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ON BOUNDARY HÖLDER LOGARITHMIC CONTINUITY OF MAPPINGS IN SOME DOMAINS

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Abstract. We study mappings satisfying some estimate of distortion of modulus of families of paths. Under some conditions on definition and mapped domains, we prove that these mappings are logarithmic Hölder continuous at boundary points.

1. Introduction

This paper is devoted to the so-called inverse Poletsky inequality established in many classes of mappings (see, e.g., [1, Lemma 2, Lemma 3], [2, Theorem B], [3, Lemma 3.1], [10, Theorem 3.2], [11, Theorem 8.5] and [15, Theorem 6.7.II]). Recall that mappings with a bounded distortion as well as quasiconformal mappings satisfy the inequalities

$$M(\Gamma) \leq N(f,D)K_O(f)M(f(\Gamma)) \tag{1}$$

for any family of paths Γ in D, where M denotes the modulus of families of paths, $1 \leq K_O(f) < \infty$ and $N(f,D)$ is a maximal multiplicity of f in D (see [10]). For more general classes of mappings, such an inequality has some more general form (see below). Note that the inequalities $M(f(\Gamma)) \leq K \cdot M(\Gamma)$, $1 \leq K < \infty$, are very similar to (1) and were established by Poletsky for quasiregular mappings, see [14, Theorem 1, § 4]. Precisely for

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this reason, we call the mappings in (1) mappings satisfying the inverse Poletsky inequality. However, we even consider some more general class of mappings.

A Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for a family Γ of paths γ in $\mathbb{R}^n$ if the relation

$$\int_{\gamma} \rho(x)\,|dx| \geq 1 \tag{2}$$

holds for any locally rectifiable path $\gamma \in \Gamma$. A *modulus* of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,. \tag{3}$$

Let $Q\colon \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies the inverse Poletsky inequality* if the relation

$$M(\Gamma) \leq \int_{f(D)} Q(y) \cdot \rho_*^n(y)\,dm(y) \tag{4}$$

holds for any family of paths Γ in D and any $\rho_* \in \operatorname{adm} f(\Gamma)$. Note that estimates (4) hold in many classes of mappings (see, e.g., [10, Theorem 3.2], [15, Theorem 6.7.II] and [11, Theorem 8.5]).

Recently we proved logarithmic Hölder continuity for mappings in (4) at the boundary of the unit ball (see [16]). In this paper, we study similar mappings between another type of domains. In particular, we deal with mappings between the so-called quasiextremal distance domains and convex domains. Note that quasiextremal distance domains (abbreviated as QED-domains) are introduced by Gehring and Martio in [5] and are structures in which the modulus of the families of paths is metrically related to the diameter of the sets. As for these domains, we have obtained logarithmic Hölder continuity at the corresponding points of the boundary. In the next parts we also study mappings of a domain with a locally quasiconformal boundary (collared domains) onto a convex domain. Besides that, we consider mappings of some regular domains which are defined as quasiconformal images of domains with a locally quasiconformal boundary. For these classes of mappings, we also prove the Hölder logarithmic continuity at the corresponding boundary points. It should be noted that for the case of regular domains, this property is formulated in terms of prime ends, and not in the Euclidean sense.

A mapping $f\colon D \to \mathbb{R}^n$ is called *discrete* if $\{f^{-1}(y)\}$ consists of isolated points for any $y \in \mathbb{R}^n$, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. A mapping f between domains D and D' is said to be *closed* if $f(E)$ is closed in D' for any closed set $E \subset D$ (see, e.g., [20, Section 3]).

In accordance with [5], a domain D in $\mathbb{R}^n$ is called a *quasiextremal distance domain* (*QED-domain for short*) if

$$M(\Gamma(E,F,\mathbb{R}^n)) \le A_0 \cdot M(\Gamma(E,F,D)) \tag{5}$$

for some finite number $A_0 \ge 1$ and all continua E and F in D. Observe that a half-space or a ball are quasiextremal distance domains, see [21, Lemma 4.3].

Subsequently, in the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x,y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, and

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}\,, \quad h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,, \quad x \ne \infty \ne y \tag{6}$$

(see e.g. [19, Definition 12.1]). In what follows, given $A, B \subset \overline{\mathbb{R}^n}$ we set

$$h(A,B) = \inf_{x\in A, y\in B} h(x,y)\,, \quad h(A) = \sup_{x,y\in A} h(x,y)\,,$$

where h is a chordal metric in (6). Similarly, we define the Euclidean distance between sets and the Euclidean diameter by the formulae

$$d(A,B) = \inf_{x\in A, y\in B} |x-y|\,, \quad d(A) = \sup_{x,y\in A} |x-y|\,.$$

Sometimes we also write $\operatorname{dist}(A,B)$ instead $d(A,B)$ and $\operatorname{diam} E$ instead $d(E)$, as well. As usually, we set

$$B(x_0,r) = \{x \in \mathbb{R}^n : |x-x_0| < r\}\,, \quad \mathbb{B}^n = B(0,1)\,,$$
$$S(x_0,r) = \{x \in \mathbb{R}^n : |x-x_0| = r\}\,.$$

Given $\delta > 0$, domains $D, D' \subset \mathbb{R}^n$, $n \ge 2$, a non-degenerate continuum $A \subset D'$ and a Lebesgue measurable function $Q\colon D' \to [0,\infty]$ we denote by $\mathfrak{S}_{\delta,A,Q}(D,D')$ a family of all open discrete and closed mappings f of D onto D' satisfying the relation (4) such that $h(f^{-1}(A), \partial D) \ge \delta$.

THEOREM 1. *Let $Q \in L^1(D')$, let D be a quasiextremal distance domain, and let D' be a convex bounded domain. Then any $f \in \mathfrak{S}_{\delta,A,Q}(D,D')$ has a continuous extension $f\colon \overline{D} \to \overline{D'}$ and for any $x_0 \in \partial D$, $x_0 \ne \infty$, there exists a neighborhood U and $C = C(n,A,D,D') > 0$ such that*

$$|\overline{f}(x) - \overline{f}(y)| \le \frac{C \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta}{|x-y|}\right)} \tag{7}$$

for any $x, y \in U \cap \overline{D}$, where $\|Q\|_1$ is a norm of the function Q in $L^1(D')$.

Consider the following definition that has been proposed by Näkki [13] (cf. [8]). The boundary of a domain D is called *locally quasiconformal* if every point $x_0 \in \partial D$ has a neighborhood U, for which there exists a quasiconformal mapping φ of U onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of the unit sphere $\mathbb{B}^n$ with a coordinate hyperplane $x_n = 0$, where $x = (x_1, \dots, x_n)$. Note that with slight differences in the definition, domains with such boundaries are also called collared domains.

THEOREM 2. *Let $Q \in L^1(D')$, let D be a domain with a locally quasiconformal boundary, and let D' be a bounded convex domain. Then any $f \in \mathfrak{S}_{\delta,A,Q}(D, D')$ has a continuous extension $f\colon \overline{D} \to \overline{D'}$, while, for any $x_0 \in \partial D$, $x_0 \neq \infty$, there exists a neighborhood V of x_0 and some numbers $C = C(n, A, D, D', x_0) > 0$ and $0 < \alpha = \alpha(x_0) \leq 1$ such that*

$$|\overline{f}(x) - \overline{f}(y)| \leq \frac{C \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta}{|x-y|^{\alpha}}\right)} \tag{8}$$

for any $x, y \in V \cap \overline{D}$, where $\|Q\|_1$ is a norm of Q in $L^1(D')$.

Using the previous theorem, it is also possible to obtain a statement about Hölder's logarithmic continuity for bad boundaries in terms of prime ends.

The definition of a prime end used below may be found in [6], cf. [8]. We say that a bounded domain D in $\mathbb{R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular. Note that the space $\overline{D}_P = D \cup E_D$ is metric, which can be demonstrated as follows. If $g\colon D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|\,, \tag{9}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of the domain D_0. The specified boundary point is unique and well-defined, see e.g. [7, Theorem 2.1, Remark 2.1], cf. [13, Theorem 4.1]. The following statement holds.

THEOREM 3. *Let $Q \in L^1(D')$, let D be regular domain, and let D' be a bounded convex domain. Then any $f \in \mathfrak{S}_{\delta,A,Q}(D, D')$ has a continuous extension $f\colon \overline{D}_P \to \overline{D'}$; in addition, for any $P_0 \in E_D$ there exists a neighborhood V of this point in $(\overline{D}_P, \rho)$ and numbers $C = C(n, A, D, D', P_0) > 0$ and $0 < \alpha = \alpha(P_0) \leq 1$ such that*

$$|\overline{f}(P_1) - \overline{f}(P_2)| \leq \frac{C \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta}{\rho^{\alpha}(P_1, P_2)}\right)} \tag{10}$$

for any $P_1, P_2 \in V$, *where* $\|Q\|_1$ *is a norm of* Q *in* $L^1(D')$.

2. Auxiliary lemmas

Before proving the main statements, we prove the following important lemma, which is proved in [16, Lemma 2.1] for the case of the unit ball.

LEMMA 4. *Let* D *and* D' *be domains satisfying the conditions of Theorem* 1, *and let* E *be a continuum in* D', $Q \in L^1(D')$. *Then there exists* $\delta_1 > 0$ *such that* $\mathfrak{S}_{\delta,A,Q} \subset \mathfrak{S}_{\delta_1,E,Q}$. *In other words, if* f *is an open discrete and closed mapping of* D *onto* D' *satisfying the condition* (4) *such that* $h(f^{-1}(A), \partial D) \geq \delta$, *then there exists* $\delta_1 > 0$, *which does not depend on* f, *such that* $h(f^{-1}(E), \partial D) \geq \delta_1$.

PROOF. We will generally use the scheme of the proof [16, Lemma 2.1]. Let us prove Lemma 4 from the opposite. Suppose that its conclusion is not true. Then there are sequences $y_m \in E$, $f_m \in \mathfrak{S}_{\delta,A,Q}$ and $x_m \in D$ such that $f_m(x_m) = y_m$ and $h(x_m, \partial D) \to 0$ as $m \to \infty$. Without loss of generality, we may assume that $x_m \to x_0$ as $m \to \infty$, where x_0 may be equal to ∞ if D is unbounded. By [18, Theorem 3.1], it follows that f_m has a continuous extension to x_0, moreover, the family $\{f_m\}_{m=1}^{\infty}$ is equicontinuous at x_0 (see, e.g., [18, Theorem 1.2]). Then, for any $\varepsilon > 0$ there is $m_0 \in \mathbb{N}$ such that $h(f_m(x_m), f_m(x_0)) < \varepsilon$ for $m \geq m_0$. On the other hand, since f_m is closed, $f_m(x_0) \in \partial D'$. Due to the compactness of the space $\overline{\mathbb{R}^n}$ and the closure of $\partial D'$, we may assume that $f_m(x_0)$ converges to some $B \in \partial D'$ as $m \to \infty$. Therefore, by the triangle inequality,

$$h(f_m(x_m), f_m(x_0)) \geq h(f_m(x_m), B) - h(B, f_m(x_0)) \geq \frac{1}{2} \cdot h(E, \partial D')$$

for sufficiently large $m \in \mathbb{N}$. Finally, we have a contradiction:

$$h(f_m(x_m), f_m(x_0)) \geq \delta_0, \quad \delta_0 := \frac{1}{2} \cdot h(E, \partial D')$$

and, at the same time, $h(f_m(x_m), f_m(x_0)) < \varepsilon$ for $m \geq m_0$. The resulting contradiction refutes the original assumption. The lemma is proved. □

The following lemma was proved in the case where the domain D' is the unit ball (see the proof of [16, Theorem 1.1]). For an arbitrary convex domain, its proof is significantly more difficult, since the previous methodology relied on the geometry of the ball.

LEMMA 5 [4]. *Let* D' *be a bounded convex domain in* $\mathbb{R}^n$, $n \geq 2$, *and let* $B(y_*, \delta_*/2)$ *be a ball centered at the point* $y_* \in D'$, *where* $\delta_* := d(y_*, \partial D')$.

Let $z_0 \in \partial D'$. Then for any points $A, B \in B(z_0, \delta_*/8) \cap D'$ there are points $C, D \in \overline{B(y_*, \delta_*/2)}$, for which the segments $[A, C]$ and $[B, D]$ are such that

$$\mathrm{dist}([A,C],[B,D]) \geq C_0 \cdot |A - C| \,, \tag{11}$$

where $C_0 > 0$ is some constant that depends only on δ_* and $d(D')$.

3. Proof of Theorem 1

The possibility of a continuous extension of the mapping f to the boundary of the domain D follows by [18, Theorem 3.1]. In particular, the weak flatness of ∂D is a consequence of the fact that D is a QED domain (see, e.g., [17, Lemma 2]), in addition, any convex domain is locally connected at its boundary because its intersection with the ball centered at the boundary point is also a convex set.

Let us prove the logarithmic Hölder continuity (7). Put $x_0 \in \partial D$, and let $y_* \in D'$ be an arbitrary point of the domain D'. Put $\delta_* := d(y_*, \partial D')$. Let $E = \overline{B(y_*, \delta_*/2)} \subset D'$. By Lemma 4 there exists $\delta_1 > 0$ such that $h(f^{-1}(E), \partial D) \geq \delta_1$ for all $f \in \mathfrak{S}_{\delta,A,Q}$. In addition, due to Theorem 1.2 in [18] the family $\mathfrak{S}_{\delta,A,Q}$ is equicontinuous in $\overline{D}$. This implies that for a number $\delta_*/8$ there is a neighborhood $U \subset B(x_0, \delta_1/2)$ of x_0 such that $|f(x) - f(x_0)| < \delta_*/8$ for all $x \in U \cap D$ and all $f \in \mathfrak{S}_{\delta,A,Q}$. Let $x, y \in U \cap D$ and

$$\varepsilon_0 := |f(x) - f(y)| < \delta_0 := \delta_*/4 \,.$$

Let us apply Lemma 5 for the points $A = f(x)$, $B = f(y)$ and $z_0 = f(x_0)$. According to this lemma, there are segments $I \supset A$ and $J \supset B$ in D' such that $I \cap E \neq \varnothing \neq J \cap E$, and

$$\mathrm{dist}(I, J) \geq C_0 \cdot |f(x) - f(y)| \,, \tag{12}$$

where C_0 is some constant which depends only on E and D'.

Let α_1, β_1 be total f-liftings of paths I and J starting at the points x and y,, respectively (they exist due to [20, Lemma 3.7], see Figure 1). By definition, $|\alpha_1| \cap f^{-1}(E) \neq \varnothing \neq |\beta_1| \cap f^{-1}(E)$. Since $h(f^{-1}(E), \partial D) \geq \delta_1$ and $x, y \in B(x_0, \delta_1/2)$, then

$$d(\alpha_1) \geq \delta_1/2 \,, \quad d(\beta_1) \geq \delta_1/2 \,. \tag{13}$$

Let $\Gamma := \Gamma(\alpha_1, \beta_1, D)$. Then, on the one hand, by the inequality (5)

$$M(\Gamma) \geq (1/A_0) \cdot M(\Gamma(\alpha_1, \beta_1, \mathbb{R}^n)) \,, \tag{14}$$

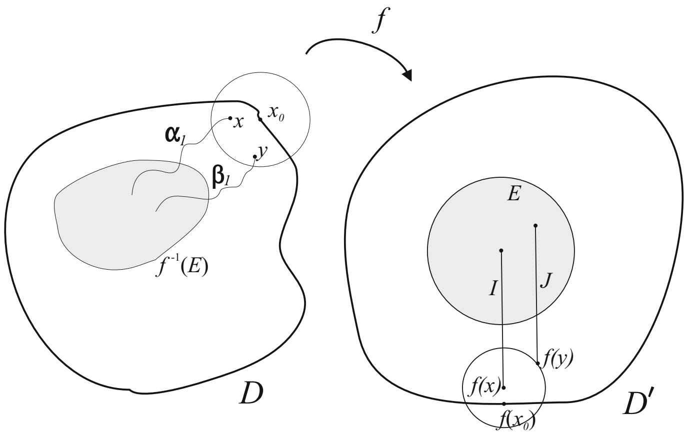

Figure 1: To proving Theorem 1

and on the other hand, by [22, Lemma 7.38],

$$M(\Gamma(\alpha_1,\beta_1,\mathbb{R}^n)) \geq c_n \cdot \log\left(1+\frac{1}{m}\right), \tag{15}$$

where $c_n > 0$ is some constant that depends only on n, and

$$m = \frac{\operatorname{dist}(\alpha_1,\beta_1)}{\min\{\operatorname{diam}(\alpha_1),\operatorname{diam}(\beta_1)\}}\,.$$

Combining (13), (14) and (15) and taking into account that $\operatorname{dist}(\alpha_1,\beta_1) \leq |x-y|$, we have

$$M(\Gamma) \geq \widetilde{c_n} \cdot \log\left(1+\frac{\delta_1}{2\operatorname{dist}(\alpha_1,\beta_1)}\right) \geq \widetilde{c_n} \cdot \log\left(1+\frac{\delta_1}{2|x-y|}\right), \tag{16}$$

where $\widetilde{c_n} > 0$ is some constant depending only on n and A_0 from the definition of the QED-domain.

Now, let us to find some upper estimate for $M(\Gamma)$. Put

$$\rho(y) = \begin{cases} \frac{1}{C_0\varepsilon_0}, & y \in D', \\ 0, & y \notin D'. \end{cases}$$

Note that ρ satisfies the relation (2) for $f(\Gamma)$ by virtue of (11). By the definition of the family $\mathfrak{S}_{\delta,A,Q}$, we obtain that

$$M(\Gamma) \leq \frac{1}{C_0^n\varepsilon_0^n}\int_{D'} Q(y)\,dm(y) = C_0^{-n}\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^n}\,. \tag{17}$$

By (16) and (17), it follows that

$$\widetilde{c_n}\cdot\log\Big(1+\frac{\delta_1}{2|x-y|}\Big)\le C_0^{-n}\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^n}\,.$$

The desired inequality (7) follows from the last relation, where $C := C_0^{-1}\cdot\widetilde{c_n}^{\,-1/n}$, taking into account that according to L'Hospital's rule,

$$\log\Big(1+\frac{1}{nt}\Big)\sim\log\Big(1+\frac{1}{kt}\Big)\quad\text{as } t\to+0$$

for any different $k, n > 0$.

We have proved Theorem 1 for the inner points $x, y\in U\cap D$. For points $x, y\in U\cap\overline{D}$, this statement follows by passing to the limit $\overline{x}\to x$ and $\overline{y}\to y$, $\overline{x},\overline{y}\in D$. $\Box$

An analogue of Theorem 1 is also valid for mappings with a fixed point in D. In order to formulate and prove the corresponding statement, let us introduce the following definition. For $a\in D$ and $b\in D'$, and a Lebesgue measurable function $Q\colon D'\to[0,\infty]$ denote by $\mathfrak{F}_{a,b,Q}$ the family of all open discrete and closed mappings f of the domain D onto D' satisfying the relation (4) such that $f(a)=b$.

THEOREM 6. *Let $Q\in L^1(D')$, let D be a domain with a locally quasiconformal boundary, and let D' be a bounded convex domain. Then any mapping $f\in\mathfrak{S}_{a,b,Q}$ has a continuous extension to the mapping $f\colon\overline{D}\to\overline{D'}$, while, for any point $x_0\in\partial D$ there is a neighborhood U of this point and a number $C=C(n,A,D,D')>0$ such that the relation* (7) *is fulfilled for some $\delta>0$.*

PROOF. The possibility of a continuous extension of f to ∂D follows by [18, Theorem 3.1]. Let us prove the logarithmic Hölder continuity of the cooresponding family of extended mappings (7). Put $E=\overline{B(b,\varepsilon_*)}$, where $\varepsilon_*<\operatorname{dist}(b,\partial D')$. Two cases are possible:

1) There exists $\delta>0$ such that $\operatorname{dist}(f^{-1}(E),\partial D)\ge\delta$ for all $f\in\mathfrak{S}_{\delta,A,Q}$. In in this case, the desired statement follows by Theorem 1.

2) There are sequences $f_m\in\mathfrak{S}_{\delta,A,Q}$ and $x_m\in D$, $y_m\in D'$, $m=1,2,\dots$, such that $f_m(x_m)=y_m$, $y_m\in E$ and $\operatorname{dist}(x_m,\partial D)\to 0$ as $m\to\infty$. Reasoning in the same way as in the proof of Lemma 4, we come to the conclusion that the family $\mathfrak{S}_{\delta,A,Q}$ is not equicontinuous at at least one point $x_0\in\partial D$, however, this contradicts the statement of [18, Theorem 7.1].

Thus, the Case 2 is impossible, and, consequently, Theorem 6 is proved. $\Box$

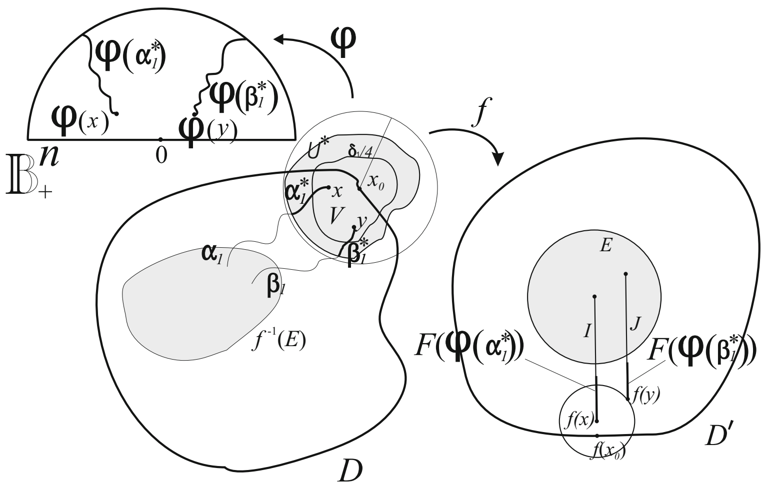

Figure 2: To the proof of Theorem 2

4. Proof of Theorem 2

The possibility of a continuous extension of the mapping f to the boundary of D follows by [18, Theorem 3.1]. In particular, locally quasiconformal boundaries are weakly flat (see [6, Proposition 2.2], see also [19, Theorem 17.10]). In addition, convex domains are obviously locally connected at its boundary.

Put $x_0 \in \partial D$. Let $y_* \in D'$ be an arbitrary point of D', $\delta_* := d(y_*, \partial D')$ and $E = \overline{B(y_*, \delta_*/2)} \subset D'$. By Lemma 4, there exists $\delta_1 > 0$ such that $h(f^{-1}(E), \partial D) \geq \delta_1$ for all $f \in \mathfrak{S}_{\delta,A,Q}$. Then $d(f^{-1}(E), \partial D) \geq \delta_1$ for any $f \in \mathfrak{S}_{\delta,A,Q}$. In addition, since by [18, Theorem 1.2] the family $\mathfrak{S}_{\delta,A,Q}$ is equicontinuous at $\overline{D}$, for $\delta_*/8$ there is a neighborhood $U \subset B(x_0, \delta_1/4)$ of x_0 such that $|f(x) - f(x_0)| < \delta_*/8$ for any $x, y \in U \cap D$ and all $f \in \mathfrak{S}_{\delta,A,Q}$.

By the definition of a locally quasiconformal boundary, there exist a neighborhood U^* of the point x_0 and a quasiconformal mapping $\varphi\colon U^* \to \mathbb{B}^n$, $\varphi(U^*) = \mathbb{B}^n$, such that $\varphi(D \cap U^*) = \mathbb{B}^n_+$, where $\mathbb{B}^n_+ = \{x \in \mathbb{B}^n : x = (x_1, \ldots, x_n), x_n > 0\}$ is a half-ball, see Figure 2. We may assume that $\varphi(x_0) = 0$ and $\overline{U^*} \subset U$ (see the proof of [19, Theorem 17.10]). Let V be any neighborhood in U^* such that $\overline{V} \subset U^*$, and let

$$\delta_2 := \mathrm{dist}(\partial V, \partial U^*)\,. \tag{18}$$

Consider the auxiliary mapping

$$F(w) := f(\varphi^{-1}(w))\,, \quad F\colon \mathbb{B}^n_+ \to U^*\,. \tag{19}$$

Let $x, y \in V \cap D$ and

$$\varepsilon_0 := |f(x) - f(y)| < \delta_0 := \delta_*/4\,.$$

Apply Lemma 5 for points $A = f(x)$, $B = f(y)$ and $z_0 = f(x_0)$. According to this lemma, there are exist segments $I \supset A$ and $J \supset B$ in D' such that $I \cap E \neq \varnothing \neq J \cap E$, moreover

$$\operatorname{dist}(I, J) \geq C_0 \cdot |f(x) - f(y)|\,, \tag{20}$$

where C_0 depends only on E and $d(D')$.

Let α_1, β_1 be the whole f-lifts of the paths I and J starting at the points x and y, respectively (they exist by [20, Lemma 3.7]). Then, by the definition, $|\alpha_1| \cap f^{-1}(E) \neq \varnothing \neq |\beta_1| \cap f^{-1}(E)$. Then

$$|\alpha_1| \cap U \neq \varnothing \neq |\alpha_1| \cap (\mathbb{R}^n \setminus U)$$

and

$$|\beta_1| \cap U \neq \varnothing \neq |\beta_1| \cap (\mathbb{R}^n \setminus U)\,.$$

Then, by [9, Theorem 1.I.5.46]

$$|\alpha_1| \cap \partial U \neq \varnothing\,, \quad |\beta_1| \cap \partial U \neq \varnothing\,. \tag{21}$$

Similarly,

$$|\alpha_1| \cap \partial V \neq \varnothing\,, \quad |\beta_1| \cap \partial V \neq \varnothing\,. \tag{22}$$

Due to (21), α_1 and β_1 contain subpaths α_1^* and β_1^* with origins at the points x and y which belong entirely in U^* and have endpoints at ∂U^*. Due to (18), (21) and (22)

$$d(\alpha_1^*) \geq \delta_2\,, \quad d(\beta_1^*) \geq \delta_2\,. \tag{23}$$

Consider the paths $\varphi(\alpha_1^*)$ and $\varphi(\beta_1^*)$. Since φ is a quasiconformal mapping, so φ^{-1} is also quasiconformal. Thus, φ^{-1} is locally Hölder continuous with some constant $\widetilde{C} > 0$ and some exponent $0 < \alpha \leq 1$ (see [15, Theorem 1.11.III]). We may consider that φ^{-1} is quasiconformal in $\mathbb{B}^n$. Then

$$\frac{1}{(\widetilde{C})^{\frac{1}{\alpha}}}|x - y|^{\frac{1}{\alpha}} \leq |\varphi^{-1}(x) - \varphi^{-1}(y)| \leq \widetilde{C} \cdot |x - y|^{\alpha} \quad \forall x, y \in \mathbb{B}^n\,. \tag{24}$$

Let $\overline{x}, \overline{y} \in U^*$ be points in α_1^* such that $d(\alpha_1^*) = |\overline{x} - \overline{y}|$. We put $x^* = \varphi(\overline{x})$ and $y^* = \varphi(\overline{y})$. Then

$$|x^* - y^*|^{\alpha} \geq \frac{1}{\widetilde{\widetilde{C}}} \cdot |\overline{x} - \overline{y}| = d(\alpha_1^*) \geq \frac{1}{\widetilde{\widetilde{C}}}\,\delta_2\,,$$

or

$$|x^* - y^*| \geq \left(\frac{1}{\widetilde{C}}\,\delta_2\right)^{1/\alpha} . \tag{25}$$

From (25), we obtain that $d(\varphi(\alpha_1^*)) \geq \left(\frac{1}{\widetilde{C}}\delta_2\right)^{1/\alpha}$. Similarly, $d(\varphi(\beta_1^*)) \geq \left(\frac{1}{\widetilde{C}}\delta_2\right)^{1/\alpha}$. Let

$$\Gamma := \Gamma\big(\varphi(\alpha_1^*), \varphi(\beta_1^*), \mathbb{B}^n_+\big) .$$

Note that $\mathbb{B}^n_+$ is a bounded convex domain, so it is a John domain (see [12, Remark 2.4]). Hence it is a uniform domain (see [12, Remark 2.13(c)]), therefore it is also a QED-domain with some $A_0^* < \infty$ in (5) (see [5, Lemma 2.18]). Then, on the one hand, by (5)

$$M(\Gamma) \geq (1/A_0^*) \cdot M\big(\Gamma(\varphi(\alpha_1^*), \varphi(\beta_1^*), \mathbb{R}^n)\big) , \tag{26}$$

and on the other hand, by [22, Lemma 7.38]

$$M(\Gamma(\varphi(\alpha_1^*), \varphi(\beta_1^*), \mathbb{R}^n)) \geq c_n \cdot \log\Big(1 + \frac{1}{m}\Big) , \tag{27}$$

where $c_n > 0$ is some constant that depends only on n,

$$m = \frac{\operatorname{dist}(\varphi(\alpha_1^*), \varphi(\beta_1^*))}{\min\{\operatorname{diam}(\varphi(\alpha_1^*)), \operatorname{diam}(\varphi(\beta_1^*))\}} .$$

Then, combining (26) and (27) and taking into account that

$$\operatorname{dist}(\varphi(\alpha_1^*), \varphi(\beta_1^*)) \leq |\varphi(x) - \varphi(y)|,$$

we obtain that

$$\begin{aligned} M(\Gamma) &\geq \widetilde{c_n} \cdot \log\left(1 + \frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha} \operatorname{dist}(\alpha_1^*, \beta_1^*)}\right) \\ &\geq \widetilde{c_n} \cdot \log\left(1 + \frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha} |\varphi(x) - \varphi(y)|}\right) , \end{aligned} \tag{28}$$

where $\widetilde{c_n} > 0$ is some constant that depends only on n and A_0^* from the definition of QED-domain.

Let us now establish an upper bound for $M(\Gamma)$. Note that F in (19) satisfies the relation (4) with the function $\widetilde{Q}(x) = K_0 \cdot Q(x)$ instead of Q, where $K_0 \geq 1$ is the constant of a quasiconformality of φ^{-1}. Let us put

$$\rho(y) = \begin{cases} \frac{1}{C_0 \varepsilon_0}, & y \in D', \\ 0, & y \notin D' , \end{cases}$$

where C_0 is the universal constant in inequality (20). Note that ρ satisfies the relation (2) for $F(\Gamma)$ due to the relation (11). Then, by the definition of $\mathfrak{S}_{\delta,A,Q}$, due to the definition of F in (19), we obtain that

$$M(\Gamma)\le\frac{1}{C_0^n\varepsilon_0^n}\int_{D'}K_0Q(y)\,dm(y)=C_0^{-n}K_0\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^n}\,. \tag{29}$$

It follows by (28) and (29) that

$$\widetilde{c_n}\cdot\log\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}|\varphi(x)-\varphi(y)|}\right)\le C_0^{-n}K_0\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^n}\,.$$

From the left side of inequality (24) it follows that

$$|\varphi(x)-\varphi(y)|\le\widetilde{C}|x-y|^{\alpha}\,,\quad x,y\in U^*\,.$$

Now, from the last two relations, it follows that

$$|f(x)-f(y)|\le C_0^{-1}\widetilde{c_n}^{\,-1/n}K_0^{1/n}\cdot\frac{(\|Q\|_1)^{1/n}}{\log^{1/n}\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}|\varphi(x)-\varphi(y)|}\right)}$$

$$\le C_0^{-1}\widetilde{c_n}^{\,-1/n}K_0^{1/n}\cdot\frac{(\|Q\|_1)^{1/n}}{\log^{1/n}\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{(1/\alpha)+1}|x-y|^{\alpha}}\right)}\,,$$

which is the desired inequality (8), where $C:=C_0^{-1}\cdot\widetilde{c_n}^{\,-1/n}\cdot K_0^{1/n}$ and $r_0=\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha+1}}$ instead of δ. However, we may replace r_0 by δ here, because, by L'Hospital's rule, $\log\left(1+\frac{1}{nt}\right)\sim\log\left(1+\frac{1}{kt}\right)$ as $t\to+0$ for any different $k,n>0$.

We proved Theorem 2 for the inner points $x,y\in V\cap D$. For $x,y\in V\cap\overline{D}$, this statement follows by means of the transition to the limit $\overline{x}\to x$ and $\overline{y}\to y$, $\overline{x},\overline{y}\in D$. $\Box$

5. Proof of Theorem 3

Let $f\in\mathfrak{S}_{\delta,A,Q}(D,D')$. It is sufficient to restrict ourselves to the case $P_1,P_2\in V\cap D$. Since D is a regular domain, there exists a quasiconformal mapping g^{-1} of the domain D onto a domain D_0 with a locally quasiconformal boundary, and, by the definition of the metric ρ in (9),

$$\rho(P_1,P_2):=\left|g^{-1}(P_1)-g^{-1}(P_2)\right|\,. \tag{30}$$

Consider the auxiliary mapping

$$F(x) = (f \circ g)(x)\,, \quad x \in D_0\,. \tag{31}$$

Since g^{-1} is quasiconformal, there is a constant $1 \le K_1 < \infty$ such that

$$\frac{1}{K_1} \cdot M(\Gamma) \le M(g(\Gamma)) \le K_1 \cdot M(\Gamma) \tag{32}$$

for any family of paths Γ in D_0. Considering inequalities (32) and taking into account that f satisfies the relation (4), we obtain that also F satisfies the relation (4) with a new function $\widetilde{Q}(x) := K_1 \cdot Q(x)$. In addition, since g is a fixed homeomorphism, then $h(F^{-1}(A), \partial D_0) \ge \delta_0 > 0$, where $\delta_0 > 0$ is some fixed number. Then Theorem 2 may be applied to the map F. Applying this theorem, we obtain that for any point $x_0 \in D_0$ there are a neighborhood U of this point and numbers $C^* = C^*(n, A, D_0, D') > 0$ and $0 < \alpha = \alpha(x_0) \le 1$ such that

$$|F(x) - F(y)| \le \frac{C^* K_1^{\frac{1}{n}} \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta_0}{|x-y|^{\alpha}}\right)} \tag{33}$$

for all $x, y \in V \cap D_0$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$. Let $U := g(V)$, $P_0 := g(x_0)$. Then, by definition, V is a neighborhood of the prime end $P_0 \in E_D$. If $P_1, P_2 \in D_P \cap V$, then $P_1 = g(x)$ and $P_2 = g(y)$ for some $x, y \in U \cap D_0$. Taking into account the relation (33) and using the relation $|x - y| = |g^{-1}(P_1) - g^{-1}(P_2)| = \rho(P_1, P_2)$, we obtain that

$$\left|F(g^{-1}(P_1)) - F(g^{-1}(P_2))\right| \le \frac{C^* K_1^{\frac{1}{n}} \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta_0}{\rho^{\alpha}(P_1,P_2)}\right)}\,,$$

or, due to (31),

$$\left|f(P_1) - f(P_2)\right| \le \frac{C^* K_1^{\frac{1}{n}} \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta_0}{\rho^{\alpha}(P_1,P_2)}\right)}\,.$$

The last ratio is desired if we put here $C := C^* K_1^{\frac{1}{n}}$. Here we also take into account that by L'Hopital's rule, $\log\left(1 + \frac{1}{nt}\right) \sim \log\left(1 + \frac{1}{kt}\right)$ as $t \to +0$ for any different $k, n > 0$. Thus, in the last relation, we may write δ instead δ_0. $\square$

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