the polynomial $T(X, Y)$.

It is sufficient to prove that the three numbers $1, \frac{\theta_3(m\tau)}{\theta_3(\tau)}, \frac{\theta_3(n\tau)}{\theta_3(\tau)}$ are linearly independent over $\overline{\mathbb{Q}}$. Suppose that these numbers are linearly dependent over $\overline{\mathbb{Q}}$. Then, there exist algebraic integers $\alpha_0, \alpha_1, \alpha_2$ not all zero such that

$$(4.6) \quad \alpha_0 + \alpha_1 \frac{\theta_3(m\tau)}{\theta_3(\tau)} + \alpha_2 \frac{\theta_3(n\tau)}{\theta_3(\tau)} = 0.$$

It is clear that neither α_1 nor α_2 vanishes, since otherwise (when $\alpha_1 = 0, \alpha_2 \neq 0$ or $\alpha_1 \neq 0, \alpha_2 = 0$) there is a contradiction to Lemma 3.8, as both the numbers $\frac{\theta_3(m\tau)}{\theta_3(\tau)}$ and $\frac{\theta_3(n\tau)}{\theta_3(\tau)}$ are transcendental. This implies that both, α_1 and α_2 , are non-zero. Then, when $\alpha_0 = 0$, we get a contradiction to [5, Theorem 1.1]. Thus, it implies that $\alpha_0\alpha_1\alpha_2 \neq 0$. Then from (4.6) and Theorem 3.3, we have

$$(4.7) \quad P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \frac{\theta_3(n\tau)}{\theta_3(\tau)} \right)^4, 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \right) = 0.$$

By the explicit form of the polynomials $P_m(X, Y)$ and $P_n(X, Y)$, we see that the polynomials

$$H_m(X) = P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \right)$$

and

$$S_n(X) = P_n \left(n^2 X^4, 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \right)$$

are non-zero. The polynomials

$$P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \frac{\theta_3(n\tau)}{\theta_3(\tau)} \right)^4, 16Y \right) \quad \text{and} \quad P_n \left(n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, 16Y \right)$$

have the same common root $Y_0 := \theta_2^4(\tau)/\theta_3^4(\tau)$. Hence, the resultant

$$(4.8) \quad R(X) := \text{Res}_Y \left(P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16Y \right), P_n(n^2 X^4, 16Y) \right)$$

is given by the determinant of a square matrix where the dimensions and elements of the corresponding Sylvester matrix depend on the degrees and on the coefficients, respectively, of the polynomials

$$P_m\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\right)X, 16Y\right)$$

and $P_n(n^2X, 16Y)$. By Lemma 3.8 we know that $\theta_3(n\tau)/\theta_3(\tau)$ is transcendental. Then, from $S_{n,d_n} \neq 0$ in (3.3), we have

$$S_{m,d_m}\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \cdot \frac{\theta_3(n\tau)}{\theta_3(\tau)}\right)^4\right) \neq 0 \quad \text{and} \quad S_{n,d_n}\left(n^2\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}\right) \neq 0.$$

Hence, there is some real number $\delta > 0$ depending on $n, m, \alpha_0, \alpha_1, \alpha_2$, and τ such that

$$S_{m,d_m}\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}X\right)^4\right) \neq 0 \quad \text{and} \quad S_{n,d_n}(n^2X^4) \neq 0$$

hold for

$$\left|X - \frac{\theta_3(n\tau)}{\theta_3(\tau)}\right| < \delta.$$

Then, for all X inside this circle, we have

$$\begin{aligned} &\deg_Y P_m\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}X\right)^4, 16Y\right) \\ &= \deg_Y P_m\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \cdot \frac{\theta_3(n\tau)}{\theta_3(\tau)}\right)^4, 16Y\right) = d_m, \end{aligned}$$

and, similarly,

$$\deg_Y P_n(n^2X^4, 16Y) = \deg_Y P_n\left(n^2\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, 16Y\right) = d_n.$$

For X restricted to the inside of the circle with radius δ mentioned above, $R(X)$ can be considered as a polynomial in X depending on the elements of a Sylvester matrix with fixed dimensions $d_n + d_m$. On the scale of things $R(X)$ is some polynomial with algebraic coefficients such that

$$(4.9) \quad R\left(\frac{\theta_3(n\tau)}{\theta_3(\tau)}\right) = 0,$$

since Y_0 is a common root of the polynomials under consideration. First we note that the polynomial $R(X)$ is not identically zero. We assume the contrary, namely that

$$(4.10) \quad R(X) \equiv 0.$$

Then, by (4.8) and (4.10),

$$\text{Res}_Y \left(P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16Y \right), P_n(n^2 X^4, 16Y) \right) = R(X) \equiv 0,$$

and so there exists a common factor $H(X, Y) \in \mathbb{C}[X, Y]$ with positive degree in Y of the polynomials

$$P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16Y \right) \quad \text{and} \quad P_n(n^2 X^4, 16Y).$$

Let

$$(4.11) \quad P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16Y \right) = H(X, Y)G(X, Y).$$

By substituting $Y = \lambda(\tau)$ defined in the introduction into the above equation, we have

$$(4.12) \quad P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16\lambda(\tau) \right) = H(X, \lambda(\tau))G(X, \lambda(\tau)).$$

Using the definition of $P_m(X, Y)$ in Theorem 3.3, we have

$$\deg_Y R_k(Y) \leq k \frac{m-1}{m} < k \quad (1 \leq k \leq \psi(m))$$

and, for $k = 0$,

$$\deg_Y R_0(Y) = 0, \quad \text{since } R_0(Y) = 1.$$

Thus, we obtain by the right-hand side of formula (3.1),

$$(4.13) \quad \begin{aligned} \deg P_m(X, Y) &= \max_{0 \leq k \leq \psi(m)} \{ \psi(m) - k + \deg_Y R_k(Y) \} \\ &= \psi(m) = \deg_X P_m(X, Y) = \deg_X P_m(X, 16\lambda(\tau)). \end{aligned}$$

The last two identities in (4.13) are a consequence of

$$P_m(X, Y) = X^{\psi(m)} + \mathcal{O}_Y(X^{\psi(m)-1}).$$

Replacing X in (4.13) by

$$m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4,$$

we obtain the identity

$$(4.14) \quad \deg_X P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16\lambda(\tau) \right) = \deg P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16Y \right).$$

Hence, by the above identities, we obtain

$$(4.15) \quad \begin{aligned} \deg_X H(X, \lambda(\tau)) + \deg_X G(X, \lambda(\tau)) &= \deg_X H(X, \lambda(\tau)) G(X, \lambda(\tau)) \\ &\stackrel{(4.12)}{=} \deg_X P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16\lambda(\tau) \right) \\ &\stackrel{(4.14)}{=} \deg P_m \left(m^2 \left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, 16Y \right) \\ &\stackrel{(4.11)}{=} \deg H(X, Y) G(X, Y) = \deg H(X, Y) + \deg G(X, Y). \end{aligned}$$

Additionally, we have the obvious inequalities

$$(4.16) \quad \deg_X H(X, \lambda(\tau)) \leq \deg H(X, Y) \quad \text{and} \quad \deg_X G(X, \lambda(\tau)) \leq \deg G(X, Y).$$

Thus, we obtain from (4.15) and (4.16) that $\deg_X H(X, \lambda(\tau)) = \deg H(X, Y)$ (and, similarly, $\deg_X G(X, \lambda(\tau)) = \deg G(X, Y)$), consequently

$$(4.17) \quad \deg_X H(X, \lambda(\tau)) \geq \deg_Y H(X, Y) \geq 1.$$

By Lemma 3.7, $\frac{\theta_2(\tau)}{\theta_3(\tau)}$ is transcendental in each of the two cases τ algebraic of degree ≥ 3 and τ such that $e^{i\pi\tau}$ is algebraic, hence the number $16\lambda(\tau)$ is transcendental over \mathbb{Q} . So, it is also transcendental over $\overline{\mathbb{Q}}$. Let

$$\beta_m := \frac{\theta_3(m\tau)}{\theta_3(\tau)}.$$

By Theorem 3.3, we know that $P_m(m^2\beta_m^4, 16\lambda(\tau)) = 0$. Moreover, it follows from [4, Lemma 2] that the polynomial $P_m(X, 16\lambda(\tau))$ is irreducible over the field $\mathbb{K} := \overline{\mathbb{Q}}(\lambda(\tau))$. Overall, we may consider β_m as an algebraic number over \mathbb{K} , where, by (3.1),

$$(4.18) \quad \deg_{\mathbb{K}} \beta_m = 4 \deg_X P_m(X, Y) = 4\psi(m).$$

Let us assume that the polynomial

$$(4.19) \quad P_m\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}X\right)^4, 16\lambda(\tau)\right)$$

is reducible over \mathbb{K} . Then we obtain the inequality

$$\deg_{\mathbb{K}}\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\beta_m\right)^4\right) < 4\psi(m).$$

But this is impossible by (4.18), and by

$$m^2, \frac{\alpha_0}{\alpha_1}, \frac{\alpha_2}{\alpha_1} \in \overline{\mathbb{Q}} \setminus \{0\},$$

since β_m is transcendental over $\overline{\mathbb{Q}}$ by the identity $P_m(m^2\beta_m^4, 16\lambda(\tau)) = 0$ and by the transcendence of $\lambda(\tau)$, as noticed earlier. The contradiction proves the irreducibility of the polynomial in (4.19).

Thus, from (4.11) and (4.17), we obtain

$$P_m\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}X\right)^4, 16\frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = \beta_1 H(X, \lambda(\tau))$$

for some non-zero complex number β_1 . Similarly, there exists a non-zero complex number β_2 such that

$$P_n(n^2X^4, 16\lambda(\tau)) = \beta_2 H(X, \lambda(\tau)),$$

and hence

$$P_m\left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}X\right)^4, 16\lambda(\tau)\right) = cP_n(n^2X^4, 16\lambda(\tau)), \quad c := \beta_1/\beta_2.$$

This polynomial identity holds for all complex numbers $\tau \in \mathbb{H}$. We know that for $\tau \rightarrow i\infty$ the function $\lambda(\tau)$ tends to zero. Hence, taking $\tau \rightarrow i\infty$ into the above equality, we have by Lemma 3.6,

$$\prod_{d|m} \left(m^2\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1}X\right)^4 - d^2\right)^{w(d,m/d)} = c \prod_{d|n} (n^2X^4 - d^2)^{w(d,n/d)}.$$

Then, comparing the multiplicity of the zero of these polynomials at $X = -(\alpha_0 + \alpha_1/\sqrt{m})/\alpha_2$ (and $d = 1$ on the left-hand side), we obtain

$$m = w(1, m) \leq \max_d w(d, n/d) \leq n,$$

which is a contradiction to the condition $n < m$ from the theorem. Hence, the polynomial $R(X)$ is non-zero. Therefore, it follows from (4.9) that the

number $\theta_3(n\tau)/\theta_3(\tau)$ is algebraic, which is a contradiction to the fact from Lemma 3.8 that the number $\theta_3(n\tau)/\theta_3(\tau)$ is transcendental. This proves the assertion. \square

PROOF OF THEOREM 2.3. Let $m = 2^a s_1$ and $n = 2^b s_2$ be two different integers with $a, b \geq 1$ and odd integers $s_1, s_2 \geq 3$. By Lemma 3.5 there exist integer polynomials $Q_m(X, Y)$ and $Q_n(X, Y)$ such that

$$Q_m\left(\frac{\theta_3^4(m\tau)}{\theta_3^4(\tau)}, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0 \quad \text{and} \quad Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0.$$

We assume that the linear equation (4.6) holds, where $\alpha_0, \alpha_1, \alpha_2$ are algebraic numbers satisfying the hypothesis in Theorem 2.3. As in the proof of Theorem 2.2, we have $\alpha_0 \alpha_1 \alpha_2 \neq 0$. By the hypotheses of the theorem, we may assume without loss of generality that $\beta := \alpha_2/\alpha_0$ satisfies $R_{n,0}(\beta^{-4}) \neq 0$. Namely, it is obvious by (3.8) and Lemma 3.6 that $R_{n,0}(\beta^{-4}) \neq 0$ holds particularly for $\beta^{-1} \notin M_{s_2}$ and for $\beta \notin \mathbb{Q}$.

Then we obtain

$$Q_m\left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \frac{\theta_3(n\tau)}{\theta_3(\tau)}\right)^4, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0.$$

By the explicit form of the polynomials $Q_m(X, Y)$ and $Q_n(X, Y)$, we see that the polynomials

$$Q_m\left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X\right)^4, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) \quad \text{and} \quad Q_n\left(X^4, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right)$$

are non-zero. Hence, the polynomials

$$Q_m\left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \frac{\theta_3(n\tau)}{\theta_3(\tau)}\right)^4, Y\right) \quad \text{and} \quad Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, Y\right)$$

have the same common root $Y_0 = \theta_2^4(\tau)/\theta_3^4(\tau)$.

Let

$$H_m(X, Y) := Q_m\left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X\right)^4, Y\right)$$

and

$$W(X) := \text{Res}_Y(H_m(X, Y), Q_n(X^4, Y)) \in \overline{\mathbb{Q}}[X].$$

From Lemma 3.5 we know that both $\deg_Y Q_m(X, Y)$ and $\deg_Y Q_n(X, Y)$ do not depend on X , since the coefficients of the leading terms with respect to Y are non-zero integers. Thus, $W(X)$ can be considered as a polynomial for all X .

In order to show that the polynomial $W(X)$ does not vanish identically, we shall prove the existence of a number η satisfying $W(\eta) \neq 0$, or, equivalently, that the polynomials $H_m(\eta, Y)$ and $Q_n(\eta^4, Y)$ are coprime. Let

$$\eta := -\frac{1}{\beta} = -\frac{\alpha_0}{\alpha_2}.$$

On the one hand, by using (3.7), we obtain

$$H_m(\eta, Y) = Q_m(0, Y) = c_m^{2^a} Y^{2^a \psi(s_1)}.$$

Therefore, $H_m(\eta, Y)$ is a nonvanishing polynomial in Y having exclusively a multiple root at $Y = 0$.

On the other hand, by applying formulas (3.5) and (3.6) in Lemma 3.5, we have

$$Q_n(\eta^4, Y) = c_n^{2^b} Y^{2^b \psi(s_2)} + \sum_{j=0}^{2^b \psi(s_2) - 1} R_{n,j}(\eta^4) Y^j$$

with $c_n \in \mathbb{Z} \setminus \{0\}$ and $R_{n,0}(X) \neq 0$ by (3.8) and Lemma 3.6. We already know by $\eta = -1/\beta$ that $R_{n,0}(\eta^4) \neq 0$. Consequently, we have $Q_n(\eta^4, 0) = R_{n,0}(\eta^4) \neq 0$.

Altogether, the polynomials $H_m(\eta, Y)$ and $Q_n(\eta^4, Y)$ have no common root. More precisely, we obtain for $W(X)$,

$$W(\eta) = \text{Res}_Y(H_m(\eta, Y), Q_n(\eta^4, Y)) \neq 0.$$

This shows that $W(X)$ does not vanish identically. By construction, we know that $W(X_0)$ vanishes for $X_0 := \frac{\theta_3(n\tau)}{\theta_3(\tau)}$, which implies the algebraicity of $\theta_3(n\tau)/\theta_3(\tau)$, a contradiction to Lemma 3.8. This finally shows that the linear relation (4.6) cannot hold. \square

PROOF OF THEOREM 2.5. Let $m = 2^a s$ and n be two integers with $a \geq 1$ and odd integers $n, s \geq 3$. By Theorem 3.3 and Lemma 3.5 there exist integer polynomials $P_n(X, Y)$ and $Q_m(X, Y)$ such that

$$Q_m\left(\frac{\theta_3^4(m\tau)}{\theta_3^4(\tau)}, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0 \quad \text{and} \quad P_n\left(n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0.$$

We assume that the linear equation (4.6) holds. As in the proof of Theorem 2.2 we have $\alpha_0 \alpha_1 \alpha_2 \neq 0$. By the hypotheses of the theorem, we may assume that $\beta := \alpha_2/\alpha_0$ satisfies either

$$\deg_{\mathbb{Q}}(\beta^4) > \psi(n), \quad \text{or} \quad S_{n,0}(n^2 \beta^{-4}) S_{n,d_n}(n^2 \beta^{-4}) \neq 0.$$

Then we obtain

$$Q_m \left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \frac{\theta_3(n\tau)}{\theta_3(\tau)} \right)^4, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \right) = 0.$$

By the explicit form of the polynomials $Q_m(X, Y)$ and $P_n(X, Y)$ given by Theorem 3.3 and Lemma 3.5, we see that the polynomials

$$Q_m \left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \right) \quad \text{and} \quad P_n \left(n^2 X^4, 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \right)$$

are non-zero. Hence, the polynomials

$$Q_m \left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \frac{\theta_3(n\tau)}{\theta_3(\tau)} \right)^4, Y \right) \quad \text{and} \quad P_n \left(n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, 16Y \right)$$

have the same common root $Y_0 = \theta_2^4(\tau)/\theta_3^4(\tau)$. Let

$$H_m(X, Y) := Q_m \left(\left(-\frac{\alpha_0}{\alpha_1} - \frac{\alpha_2}{\alpha_1} X \right)^4, Y \right)$$

and

$$W(X) := \text{Res}_Y (H_m(X, Y), P_n(n^2 X^4, 16Y)).$$

From Lemma 3.5, formula (3.5), we know that $\deg_Y Q_m(X, Y)$ (and, consequently, $\deg_Y H_m(X, Y)$) does not depend on X , since the coefficient of the leading term with respect to Y is the non-zero integer $c_m^{2^a}$. For all real numbers X which are not a root of the polynomial $S_{n,d_n}(X)$ in (3.2), the leading term of $P_n(X, Y)$ with respect to Y does not vanish. Consequently, $W(X)$ is given by the same polynomial for all these X , since the degrees $\deg_Y H_m(X, Y)$ and $\deg_Y P_n(n^2 X^4, 16Y)$ do not depend on all X satisfying $S_{n,d_n}(n^2 X^4) \neq 0$. Note that $S_{n,d_n}(X) \neq 0$ by (3.3).

In order to show that $W(X)$ does not vanish identically for X with $S_{n,d_n}(n^2 X^4) \neq 0$, we shall prove the existence of a number η satisfying $W(\eta) \neq 0$, or, equivalently, that the polynomials $H_m(\eta, Y)$ and $P_n(n^2 \eta^4, 16Y)$ are coprime. Let

$$\eta := -\frac{1}{\beta} = -\frac{\alpha_0}{\alpha_2}.$$

On the one hand, by using (3.7), we obtain

$$H_m(\eta, Y) = Q_m(0, Y) = c_m^{2^a} Y^{2^a \psi(s)}.$$

Therefore, $H_m(\eta, Y)$ is a nonvanishing polynomial in Y having only a multiple root at $Y = 0$.

On the other hand, by the hypothesis on β in Theorem 2.5 and by

$$\deg_X S_{n,d_n}(X) \leq \deg_X P_n(X, Y) = \psi(n)$$

(cf. (3.2) in Theorem 3.3), we know that

$$S_{n,d_n}(n^2\eta^4) = S_{n,d_n}\left(\frac{n^2}{\beta^4}\right) = S_{n,d_n}\left(\frac{n^2\alpha_0^4}{\alpha_2^4}\right) \neq 0.$$

This shows that the degree with respect to Y of the polynomial on the right-hand side of (3.2) does not change for the particular choice of $X = n^2\eta^4$. Moreover, it follows from (3.4) that $S_{n,0} \neq 0$, and therefore the inequality

$$\deg_X S_{n,0}(X) \leq \deg_X P_n(X, Y) = \psi(n)$$

and the conditions on β imply that

$$S_{n,0}(n^2\eta^4) = S_{n,d_n}\left(\frac{n^2}{\beta^4}\right) = S_{n,0}\left(\frac{n^2\alpha_0^4}{\alpha_2^4}\right) \neq 0.$$

Thus, again the application of (3.2) gives $P_n(n^2\eta^4, 0) \neq 0$. Altogether, the polynomials $H_m(\eta, Y)$ and $P_n(n^2\eta^4, 16Y)$ have no common root. More precisely, we obtain that

$$W(\eta) = \text{Res}_Y(H_m(\eta, Y), P_n(n^2\eta^4, 16Y)) \neq 0.$$

This shows that $W(X)$ does not vanish identically for all X satisfying $S_{n,d_n}(n^2X^4) \neq 0$. By construction, we know that $W(X_0)$ vanishes for $X_0 := \frac{\theta_3(n\tau)}{\theta_3(\tau)}$, and since $\theta_3(n\tau)/\theta_3(\tau)$ is transcendental by Lemma 3.8, we have by (3.3) that

$$S_{n,d_n}\left(n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}\right) \neq 0.$$

Thus, $X = X_0$ is a zero of the function $W(X)$, which restricted to all values X satisfying $S_{n,d_n}(n^2X^4) \neq 0$ results in the same nonvanishing polynomial $W(X)$. This implies the algebraicity of $\theta_3(n\tau)/\theta_3(\tau)$, a contradiction to Lemma 3.8. This finally shows that the linear relation (4.6) cannot hold. \square

PROOF OF PROPOSITION 2.6. Replacing τ by $2^m\tau$, it suffices to prove the assertion for the three numbers

$$\theta_3(\tau), \quad \theta_3(2\tau), \quad \theta_3(4\tau).$$

We have the following identities:

$$(4.20) \quad 2\theta_3^2(2\tau) = \theta_3^2 + \theta_4^2, \quad 2\theta_3(4\tau) = \theta_3 + \theta_4.$$

Suppose there exist algebraic numbers α, β, γ not all zero such that

$$(4.21) \quad 2\alpha\theta_3(\tau) + 2\beta\theta_3(2\tau) + 2\gamma\theta_3(4\tau) = 0.$$

Substituting (4.20) into (4.21), we get

$$2\alpha\theta_3 + 2\beta\sqrt{\frac{\theta_3^2 + \theta_4^2}{2}} + \gamma(\theta_3 + \theta_4) = 0.$$

By rearranging this formula, we get

$$(4.22) \quad ((2\alpha + \gamma)^2 - 2\beta^2)^2\theta_3^2 + (\gamma^2 - 2\beta^2)\theta_4^2 + 2\gamma(2\alpha + \gamma)\theta_3\theta_4 = 0.$$

Dividing (4.22), by θ_3^2 , we obtain

$$((2\alpha + \gamma)^2 - 2\beta^2)^2 + (\gamma^2 - 2\beta^2)\left(\frac{\theta_4}{\theta_3}\right)^2 + 2\gamma(2\alpha + \gamma)\frac{\theta_4}{\theta_3} = 0.$$

Hence, by Lemma 3.7, we have

$$(2\alpha + \gamma)^2 - 2\beta^2 = 0, \quad \gamma^2 - 2\beta = 0, \quad 2\gamma(2\alpha + \gamma) = 0.$$

Thus, we conclude that $\alpha = \beta = \gamma = 0$. \square

5. Concluding remarks

In the case when $\tau \in \mathbb{H}$ such that $e^{i\pi\tau}$ is algebraic, the number $\theta_3(\tau)$ is transcendental due to the algebraic independence of the values $\theta_3(m\tau)$ and $\theta_3(n\tau)$ for distinct positive integers m, n . By our Theorem 2.2, we know that at least two of the numbers among $\theta_3(\tau), \theta_3(m\tau)$ and $\theta_3(n\tau)$ are transcendental for any $\tau \in \mathbb{H}$ such that either τ is algebraic of degree ≥ 3 or $e^{i\pi\tau}$ is algebraic. In this context, it is interesting to consider the following problem:

PROBLEM 1. *Let τ and m, n be as in Theorem 2.2. Then 1, $\theta_3(\tau), \theta_3(m\tau)$ and $\theta_3(n\tau)$ are $\overline{\mathbb{Q}}$ -linearly independent.*

As a consequence of this problem, one can conclude the transcendence of $\theta_3(\tau)$ for algebraic τ of degree ≥ 3 , which is not known in this case.

In this paper we considered linear forms in three values of theta constant θ_3 . It is natural to consider the following more general problem:

PROBLEM 2. Let a_1, a_2, \dots, a_m be distinct positive integers. Let $\alpha_0, \dots, \alpha_m$ be non-zero algebraic numbers. For what values of $\tau \in \mathbb{H}$, will the linear form

$$L := \alpha_1 \theta_3(a_1 \tau) + \dots + \alpha_m \theta_3(a_m \tau)$$

vanish? In the case α_i 's are rational and $\tau = \frac{i \log b}{2\pi}$, where $b \geq 2$ is an integer, $L \neq 0$ by the result from [6] and [8].

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References

- [1] B.C. Berndt, *Ramanujan's Notebook*, part V, Springer-Verlag (New York, 1998).
- [2] D. Bertrand, Theta functions and transcendence, *Ramanujan J.* **1** (1997), 339–350.
- [3] C. Elsner and Y. Tachiya, Algebraic results for certain values of the Jacobi theta-constant $\theta_3(\tau)$, *Math. Scand.*, **123**, (2018), 249–272.
- [4] C. Elsner, F. Luca and Y. Tachiya, Algebraic results for the values $\theta_3(m\tau)$ and $\theta_3(n\tau)$ of the Jacobi-theta constant, *Mosc. J. Comb. Number Theory*, **8** (2019), 71–79.
- [5] C. Elsner, M. Kaneko and Y. Tachiya, Algebraic independence results for the values of the theta-constants and some identities, *J. Ramanujan Math. Soc.*, **35** (2020), 71–80.
- [6] D. Karmakar, V. Kumar and R. Thangadurai, Linear independence of special values of Jacobi theta-constants (submitted).
- [7] Yu.V. Nesterenko, On some identities for theta-constants, in: *Diophantine Analysis and Related Fields 2006*, Sem. Math. Sci., 35, Keio University (Yokohama, 2006), pp. 151–160.
- [8] M. Shintaro and Y. Tachiya, Linear independence of certain numbers in the base- b number system, *Arch. Math. (Basel)* **122** (2024), 31–40.
- [9] Th. Schneider, Arithmetische Untersuchungen elliptischer Integrale, *Math. Ann.*, **113** (1937), 1–13.

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