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ELLIPSEPHIC HARMONIC SERIES REVISITED

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Abstract. Ellipsephic or Kempner-like harmonic series are series of inverses of integers whose expansion in base B, for some $B \geq 2$, contains no occurrence of some fixed digit or some fixed block of digits. A prototypical example was proposed by Kempner in 1914, namely the sum inverses of integers whose expansion in base 10 contains no occurrence of a nonzero given digit. Results about such series address their convergence as well as closed expressions for their sums (or approximations thereof). Another direction of research is the study of sums of inverses of integers that contain only a given finite number, say k, of some digit or some block of digits, and the limits of such sums when k goes to infinity. Generalizing partial results in the literature, we give a complete result for any digit or block of digits in any base.

1. Introduction

While the harmonic series $\sum \frac{1}{n}$ is divergent, restricting the indices in the sum to integers satisfying innocent-looking conditions can yield convergent series. One of the first such examples is probably the 1914 result of Kempner [20] stating that the sum of inverses of integers whose expansion in base 10 contains no occurrence of a given digit ($\neq 0$) converges. After Kempner's paper and the 1916 paper of Irwin [19], several papers addressed extensions or generalizations of this result, as well as closed forms of numerical computations of the sums of the corresponding series: see, e.g., [1,6–11,13–16,21–32] and the references therein.

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We will revisit harmonic series with missing digits or missing blocks of digits: these series are called Kempner-like harmonic series or ellipsephic harmonic series in the literature (for the origin of the term *ellipsephic* coined by C. Mauduit, one can look at [12, p. 12] and [17, Footnote, p. 6]; also see the discussion in [5]). More precisely, let B an integer ≥ 2 and let $a_{w,B}(n)$ denote the number of occurrences of the block (word / string) w in the base B expansion of the integer n (where $|w|$ is the length of the block w), and $s_B(n)$ the sum of the digits of the base B expansion of n. It was proven by Farhi [14] that, for any digit j in $\{0, 1, \ldots, 9\}$, one has

$$\lim_{k\to\infty} \sum_{\substack{n\geq 1\\ a_{j,10}(n)=k}} \frac{1}{n} = 10 \log 10.$$

As explained in [5], a post on `mathfun` asked for the value of

$$\lim_{k\to\infty} \sum_{\substack{n\geq 1\\ s_2(n)=k}} \frac{1}{n}.$$

It was proved in [5] that

$$\lim_{k\to\infty} \sum_{\substack{n\geq 1\\ s_B(n)=k}} \frac{1}{n} = \frac{2\log B}{B-1}. \tag{1.1}$$

Thus we have of course

$$\lim_{k\to\infty} \sum_{\substack{n\geq 1\\ a_{1,2}(n)=k}} \frac{1}{n} = \lim_{k\to\infty} \sum_{\substack{n\geq 1\\ s_2(n)=k}} \frac{1}{n} = 2\log 2.$$

Here we will evaluate all such series, where $a_{1,B}(n)$ is replaced with $a_{w,B}(n)$, where w is any block (string) of digits in base B, by proving the following theorem.

THEOREM 1. *Let w any block of digits in base B. Let $a_{w,B}(n)$ be the number of (possibly overlapping) occurrences of w in the base B expansion of n. Then*

$$\lim_{k\to\infty} \sum_{\substack{n\geq 1\\ a_{w,B}(n)=k}} \frac{1}{n} = B^{|w|} \log B. \tag{1.2}$$

To prove this result, we will replace $1/n$ with the seemingly more complicated function $\log_B(\frac{n}{n+1})$, and make use of results proved in (or inspired by) [3,18]. In passing we will generalize [3,5] and re-prove [18].

2. “Reducing” the problem

Let B be an integer ≥ 2. Let w be a string of letters in $\{0, 1, \ldots, B - 1\}$. Let $a_{w,B}(n)$ be the number of (possibly overlapping) occurrences of the string w in the base B expansion of n. Recall that the base-B expansion of 0 is the empty string. In particular $a_{w,B}(0) = 0$. First we note that the series

$$\sum_{\substack{n\geq 1 \\ a_{w,B}(n)=k}} \frac{1}{n}$$

converges: the proof is the same as in [3, Lemma 1, p. 194], namely one uses a counting argument for the case of a single digit, and one replaces the base with some of its powers for the case of a block of digits. Now, to evaluate the series, the idea is to replace it with a convergent series

$$\sum_{\substack{n\geq 1 \\ a_{w,B}(n)=k}} b_w(n)$$

whose sum, say $A_w(k)$, tends to a limit, say A_w when $k \to \infty$. Furthermore, if we have the property $b_w(n) - 1/(B^{|w|}n) = \mathcal{O}_w(1/n^2)$ when n tends to infinity, then we obtain

$$\sum_{\substack{n\geq 1 \\ a_{w,B}(n)=k}} \frac{1}{n} \text{ converges, and } \lim_{k\to\infty} \sum_{\substack{n\geq 1 \\ a_{w,B}(n)=k}} \frac{1}{n} = B^{|w|}A_w.$$

(Note that if $a_{w,B}(n)$ tends to infinity, then n must also tend to infinity.)

Thus we have just showed the following result.

Proposition 2. *Let B be an integer ≥ 2 and w a string of letters in $\{0, 1, \ldots, B - 1\}$. Let $a_{w,B}(n)$ be the number of (possibly overlapping) occurrences of the string w in the base B expansion of n. To prove Theorem 1 it suffices to find a sequence $(b_w(n))_n$ such that*

* $\sum_{n\geq 1, a_{w,B}(n)=k} b_w(n)$ *converges. Let $A_w(k)$ be its sum;*
* $A_w(k)$ *tends to a limit, say A_w when k goes to infinity;*
* $b_w(n) - 1/(B^{|w|}n) = \mathcal{O}_w(1/n^2)$ *when n tends to infinity.*

Then

$$\sum_{\substack{n\geq 1 \\ a_{w,B}(n)=k}} \frac{1}{n} \textit{ converges, and } \lim_{k\to\infty} \sum_{\substack{n\geq 1 \\ a_{w,B}(n)=k}} \frac{1}{n} = B^{|w|}A_w.$$

Inspired by [3,18], we define $L(n)$ by $L(0) := 0$ and

$$L(n) := \log_B\Big(\frac{n}{n+1}\Big)$$

for $n \geq 1$. For a string w over the alphabet $\{0, 1, \ldots, B-1\}$, let $v(w)$ denote the integer whose expansion in base B is w (with possible leading 0's if $w \neq 0^j$).

Proposition 3. *Let w be a nonempty string over the alphabet $\{0, 1, \ldots, B-1\}$,*

$$g = B^{|w|-1}, \quad h = \Big\lfloor \frac{v(w)}{B} \Big\rfloor.$$

Then, for all $k \geq 0$,

$$\sum_{\substack{n \\ a_{w,B}(gn+h)=k}} L(Bgn + v(w)) = -1, \tag{2.1}$$

where the sum is over $n \geq 1$ if $w = 0^j$ and $n \geq 0$ otherwise.

Proof. Let c be the last letter of w. Let $d_w(k)$ be defined by

$$d_w(k) = \sum_{\substack{n \geq 0 \\ a_{w,B}(n)=k}} L(Bn + c).$$

(Note that this series converges since $L(n) \sim \frac{1}{n \log B}$ when n goes to infinity.) By writing $n = gr + m$, with $r \geq 0$ and $0 \leq m \leq g-1$, we see that

$$d_w(k) = \sum_{m=0}^{g-1} \sum_{\substack{r \geq 0 \\ a_{w,B}(gr+m)=k}} L(Bgr + Bm + c).$$

Similarly, if we let

$$e_w(k) = \sum_{\substack{n \geq 0 \\ a_{w,B}(Bn+c)=k}} L(Bn + c)$$

(which is convergent, like $d_w(k)$), then

$$e_w(k) = \sum_{m=0}^{g-1} \sum_{\substack{r \geq 0 \\ a_{w,B}(Bgr+Bm+c)=k}} L(Bgr + Bm + c).$$

Note that

$$a_{w,B}(Bgr+Bm+c)-a_{w,B}(gr+m)=\begin{cases}1 & \text{if } m=h,\\ 0 & \text{otherwise,}\end{cases}$$

for $r\ge 0$ if $w\ne 0^j$ and for $r\ge 1$ if $w=0^j$. For $w=0^j$ and $r=0$, the above difference is 0 for all m because we do not pad leading 0's in this case.

Therefore

$$d_w(k)-e_w(k)=\sum_{\substack{r\\ a_{w,B}(gr+h)=k}} L(Bgr+v(w))-\sum_{\substack{r\\ a_{w,B}(gr+h)=k-1}} L(Bgr+v(w)) \tag{2.2}$$

where the sum is over $r\ge 1$ if $w=0^j$ and $r\ge 0$ otherwise.

If we could show that $d_w(k)=e_w(k)$ for $k>0$, then it would follow from equation (2.2) that the value of the sum

$$\sum_{\substack{r\\ a_{w,B}(gr+h)=k}} L(Bgr+v(w))$$

is independent of k and hence equal to $d_w(0)-e_w(0)$. To prove this, notice that

$$L(n)-\sum_{j=0}^{B-1} L(Bn+j)=\begin{cases}0 & \text{if } n>0,\\ 1 & \text{if } n=0,\end{cases} \tag{2.3}$$

and

$$\sum_{\substack{n\ge 0\\ a_{w,B}(n)=k}} L(n)=\sum_{j=0}^{B-1}\sum_{\substack{n\ge 0\\ a_{w,B}(Bn+j)=k}} L(Bn+j)$$

$$=\sum_{\substack{n\ge 0\\ a_{w,B}(Bn+c)=k}} L(Bn+c)+\sum_{\substack{j=0\\ j\ne c}}^{B-1}\sum_{\substack{n\ge 0\\ a_{w,B}(n)=k}} L(Bn+j).$$

Hence

$$e_w(k)=\sum_{\substack{n\ge 0\\ a_{w,B}(Bn+c)=k}} L(Bn+c)=\sum_{\substack{n\ge 0\\ a_{w,B}(n)=k}} \Big(L(n)-\sum_{\substack{j=0\\ j\ne c}}^{B-1} L(Bn+j)\Big)$$

$$= \sum_{\substack{n\geq 0 \\ a_{w,B}(n)=k}} \Big(L(n) - \sum_{j=0}^{B-1} L(Bn+j)\Big) + \sum_{\substack{n\geq 0 \\ a_{w,B}(n)=k}} L(Bn+c).$$

By (2.3), the first sum is 1 if $k=0$, and 0 if $k>0$. The second sum is the definition of $d_w(k)$. □

LEMMA 4. *Let t be an integer whose expansion in base B is $t = b_1 b_2 \dots b_s$. For $1 \leq r \leq s$,*

If $b_1 \dots b_r$ is not a suffix of w, then

$$\sum_{\substack{n \\ a_{w,B}(B^r n + v(b_1 \dots b_r)) = k}} L(B^s n + t) = \sum_{\substack{n \\ a_{w,B}(B^{r-1} n + v(b_1 \dots b_{r-1})) = k}} L(B^s n + t).$$

If $b_1 \dots b_r$ is a suffix of w, then

$$\sum_{\substack{n \\ a_{w,B}(B^r n + v(b_1 \dots b_r)) = k}} L(B^s n + t) \tag{2.4}$$

$$= \sum_{\substack{n \\ a_{w,B}(B^{r-1} n + v(b_2 \dots b_r)) = k}} L(B^{s-1} n + t') - \sum_{\substack{j=0 \\ j \neq b_1}}^{B-1} \sum_{\substack{n \\ a_{w,B}(B^{r-1} n + v_j) = k}} L(B^s n + t_j)$$

where $t' = v(b_2 \cdots b_s)$, $t_j = v(j b_2 \cdots b_s)$, and $v_j = v(j b_2 \dots b_{r-1})$ if $r \geq 2$ and $v_j = 0$ if $r = 1$.

The proof is the same as in [3].

THEOREM 5. *There is a rational function $b_w(n)$ such that for all $k \geq 0$ we have*

$$\sum_{\substack{n \\ a_{w,B}(n)=k}} \log_B(b_w(n)) = -1. \tag{2.5}$$

(*the summation is over $n \geq 1$ for $w = 0^j$ and $n \geq 0$ otherwise*) *and*

$$\log(b_w(n)) = -\frac{1}{B^{|w|} n} + \mathcal{O}(1/n^2). \tag{2.6}$$

PROOF. The existence of $b_w(n)$ follows from Proposition 3 and iterated applications of Lemma 4. The process of obtaining $b_w(n)$ can be visualized by a tree T whose root is

$$\sum_{\substack{n \\ a_{w,B}(gn+h)=k}} L(Bgn + v(w))$$

and a node is a leaf if the condition of n for sum is $a_{w,B}(n) = k$, has a child corresponding to the right side if we are in the first case in Lemma 4, and has B children corresponding to the B terms (minus signs are included in the terms) of the right side if we are in the second case. Then $b_w(n)$ is the sum of the summands of the leaves of this tree. To prove (2.6), we first notice that

$$\log B \cdot L(an+b) = \log\Big(1 - \frac{1}{an+b+1}\Big) = -\frac{1}{an} + \mathcal{O}(1/n^2)$$

where a and b are positive constants. In particular, in (2.1),

$$L(Bgn + v(w)) = -\frac{1}{\log B \cdot B^{|w|}n} + \mathcal{O}(1/n^2).$$

Then, we note that, in T, the first-order term in the summand of each node that is not a leaf is the sum of the first-order term in its children. For example, when a node has B children and the first-order term of the summand is $-\frac{1}{\log B \cdot B^s n}$, then the sum of first-order terms of the summands of its children is

$$-\frac{1}{\log B \cdot B^{s-1}n} - \sum_{\substack{j=0\\ j\neq b_1}}^{B-1}\Big(-\frac{1}{\log B \cdot B^s n}\Big) = -\frac{1}{\log B \cdot B^s n}.$$

By induction we conclude that

$$\log_B(b_w(n)) = -\frac{1}{\log B \cdot B^{|w|}n} + \mathcal{O}(1/n^2). \quad \square$$

Remark 6. Theorem 5 above generalizes the case $B = 2$ in [3] (also see [4]). It can also give another proof of [18, Theorem 3].

Corollary 7. *Theorem 1 (i.e., equality (1.2)) is true.*

Proof. It suffices to apply Theorem 5 and Proposition 2. $\square$

Remark 8. Actually the same "reducing trick" can be used to re-prove equation (1.1) by using a result in [2]. Namely, up to notation, it was proved in [2, Lemme, p. 142] that, for all $k \geq 0$,

$$\sum_{s_B(n)=k} \log\Big(\frac{n+1}{B\lfloor n/B\rfloor + B}\Big) = -\log B. \tag{2.7}$$

Define the fractional part of n/B by $\{n/B\} := n/B - \lfloor n/B \rfloor$. Then, we have when n tends to infinity,

$$\log\Big(\frac{n+1}{B\lfloor n/B\rfloor + B}\Big) = \log\Big(1 + \frac{1-B+B\{n/B\}}{n+B(1-\{n/B\})}\Big)$$

$$= \log\Big(1+\frac{1-B+B\{n/B\}}{n}\Big) + O(1/n^2) = \frac{1-B}{n} + \Big(\frac{B\{n/B\}}{n}\Big) + O(1/n^2).$$

Thus (convergences are consequences of, e.g., equation 2.7, see [2]):

$$-\log B = \sum_{s_B(n)=k} \log\Big(\frac{n+1}{B\lfloor n/B\rfloor + B}\Big)$$

$$= (1-B) \sum_{s_B(n)=k} \frac{1}{n} + \sum_{s_B(n)=k} \frac{B\{n/B\}}{n} + O(1/n^2).$$

Hence (note that the term $O(1/n^2)$ below can be chosen independent of k),

$$-\log B = (1-B) \sum_{s_B(n)=k} \frac{1}{n} + \sum_{0\le j\le B-1} \sum_{s_B(n)=k-j} \frac{j}{Bn+j} + O(1/n^2)$$

$$= (1-B) \sum_{s_B(n)=k} \frac{1}{n} + \sum_{0\le j\le B-1} \sum_{s_B(n)=k-j} \frac{j}{Bn} + O(1/n^2).$$

Now, if k tends to infinity, we have that $k-j$ tends to infinity for $j \in [0, B-1]$, and also that n must tend to infinity, hence, letting $\lim_{k\to\infty} \sum_{s_B(n)=k} \frac{1}{n} := \ell$,

$$-\log B = (1-B)\,\ell + \sum_{0\le j\le B-1} \frac{j}{B}\ell = -\frac{B-1}{2}\,\ell, \quad \text{thus } \ell = \frac{2\log B}{B-1}.$$

3. A few words about numerical verification

A referee asked whether it is possible to check numerically our main equality (1.2), pointing that this seems to be a difficult task. Indeed, the numerical computation of ellipsephic (or Kempner-like) sums is far from easy: such series converge very slowly as indicated by the title of [28]. In the same paper [28] there is a subtle algorithm to obtain close approximations of some sums of such series; interestingly enough a step consists of replacing $1/n$ with $1/n^k$ (this idea was already present in [8]). Many more details are given in [9]. Now, having precise values of such sums makes it possible to compute their limits. We do not know whether there is a quicker or more precise way of computing such limits.

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Added in text (April 11, 2024). We would like to mention five very recent papers, where—in particular—the introduction of measures gives nice results, both computational and theoretical, namely:

J.-F. Burnol, Moments in the exact summation of the curious series of Kempner type, preprint, arXiv: 2402.08525 (2024).

J.-F. Burnol, Moments for the summation of Irwin series, preprint, arXiv: 2402.09083 (2024).

J.-F. Burnol, Summing the "exactly one 42" and similar subsums of the harmonic series, preprint, arXiv: 2402.14761 (2024).

J.-F. Burnol, Sur l'asymptotique des sommes de Kempner pour de grandes bases, preprint, arXiv: 2403.01957 (2024).

J.-F. Burnol, Digamma function and general Fischer series in the theory of Kempner sums, preprint, arXiv: 2403.03912 (2024).

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