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THE HAUSDORFF–YOUNG INEQUALITY AND FREUD WEIGHTS

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Abstract. We discuss a sharpened Hausdorff–Young inequality and estimate the maximal coefficients of orthogonal expansions in terms of Freud polynomials when $1 < p < 2$ and $2 < p < \infty$. We also consider n-dimensional expansions by orthogonal functions associated to Freud-type weights when $1 < p < 2$.

In his influential work on orthogonal polynomials for general weights on the real line, G. Freud considered weights $w(x)$ of the form

$$w(x) = e^{-Q(x)}, \quad x \in \mathbb{R}\,,$$

where $Q(x)$ is nonnegative, even, convex and of smooth polynomial growth at infinity [8,12]. By a *Freud weight* we mean a function $w(x)$ on $\mathbb{R}$ that satisfies these conditions.

Given polynomials p_m, each of degree exactly equal to m, $m = 0, 1, 2, \dots$, we say that the family $\{p_m\}$ is *associated to the Freud weight* $w(x)$, provided that the p_m's satisfy the orthogonality relation

$$\text{(0.1)} \qquad \int_{\mathbb{R}} p_m(x)w(x)\,p_k(x)w(x)\,dx = \delta_{m,k}\,, \quad m, k = 0, 1, 2, \dots .$$

An important class of Freud weights is given by

$$W_\alpha(x) = e^{-\frac{1}{2}\,|x|^\alpha}\,, \quad \alpha > 1\,,$$

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corresponding to the functions

$$Q_\alpha(x) = \frac{1}{2}\,|x|^\alpha\,, \quad \alpha > 1\,.$$

The associated family of orthogonal polynomials is then denoted $\{p_{m,\alpha}\}$, and includes the Hermite polynomials on $\mathbb{R}$, which correspond to $W_2(x)$; in general, no explicit expression for these (uniquely determined) polynomials is available.

Let the Freud functions, $\mathcal{F}_{m,\alpha}(x)$, be given by

$$\mathcal{F}_{m,\alpha}(x) = p_{m,\alpha}(x)\,W_\alpha(x)\,, \quad m = 0, 1, 2, \dots,$$

and the Freud coefficients of $f(x)$, $c_m(\alpha)$, by

$$c_m(\alpha) = \int_{\mathbb{R}} f(x)\,\mathcal{F}_{m,\alpha}(x)\,dx\,, \quad m = 0, 1, 2, \dots. \tag{0.2}$$

We indicate this correspondence by $f(x) \sim \sum_m c_m(\alpha)\,\mathcal{F}_{m,\alpha}(x)$, and note that the $\{\mathcal{F}_{m,\alpha}\}$ constitute an ONS on $L^2(\mathbb{R})$, and that for f in $L^2(\mathbb{R})$ the Parseval–Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 dx = \sum_{m=0}^{\infty} |c_m(\alpha)|^2 \tag{0.3}$$

holds.

When f is in $L^p(\mathbb{R})$ for $1 < p < 2$ and $\alpha > 1$, expansions in terms of the $\{p_{m,\alpha}\}$ satisfy what Ditzian calls an analogue of the Hausdorff–Young inequality; more precisely, the Freud coefficients of f belong to a weighted ℓ^q space, where $q = p'$, the conjugate index to p, with the corresponding norm inequality [7, Theorem 2.2, p. 583]. On the other hand, a sharpened Hausdorff–Young inequality holds for the Hermite expansions [4, Theorem 4.1], [5]; we refer to the estimate as sharpened because it is of type (p, q) with $q < p'$.

This note concerns a sharpened Hausdorff–Young inequality for the orthogonal expansions in terms of the $\{\mathcal{F}_{m,\alpha}\}$ associated to the weights $W_\alpha(x)$ when $\alpha < 3$, including Lorentz and Orlicz space estimates, and n-dimensional expansions. We also estimate the maximal coefficients of orthogonal expansions in terms of the Freud polynomials when $1 < p < \infty$ and $p \neq 2$.

The paper is organized as follows. Section 1 contains the necessary background material, including the interpolation results that form the basis for our estimates. In Section 2 we discuss the case $n = 1$, including maximal results when $1 < p < 2$, and in Section 3 we consider n-dimensional estimates. We close in Section 4 with the Hausdorff–Young inequality and estimation of the maximal coefficients of orthogonal expansions in terms of Freud polynomials when $2 < p < \infty$.

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1. Preliminaries

Given a function f defined on $\mathbb{R}^n$, with ν the Lebesgue measure on $\mathbb{R}^n$, let $m(f,\lambda)$ denote the *distribution function* of f,

$$m(f,\lambda) = \nu\big(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big), \quad \lambda > 0 .$$

$m(f,\lambda)$ is nonincreasing and right continuous, and the *nonincreasing rearrangement* f^* of f defined for $t > 0$ by

$$f^*(t) = \inf\big\{\, \lambda : m(f,\lambda) \le t \big\} , \quad \inf \emptyset = 0 ,$$

is informally its inverse (this statement is made precise in [13, p. 43]). f^* is nonincreasing and right continuous and, at its points of continuity t, $f^*(t) = \lambda$ is equivalent to $m(f,\lambda) = t$.

The *Lorentz space* $L^{p,q}(\mathbb{R}^n) = L(p,q)$, $0 < p < \infty$, $0 < q \le \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty \big(t^{1/p} f^*(t)\big)^{q}\, \frac{dt}{t}\right)^{1/q}, \quad 0 < q < \infty ,$$

and,

$$\|f\|_{p,\infty} = \sup_{t>0} \big(t^{1/p} f^*(t)\big) = \sup_{\lambda>0} \lambda\, m(f,\lambda)^{1/p} , \quad q = \infty .$$

The Lorentz spaces are monotone with respect to the second index, that is, if $0 < q < q_1 \le \infty$, then $L(p,q) \subset L(p,q_1)$, and $\|f\|_{p,q_1} \lesssim \|f\|_{p,q}$, with $L(p,p)$ being the Lebesgue space $L^p(\mathbb{R}^n)$, and $L(p,\infty)$ the space weak-$L^p(\mathbb{R}^n)$.

As for the Lorentz sequence spaces, given n-tuples of nonnegative integers m, and a sequence $c = \{c_m\}$, let $\{c_k^*\}$ denote the sequence obtained by ordering $\{|c_m|\}$ in a nonincreasing fashion. The *Lorentz sequence space* $\ell^{p,q}$, $1 \le p < \infty$, $1 \le q \le \infty$, consists of those sequences $c = \{c_m\}$ with finite quasinorm $\|c\|_{\ell^{p,q}}$ given by

$$\|c\|_{\ell^{p,q}} = \left(\sum_{k=1}^{\infty} \big(k^{1/p} c_k^*\big)^{q}\, \frac{1}{k}\right)^{1/q}, \quad 1 \le q < \infty ,$$

and, with μ the atomic measure concentrated on the lattice of n-tuples of nonnegative integer atoms m taking the value $\mu(m) = 1$ on each such atom,

$$\|c\|_{\ell^{p,\infty}} = \sup_{k\ge 1} k^{1/p} c_k^* = \sup_{\lambda>0} \lambda\mu\big(\{m : |c_m| > \lambda\}\big)^{1/p}, \quad q = \infty .$$

Now, for the Orlicz spaces, the letters A, B are reserved for *Young's functions*, i.e., for functions $A(t)$ defined for $t \ge 0$ that are zero at zero, increasing, and convex, or, more generally, $A(t)/t$ increasing to ∞ as $t \to \infty$. The *Orlicz space* $L^A(\mathbb{R}^n)$ consists of those measurable functions f (modulo equality a.e.) such that $\int_{\mathbb{R}^n} A\big(|f(x)|/M\big)\, dx < \infty$ for some M, normed by

$$\|f\|_A = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} A\Big(\frac{|f(x)|}{\lambda}\Big)\, dx \le 1\Big\}.$$

The *Orlicz sequence space* ℓ^A consists of those sequence $c = \{c_m\}$ such that for some M,

$$\sum_m A\big(|c_m|/M\big) < \infty ,$$

normed by

$$\|c\|_{\ell^A} = \inf\Big\{\lambda > 0 : \sum_m A\Big(\frac{|c_m|}{\lambda}\Big) \le 1\Big\}.$$

Finally, a mapping T of a class of functions f on (X, μ) into a class of functions on (Y, ν) is said to be *sublinear* provided that,

(i) If T is defined for f_0, f_1, then T is defined for $f_0 + f_1$, and

$$\big|T(f_0 + f_1)(x)\big| \le \big|T(f_0)(x)\big| + \big|T(f_1)(x)\big| .$$

(ii) $\big|T(\lambda f)(x)\big| = |\lambda|\, \big|T(f)(x)\big|$, for any scalar λ .

Clearly a linear operator T is sublinear.

A sublinear operator T defined for $f \in L^A(\mathbb{R}^n)$ and taking values $T(f) = \{c_m\}$ in ℓ^B is said to be *bounded* if there is a constant $K > 0$ such that

$$\sum_m B\Big(\frac{|c_m|}{K}\Big) \le 1 \quad \text{whenever} \quad \int_{\mathbb{R}^n} A\big(|f(x)|\big)\, dx \le 1.$$

Bounded sublinear operators T from ℓ^A to $L^B(\mathbb{R}^n)$ or from $L^A(\mathbb{R}^n)$ into $L^B(\mathbb{R}^n)$ are defined similarly. In either case, the smallest K above is called the *norm* of T, is denoted by $\|T\|$, and the operator is said to be of *type* (A, B). In the former case T satisfies $\|T(f)\|_B \lesssim \|T\|\, \|f\|_{\ell^A}$, and similar norm inequalities in the other cases. When $A(t) = t^p$ and $B(t) = t^q$, we say that T is of *type* (p, q). A mapping T from $L^p(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$, or $\ell^{q,\infty}$, is said to be of *weak-type* (p,q). Similarly for mappings from ℓ^p into weak-$L^q(\mathbb{R}^n)$ spaces.

To interpolate the Lorentz spaces we will use A. P. Calderón's theorem that asserts that if T is a sublinear mapping which simultaneously maps $L(p_0,1)$ into $L(q_0,\infty)$ and $L(p_1,1)$ into $L(q_1,\infty)$, with $1 \le p_0 \ne p_1 \le \infty$, then T maps $L(p,s)$ into $L(q,s)$ where, $1 \le s \le \infty$, $0<\theta<1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, and $1/q = (1-\theta)/q_0 + \theta/q_1$, [3, Corollary to Theorem 10, p. 293].

The underlying principle to interpolate the Orlicz spaces is the following [18]. If a sublinear mapping T is of type, or weak-type, or mixed types, (p_0,q_0) and (p_1,q_1), with $1 \le p_0 \ne p_1 \le \infty$, and the equation of the straight line passing through the points $(1/p_0,1/q_0)$, $(1/p_1,1/q_1)$ is given by $y = \varepsilon x + \delta$, then, under appropriate growth conditions on the Young's functions A, B, the mapping T is of type (A,B) provided that $B^{-1}(t) = t^{\delta} A^{-1}(t^{\varepsilon})$.

For further consideration of the Lorentz and Orlicz spaces the reader may consult [1,9,11,13,17].

2. Sharpened Hausdorff–Young Inequality on the line

Hille's remarkable estimate for the Hermite functions, $\mathcal{H}_m(x)$, to wit,

$$\big|\,\mathcal{H}_m(x)\big| \lesssim m^{-1/12}, \quad m = 1,2,\dots,$$

is the key ingredient in proving the sharpened Hausdorff–Young inequality for the Hermite expansions [4, Theorem 4.1]. Ditzian established similar estimates for the Freud coefficients corresponding to $W_\alpha(x)$, $\alpha > 1$, namely,

$$|c_0(\alpha)| \lesssim_\alpha \|f\|_1, \quad |c_m(\alpha)| \lesssim_\alpha m^{\frac{1}{6}\left((\alpha-3)/\alpha\right)} \|f\|_1, \quad m = 1,2,\dots$$

Before stating our results we find it convenient to introduce the notation

$$\frac{1}{\gamma} = \frac{1}{6}\cdot\frac{3-\alpha}{\alpha}. \tag{2.1}$$

Note that $0 < 1/\gamma < 1/3$ for $1 < \alpha < 3$, and that, in that case, Ditzian's estimates [7, (2.1), p. 583] assert that

$$|c_0(\alpha)| \lesssim_\alpha \|f\|_1, \quad |c_m(\alpha)| \lesssim_\alpha m^{-1/\gamma} \|f\|_1, \quad m = 1,2,\dots \tag{2.2}$$

We then have:

THEOREM 2.1. *With* $1<\alpha<3$, *let* γ *be given by* (2.1). *Suppose that* f *has the expansion* $f(x) \sim \sum_m c_m(\alpha)\,\mathcal{F}_{m,\alpha}(x)$, *where the coefficients are defined as in* (0.2) *above, and let* T *denote the mapping that assigns to* f *the* *sequence $\{c_m(\alpha)\}$ of its Freud coefficients. Then, if $1 \le s \le \infty$, and p, q verify*

$$1 < p < 2 \quad \text{and} \quad \Big(1 - \frac{2}{\gamma}\Big)\frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{\gamma}, \tag{2.3}$$

we have

$$\|T(f)\|_{\ell^{q,s}} = \|\{c_m(\alpha)\}\|_{\ell^{q,s}} \lesssim_{\alpha,p,s} \|f\|_{p,s}, \tag{2.4}$$

and, in particular,

$$\|T(f)\|_{\ell^q} = \|\{c_m(\alpha)\}\|_{\ell^q} \lesssim_{\alpha,p} \|f\|_p. \tag{2.5}$$

Moreover, if A, B are Young's functions such that $B(t)/t^2$ increases, $B(t)/t^\gamma$ decreases, and $\int_t^\infty \big(B(s)/s^\gamma\big)\, ds/s \lesssim B(t)/t^\gamma$, T maps $L^A(\mathbb{R})$ continuously into the Orlicz sequence space ℓ^B provided that A, B verify

$$B^{-1}(t) = t^{(\gamma-1)/\gamma}\, A^{-1}(t^{(2-\gamma)/\gamma}), \quad t > 0. \tag{2.6}$$

Furthermore, if the maximal coefficients $C_m(\alpha)$ of f are given by

$$C_m(\alpha) = \sup_{\beta>0}\Big|\int_{-\beta}^{\beta} f(x)\mathcal{F}_{m,\alpha}(x)\, dx\Big|, \quad m = 0, 1, 2, \dots, \tag{2.7}$$

all norm inequalities above hold with $C_m(\alpha)$ in place of $c_m(\alpha)$ there.

Proof. Let μ denote the atomic measure concentrated on the integer atoms $m = 0, 1, 2, \dots$, taking the value $\mu(m) = 1$ on each such atom. Given $\lambda > 0$, let $\mathcal{I}_\lambda = \{m : |c_m(\alpha)| > \lambda\}$; we are interested in estimating $\mu\big(\mathcal{I}_\lambda\big)$.

Now, if $0 \ne m \in \mathcal{I}_\lambda$, on account of (2.2) we have

$$\lambda < |c_m(\alpha)| \lesssim_\alpha m^{-1/\gamma}\|f\|_1,$$

and, consequently, such m verify

$$m \lesssim_\alpha \Big(\frac{\|f\|_1}{\lambda}\Big)^\gamma.$$

Hence, it readily follows that

$$\lambda^\gamma\, \mu\big(\{m \ne 0 : |c_m(\alpha)| > \lambda\}\big) \lesssim_\alpha \|f\|_1^\gamma, \tag{2.8}$$

which gives the desired estimate for $\mu(\mathcal{I}_\lambda)$ when $0 \notin \mathcal{I}_\lambda$. And, if $0 \in \mathcal{I}_\lambda$, since $|c_0(\alpha)| \lesssim_\alpha \|f\|_1$, it follows that $\lambda < |c_0(\alpha)| \lesssim_\alpha \|f\|_1$, and so

$$\lambda^\gamma\, \mu(0) = \lambda^\gamma \lesssim_\alpha \|f\|_1^\gamma,$$

which combined with (2.8) above gives that, also in this case, $\lambda^\gamma \mu\big(\mathcal{I}_\lambda \big) \lesssim_\alpha \|f\|_1^\gamma$. Hence,

$$\|\{c_m(\alpha)\}\|_{\ell^{\gamma,\infty}} = \sup_{\lambda>0} \lambda\, \mu\big(\{ m : |c_m(\alpha)| > \lambda \} \big)^{1/\gamma} \lesssim_\alpha \|f\|_1 \,, \tag{2.9}$$

and T is continuous from $L(1,1) = L^1(\mathbb{R})$ into the weak sequence space $\ell^{\gamma,\infty}$.

Also, since by (0.3) above T is of type $(2,2)$ and the Lorentz norms are monotone with respect to the second index, we have

$$\|\{c_m(\alpha)\}\|_{\ell^{2,\infty}} \lesssim \|\{c_m(\alpha)\}\|_{\ell^2} \lesssim \|f\|_2 \lesssim \|f\|_{2,1} \,,$$

and T maps $L(2,1)$ continuously into $\ell^{2,\infty}$.

We are thus in the right framework to interpolate for Lorentz spaces, and, consequently, it follows that T maps the Lorentz space $L(p,s)$ continuously into the Lorentz sequence space $\ell(q,s)$, $1 \le s \le \infty$, where, for $0 < \theta < 1$,

$$\frac{1}{p} = \theta + \frac{1-\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{\gamma} + \frac{1-\theta}{2} \,.$$

Now, from the above relations it follows that

$$\frac{1}{p} - \frac{1}{q} = \theta\Big(1 - \frac{1}{\gamma}\Big), \quad \theta = \frac{2}{p} - 1 \,,$$

which, upon eliminating θ, imply that

$$\Big(1 - \frac{2}{\gamma}\Big)\frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{\gamma} \,,$$

and (2.3) above holds.

Moreover, on account of the monotonicity of the Lorentz norms with respect to the second index, since for p, q verifying (2.3) we have $p < 2 < q$, setting $s = q$ in (2.4), it follows that

$$\|\{c_m(\alpha)\}\|_{\ell^q} \lesssim \|\{c_m(\alpha)\}\|_{\ell^{q,q}} \lesssim_p \|f\|_{p,q} \lesssim_p \|f\|_{p.p} \lesssim_p \|f\|_p \,,$$

(2.5) holds, and T is of type (p,q) .

Turning now to the Orlicz spaces, observe that the equation of the line passing through $(1, 1/\gamma)$ and $(1/2, 1/2)$ is given by

$$y = \Big(\frac{2}{\gamma} - 1\Big)\, x + \Big(1 - \frac{1}{\gamma}\Big) \,,$$

and, consequently, (2.6) follows now by interpolation [18, Theorem 2.8, p. 184].

To proceed with the maximal estimates, we transfer the results from the atomic measure to the Lebesgue measure on $\mathbb{R}$ by means of a technique introduced in [3], and conclude that (2.5) holds with $\{C_m(\alpha)\}$ in place of $\{c_m(\alpha)\}$ there.

More precisely, let

$$\mathcal{F}_\alpha(u,x) = \mathcal{F}_{m,\alpha}(x)\,, \quad m \le u < m+1,\ m = 0,1,2,\dots\,,$$

and from

$$c_m(\alpha) = \int_{-\infty}^{\infty} f(x)\,\mathcal{F}_{m,\alpha}(x)\,dx, \quad m = 0,1,2,\dots\,,$$

pass to

$$c_\alpha(f)(u) = \int_{-\infty}^{\infty} f(x)\,\mathcal{F}_\alpha(u,x)\,dx, \quad u \in \mathbb{R}^+\,.$$

Now, if p, q satisfy the relation (2.3) above, it follows that

$$\|c_\alpha(f)\|_q^q = \int_0^\infty \Big|\int_{-\infty}^{\infty} f(x)\,\mathcal{F}_\alpha(u,x)\,dx\Big|^q du$$

$$= \sum_{m=0}^{\infty} \int_m^{m+1} \Big|\int_{-\infty}^{\infty} f(x)\,\mathcal{F}_{m,\alpha}(x)\,dx\Big|^q du = \sum_{m=0}^{\infty} \big|c_m(\alpha)\big|^q,$$

and, consequently, by (2.5),

$$\|c_\alpha(f)\|_q = \|\{c_m(\alpha)\}\|_{\ell^q} \lesssim_{\alpha,p} \|f\|_p\,.$$

Now, since $p < q$ and $\{\chi_{[-\beta,\beta]}\}$ are filtrations in the sense of Christ–Kiselev, the conditions of the maximal inequality are satisfied [6], [16, Theorem 2.11.1, p. 169], and so, with

$$C_\alpha(f)(u) = \sup_\beta \big|c_\alpha(f\chi_\beta)(u)\big| = \sup_\beta \Big|\int_{-\beta}^{\beta} f(x)\,\mathcal{F}_\alpha(u,x)\,dx\Big|,$$

it follows that $\|C_\alpha(f)\|_q \lesssim_{\alpha,p} \|f\|_p$. Again, as above,

$$\|C_\alpha(f)\|_q^q = \sum_{m=0}^{\infty} \int_m^{m+1} \Big(\sup_\beta \Big|\int_{-\beta}^{\beta} f(x)\,\mathcal{F}_{m,\alpha}(x)\,dx\Big|\Big)^q du = \sum_{m=0}^{\infty} C_m(\alpha)^q,$$

and, consequently,

$$\|\{C_m(\alpha)\}\|_{\ell^q} \lesssim_{\alpha,p} \|f\|_p\,, \tag{2.10}$$

and (2.5) holds with $\{C_m(\alpha)\}$ in place of $\{c_m(\alpha)\}$ there.

Let now S be the sublinear mapping that assigns to f the sequence $\{C_m(\alpha)\}$ of its maximal Freud coefficients. Then (2.10) holds for those p, q that verify (2.3) above. The estimates for $\{C_m(\alpha)\}$ in the Lorentz and Orlicz spaces follow now by interpolation; in the case of Lorentz spaces we use [3, Corollary to Theorem 10, p. 293], and for the Orlicz spaces we essentially repeat the argument for the $\{c_m(\alpha)\}$. The proof is thus finished. □

A companion result to the Hausdorff–Young inequality addresses under what conditions $\{c_m\}$ is the sequence of Fourier coefficients of a function f in the Hausdorff–Young range [2], [20, Vol. 2, Theorem 2.3, p. 101]. For the Hermite expansions in $\mathbb{R}$, this is done in [4, Theorem 4.2].

And, for the Freud expansions we have:

THEOREM 2.2. *With* $1 < \alpha < 3$, *let* γ *be given by* (2.1). *Let* $\gamma/(\gamma-1) < p < 2$, *and suppose that* q *is such that*

$$\frac{1}{p} + \Big(1 - \frac{2}{\gamma}\Big)\frac{1}{q} = 1 - \frac{1}{\gamma}\,. \tag{2.11}$$

Then, given $\{c_m\} \in \ell^{p,s}$, *there is* $f \in L(q,s)$ *such that* $c_m(\alpha) = c_m$, *and*

$$\|f\|_{q,s} \lesssim_{\alpha,p,s} \|\{c_m\}\|_{\ell^{p,s}}\,,$$

and, in particular, $\|f\|_q \lesssim_{\alpha,p} \|\{c_m\}\|_{\ell^p}$. *Thus, if* τ *denotes the mapping that assigns* f *to the sequence* $\{c_m\}$, τ *is of type* (p,q) *whenever* (2.11) *holds.*

Moreover, if A, B *are Young's functions such that* $B(t)/t^2$ *increases, and for some* $r > 2$, $B(t)/t^r$ *decreases and* $\int_t^\infty \big(B(s)/s^r\big)\,ds/s \lesssim B(t)/t^r$, *then* τ *maps the Orlicz sequence space* ℓ^A *continuously into the Orlicz space* $L^B(\mathbb{R})$, *provided that* A, B *verify*

$$B^{-1}(t) = t^{(\gamma-1)/(\gamma-2)}\,A^{-1}(t^{\gamma/(2-\gamma)})\,, \quad t > 0\,. \tag{2.12}$$

Furthermore, the maximal operator τ^* *associated to* τ *is of type* (A,B), *and for* $f = \tau(\{c_m\})$ *we have*

$$f(x) = \lim_M \sum_{m=0}^{M} c_m \mathcal{F}_{m,\alpha}(x) \quad a.e.$$

PROOF. Let $b(x) = \{F_{m,\alpha}(x)\}$. Then, by (2.2), as in (2.9) it follows that $b(x) \in \ell^{\gamma,\infty}$ uniformly in x, and so, for a sequence $\{c_m\}$ in $\ell^{\gamma/(\gamma-1),1}$ we have

$$\Big|\sum_m c_m\,\mathcal{F}_{m,\alpha}(x)\Big| \lesssim_\alpha \|\{c_m\}\|_{\ell^{\gamma/(\gamma-1),1}}\,, \quad \text{uniformly in } x\,.$$

Hence, if $f(x) \sim \sum_{m=0}^\infty c_m\mathcal{F}_{m,\alpha}(x)$, then $f \in L^\infty(\mathbb{R})$, and

$$\|f\|_{\infty,\infty} = \|f\|_\infty \lesssim_\alpha \|\{c_m\}\|_{\ell^{\gamma/(\gamma-1),1}}\,. \tag{2.13}$$

And, by a now familiar argument, since τ is of type $(2,2)$ we have $\|f\|_{2,\infty} \lesssim \|\{c_m\}\|_{\ell^{2,1}}$. We are thus in the right framework to interpolate for Lorentz spaces, and, consequently, T maps the Lorentz space $L(p,s)$ continuously into the Lorentz sequence space $\ell^{q,s}$, $1 \le s \le \infty$, where, for $0<\theta<1$,

$$\frac{1}{p} = \frac{\gamma-1}{\gamma}\,\theta + \frac{1-\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{2}\,.$$

Now, from the above relations it follows that

$$\frac{1}{p} - \frac{1}{q} = \theta\Big(1-\frac{1}{\gamma}\Big), \quad \theta = 1-\frac{2}{q}\,,$$

which, upon eliminating θ, imply that

$$\frac{1}{p} + \Big(1-\frac{2}{\gamma}\Big)\frac{1}{q} = 1-\frac{1}{\gamma}\,,$$

which gives (2.11) above, and, provided that (2.11) holds, we get that

$$\|f\|_{q,s} \lesssim_{p,s} \|\{c_m\}\|_{\ell^{p,s}}\,, \quad 1 \le s \le \infty\,.$$

And, since $p<q$, setting $s=q$ gives that τ is of type (p,q) when (2.11) holds.

As for the Orlicz spaces, since the equation of the line passing through $((\gamma-1)/\gamma, 0)$ and $(1/2,1/2)$ is given by

$$y = \frac{\gamma}{2-\gamma}\,x + \frac{\gamma-1}{\gamma-2}\,,$$

(2.12) follows now by interpolation [18, Theorem 2.8, p. 184].

And, we can say more. Let

$$\tau^*\big(\{c_m\}\big) = \sup_M \Big|\sum_{m=0}^{M} c_m \mathcal{F}_{m,\alpha}(x)\Big|\,.$$

Then, by the Orlicz spaces discrete Christ–Kiselev maximal inequality established in [4, Theorem 5.1], it follows that τ^* maps ℓ^A continuously into $L^B(\mathbb{R})$ whenever τ is of type (A,B). We will verify next that the conditions for the Orlicz space convergence result [4, Corollary 3.2] are also met.

Let $f_M = \sum_{m=1}^{M} c_m \mathcal{F}_{m,\alpha}(x)$, and observe that by the linearity and boundedness of τ, with $c_{M_1}^{M_2}$ denoting the sequence with terms c_m for $M_1+1 \le m \le M_2$ and 0 otherwise, we have

$$\|f_{M_2} - f_{M_1}\|_B \lesssim_A \|c_{M_1}^{M_2}\|_{\ell^A} \to 0 \quad \text{as } M_1, M_2 \to \infty,$$

and, consequently, $\{f_M\}$ is Cauchy in $L^B(\mathbb{R})$. If we denote the limit of this sequence by f, then $f(x) \sim \sum_{m=0}^{\infty} c_m\, \mathcal{F}_{m,\alpha}(x)$, $\|f\|_B \lesssim_A \|\{c_m\}\|_{\ell^A}$, and

$$\lim_M \Big\| f - \sum_{m=0}^{M} c_m \mathcal{F}_{m,\alpha} \Big\|_B = 0\,.$$

Also, for a dense subset of ℓ^A, namely, those sequences with finitely many nonzero terms, $\sum_{m=0}^{\infty} c_m \mathcal{F}_{m,\alpha}(x)$ is actually a finite sum, and so,

$$\lim_{M\to\infty} \sum_{m=0}^{M} c_m \mathcal{F}_{m,\alpha}(x) = \sum_{m=0}^{\infty} c_m \mathcal{F}_{m,\alpha}(x)\,, \quad \text{all } x \in \mathbb{R}\,.$$

Hence, all the conditions for the Orlicz spaces pointwise convergence result are met, it follows that

$$f(x) = \lim_M \sum_{m=0}^{M} c_m \mathcal{F}_{m,\alpha}(x) \quad \text{a.e.},$$

and the proof is finished. □

3. Sharpened Hausdorff–Young inequality in $\mathbb{R}^n$

The n-dimensional Hermite functions are obtained as products of the 1-dimensional Hermite functions [5,15,19], and constitute an ONS in $\mathbb{R}^n$ with respect to the Lebesgue measure there. The same is true for general n-dimensional expansion in terms of the orthogonal functions $\mathcal{F}_{m,\alpha}$.

To the point, having fixed an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, where $1 < \alpha_k < 3$ for $k = 1, \dots, n$, and given $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ and an n-tuple of nonnegative integers $m = (m_1, \dots, m_n)$, let the Freud functions, $\mathcal{F}_{m,\alpha}(x)$, be given by

$$\mathcal{F}_{m,\alpha}(x) = \mathcal{F}_{m_1,\alpha_1}(x_1) \cdots \mathcal{F}_{m_n,\alpha_n}(x_n)\,,$$

and the coefficients of $f(x)$, $c_m(\alpha)$, by

$$c_m(\alpha) = \int_{\mathbb{R}^n} f(x)\, \mathcal{F}_{m,\alpha}(x)\, dx\,.$$

We indicate this correspondence by $f(x) \sim \sum_m c_m(\alpha) \mathcal{F}_{m,\alpha}(x)$, and note that, in particular, the n-dimensional Freud expansions satisfy the Parseval–Plancherel formula in $\mathbb{R}^n$, to wit,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \sum_m \big| c_m(\alpha) \big|^2.$$

In order to simplify the statement of our results, with $1 < \alpha_k < 3$, $1 \le k \le n$, we introduce the notation

$$\frac{1}{\gamma_k} = \frac{1}{6} \cdot \frac{3-\alpha_k}{\alpha_k}\,, \quad 1 \le k \le n\,, \quad \text{and,} \quad \Gamma = \gamma_1 + \cdots + \gamma_n\,. \tag{3.1}$$

We then have:

THEOREM 3.1. *Let* Γ *be given by* (3.1). *If* $f(x) \sim \sum_m c_m \mathcal{F}_{m,\alpha}(x)$ *denotes the expansion of a function* f *defined on* $\mathbb{R}^n$ *in a Freud series, let* T *denote the mapping that assigns to* f *its sequence of Freud coefficients* $\{c_m(\alpha)\}$. *Then,* T *maps the Lorentz space* $L(p,s)$ *continuously into the Lorentz sequence space* $\ell^{q,s}$, $1 \le s \le \infty$, *provided that* p, q *verify*

$$1 < p < 2, \quad \textit{and,} \quad \Big(1 - \frac{2}{\Gamma}\Big)\frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{\Gamma}\,. \tag{3.2}$$

In particular,

$$\|T(f)\|_{\ell^q} = \|\{c_m(\alpha)\}\|_{\ell^q} \lesssim \|f\|_p, \tag{3.3}$$

and T *is of type* (p,q) *whenever* (3.2) *holds.*

Moreover, if A, B *are Young's functions such that* $B(t)/t^2$ *increases and* $\int_0^t \big(B(s)/s^\Gamma\big)\, ds/s \lesssim B(t)/t^\Gamma$, T *is of type* (A,B), *provided that* A, B *verify*

$$B^{-1}(t) = t^{\big((\Gamma-1)/\Gamma\big)}\, A^{-1}\Big(t^{\big((2-\Gamma)/\Gamma\big)}\Big), \quad t > 0\,. \tag{3.4}$$

PROOF. For simplicity, since no new ideas are required for general n, we will carry out the proof for $n = 2$. With $\alpha = (\alpha_1, \alpha_2)$, let $f(x) \sim \sum_m c_m(\alpha)\mathcal{F}_{m,\alpha}(x)$ denote the Freud expansion of f.

Now, by (2.2), it readily follows that

$$|c_{m_0}(\alpha)| \lesssim_\alpha \|f\|_1\,, \quad m_0 = (0,0)\,,$$

and also

$$\big|c_m(\alpha)\big| \lesssim_\alpha {m_1}^{-1/\gamma_1} \|f\|_1\,, \quad m = (m_1, 0)\,, m_1 \ge 1, \tag{3.5}$$

and,

$$\big|c_m(\alpha)\big| \lesssim_\alpha {m_2}^{-1/\gamma_2}\, \|f\|_1\,, \quad m = (0, m_2)\,, m_2 \ge 1\,. \tag{3.6}$$

And, for $m = (m_1, m_2)$ with $m_1 \cdot m_2 \ne 0$, we have

$$\big|c_m(\alpha)\big| \lesssim_\alpha m_1^{-1/\gamma_1}\, m_2^{-1/\gamma_2}\, \|f\|_1\,, \quad m = (m_1, m_2)\,, m_1 \cdot m_2 \ne 0\,. \tag{3.7}$$

Let μ denote the atomic measure concentrated on the lattice of 2-tuples of integer atoms $m = (m_1, m_2)$ with $m_1, m_2 = 0, 1, 2, \ldots$, taking the value $\mu(m) = 1$ on each such atom.

Given $\lambda > 0$, let $\mathcal{I}_\lambda = \{m : |c_m| > \lambda\}$; we are interested in estimating $\mu(\mathcal{I}_\lambda)$. Now, if $m = (m_1, m_2)$ is in $\mathcal{I}_\lambda$ and $m_1 \cdot m_2 \neq 0$, by (3.6) we have

$$\lambda < |c_m(\alpha)| \lesssim_\alpha m_1^{-1/\gamma_1}\, m_2^{-1/\gamma_2} \|f\|_1 ,$$

and, consequently,

$$m_1^{1/\gamma_1} m_2^{1/\gamma_2} \lesssim_\alpha \|f\|_1/\lambda ,$$

which, since $m_1, m_2 \geq 1$ implies that $m_1^{1/\gamma_1} \lesssim_\alpha \big(\|f\|_1/\lambda\big)$ and that $m_2^{1/\gamma_2} \lesssim_\alpha \big(\|f\|_1/\lambda\big)$, and so,

$$m_1 \lesssim_\alpha \big(\|f\|_1/\lambda\big)^{\gamma_1}, \quad m_2 \lesssim_\alpha \big(\|f\|_1/\lambda\big)^{\gamma_2}.$$

Hence,

$$\mu\big(\{m = (m_1, m_2) \in \mathcal{I}_\lambda : m_1 \cdot m_2 \neq 0\}\big) \lesssim_\alpha \big(\|f\|_1/\lambda\big)^{\gamma_1}\big(\|f\|_1/\lambda\big)^{\gamma_2} =_\alpha \big(\|f\|_1/\lambda\big)^{\gamma_1+\gamma_2} . \tag{3.8}$$

Also, since from (3.5) and (3.6) above

$$\big|c_m(\alpha)\big| \lesssim_\alpha m_1^{-1/\gamma_1} \|f\|_1 \lesssim_\alpha m_1^{-1/(\gamma_1+\gamma_2)} \|f\|_1 , \quad m = (m_1, 0) ,$$

and

$$\big|c_m(\alpha)\big| \lesssim_\alpha m_2^{-1/\gamma_2} \|f\|_1 \lesssim_\alpha m_1^{-1/(\gamma_1+\gamma_2)} \|f\|_1 , \quad m = (0, m_2) ,$$

it follows that

$$\mu\big(\{m = (m_1, m_2) \in \mathcal{I}_\lambda : m_1 = 0 \text{ or } m_2 = 0\}\big) \lesssim_\alpha \big(\|f\|_1/\lambda\big)^{\gamma_1+\gamma_2} ,$$

which combined with (3.8) above yields

$$\lambda^{\gamma_1+\gamma_2}\, \mu\big(\big\{ m = (m_1, m_2), (m_1, m_2) \neq (0,0) : |c_m(\alpha)| > \lambda\big\}\big) \lesssim_\alpha \|f\|_1^{\gamma_1+\gamma_2}. \tag{3.9}$$

Now, if $m_0 = (0, 0) \in \mathcal{I}_\lambda$, since as observed above $|c_{m_0}(\alpha)| \lesssim_\alpha \|f\|_1$, it follows that $\lambda < |c_{m_0}(\alpha)| \lesssim_\alpha \|f\|_1$, and so

$$\lambda^{\gamma_1+\gamma_2}\, \mu(m_0) = \lambda^{\gamma_1+\gamma_2} \lesssim_\alpha \|f\|_1^{\gamma_1+\gamma_2} ,$$

which combined with (3.9) above gives that

$$\lambda^{\gamma_1+\gamma_2}\, \mu\big(\, \mathcal{I}_\lambda\big) \lesssim_\alpha \|f\|_1^{\gamma_1+\gamma_2} .$$

Therefore, as in (2.9), it follows that, with $\Gamma = \gamma_1 + \gamma_2$,

$$\|\{c_m(\alpha)\}\|_{\ell^{\Gamma,\infty}} = \sup_{\lambda>0} \lambda\, \mu\big(\{\, m : |c_m(\alpha)| > \lambda\}\big)^{1/\Gamma} \lesssim_\alpha \|f\|_1\,, \tag{3.10}$$

and T is continuous from $L(1,1) = L^1(\mathbb{R}^2)$ into the weak sequence space $\ell^{\Gamma,\infty}$.

This estimate is precisely (2.9) above with γ replaced by Γ there. Also, T is of type $(2,2)$ as established by the Parseval–Plancherel formula, and so we have $\|\{c_m(\alpha)\}\|_{\ell^{2,\infty}} \lesssim \|f\|_{2,1}$. The proof now proceeds mutatis mutandis as that of Theorem 2.1 replacing γ by Γ there. The details are left to the reader. □

As for the companion result to the Hausdorff–Young inequality, for the Freud expansions in n dimensions we have:

THEOREM 3.2. *Let Γ be given by* (3.1), *and suppose that p, q verify*

$$\frac{\Gamma}{\Gamma-1} < p < 2, \quad \textit{and,} \quad \frac{1}{p} + \Big(1 - \frac{2}{\Gamma}\Big)\frac{1}{q} = 1 - \frac{1}{\Gamma}\,. \tag{3.11}$$

Then, given $\{c_m\}$ in the Lorentz sequence space $\ell^{p,s}$, there is f in the Lorentz space $L(q,s)$, $1 \le s \le \infty$, such that $f(x) \sim \sum_m c_m\, \mathcal{F}_{m,\alpha}(x)$, and

$$\|f\|_{q,s} \lesssim_{\alpha,p,s} \|\{c_m\}\|_{\ell^{p,s}}\,.$$

In particular, if τ denotes the mapping that assigns f to the sequence $\{c_m\}$, τ is of type (p,q) whenever (3.11) *holds.*

Moreover, if A, B are Young's functions such that $B(t)/t^2$ increases, and for some $r > 2$, $B(t)/t^r$ decreases and $\int_t^\infty \big(B(s)/s^r\big)\, ds/s \lesssim B(t)/t^r$, then τ is of type (A,B), provided that A, B verify

$$B^{-1}(t) = t^{(\Gamma-1)/(\Gamma-2)}\, A^{-1}(t^{\Gamma/(2-\Gamma)})\,, \quad t > 0\,. \tag{3.12}$$

PROOF. For simplicity we argue the case $n = 2$ as no new ideas are required for general n. Let $b(x) = \{\mathcal{F}_{m,\alpha}(x)\}$. Then, as it was shown in the argument leading to (3.10), with $\Gamma = \gamma_1 + \gamma_2$ now, $b(x)$ is in the Lorentz sequence space $\ell^{\Gamma,\infty}$, uniformly in x. Therefore, for a sequence $\{c_m\}$ in its conjugate Lorentz sequence space, $\ell^{\Gamma/(\Gamma-1),1}$, it follows that

$$\Big|\sum_m c_m\, \mathcal{F}_{m,\alpha}(x)\Big| \lesssim_\alpha \|\{c_m\}\|_{\ell^{\Gamma/(\Gamma-1),1}}\,, \quad \text{uniformly in } x \in \mathbb{R}^2\,.$$

Hence, if $f(x) \sim \sum_m c_m \mathcal{F}_{m,\alpha}(x)$, then $f \in L^\infty(\mathbb{R}^2)$, and

$$\|f\|_{\infty,\infty} = \|f\|_\infty \lesssim_\alpha \|\{c_m\}\|_{\ell^{\Gamma/(\Gamma-1),1}}\,,$$

which is estimate (2.13) with Γ in place of γ there.

And, since by the Parseval–Plancherel formula τ is of type $(2,2)$ and we have $\|f\|_{2,\infty} \lesssim \|\{c_m\}\|_{\ell^{2,1}}$, the proof proceeds mutatis mutandis as that of Theorem 2.2 replacing γ with Γ there. The details are left to the reader. □

A couple of remarks in this context. From (2.1), with $\alpha = 2$ there, it follows that $\gamma = 12$, which is Hille's estimate, and so our results include the Hermite expansions. And, if all the α_k's are equal to $1 < \alpha < 3$, say, then, $\Gamma = n\,\alpha$, and so, as $n \to \infty$, the expressions (3.2) and (3.11) above relating p, q, become $1/p + 1/q = 1$, which is precisely the Hausdorff–Young range in the case of Fourier expansions. And, naturally, the expressions (3.4) and (3.12) above approach the formula $B^{-1}(t) = t\,A^{-1}(1/t)$, which is the condition for the Hausdorff–Young inequality to hold for the Fourier transform in the case of Orlicz spaces [10].

4. Hausdorff–Young and maximal coefficients estimates, $2 < p < \infty$

In this section we complement the results for the Hausdorff–Young inequality and the estimation of the maximal Freud coefficients obtained in Theorem 2.1 for $1 < p < 2$, and consider the values of p between 2 and ∞.

We begin by noting the pointwise estimates for the Freud polynomials $\{p_{m,\alpha}\}$ obtained by Ditzian in [7, (2.1), p. 583], to wit,

$$\big|\, p_{m,\alpha}(x) \big| \lesssim_\alpha m^{-1/\gamma}\, e^{\frac{1}{2}\,|x|^\alpha}\,. \tag{4.1}$$

Now, the relation (0.1) with $p_m = p_{m,\alpha}$ and $w = W_\alpha$ there may be restated as

$$\int_{\mathbb{R}} p_{m,\alpha}(x)\, p_{k,\alpha}(x) W_\alpha(x)^2\, dx = \delta_{m,k}, \quad m, k = 0, 1, 2, \dots,$$

and the $\{p_{m,\alpha}\}$ may be considered as an ONS with respect to the measure $d\mu(x) = W_\alpha(x)^2\, dx$, or the weight $W_\alpha(x)^2$, in $\mathbb{R}$.

The L^p spaces in this setting are denoted by $L^p_\mu(\mathbb{R})$, and the Lorentz spaces, which are defined with the measure ν in the Preliminaries replaced by the measure μ in all the relevant definitions there, by $L_\mu(p,q)$.

We are particularly interested in $L_\mu(\infty, 1)$. Now, by [3, p. 284],

$$\lim_{p\to\infty} \|f\|_{L_\mu(p,1)} = \|f\|_{L_\mu(\infty,1)}\,,$$

and we are interested in evaluating the limit.

Recall that by [3, p. 283], an equivalent Lorentz norm is given by

$$\|f\|_{L\mu(p,q)} = \left(\frac{p-1}{p^2} \int_0^\infty \big(f^{**}(t)\, t^{1/p} \big)^{\,q}\, \frac{dt}{t} \right)^{1/q}, \quad 1 \le p, q < \infty\,,$$

where

$$t\, f^{**}(t) = \int_0^t f^*(s)\, ds = \sup_{\mu(E)\le t} \int_E |f(x)|\, d\mu\,.$$

Then, to calculate the limit, note that since $A = \mu(\mathbb{R}) < \infty$, we have

$$\|f\|_{L_\mu(p,1)} = \frac{p-1}{p^2} \int_0^A f^{**}(t)\, t^{1/p}\, \frac{dt}{t} + \frac{p-1}{p^2} \int_A^\infty f^{**}(t)\, t^{1/p}\, \frac{dt}{t}\,,$$

where the second integral tends to 0 as $p \to \infty$, and, since $(p-1)p^{-2}t^{(1/p)-1}$ is the kernel of an approximate identity as $p \to \infty$, it follows that

$$\lim_{p\to\infty} \|f\|_{L_\mu(p,1)} = \lim_{p\to\infty} \frac{p-1}{p^2} \int_0^A f^{**}(t)\, t^{1/p}\, \frac{dt}{t} = f^{**}(0^+)\,.$$

Likewise, as is discussed in [13, Teorema 6, pp. 69-70],

$$f^{**}(0^+) = \|f\|_\infty\,,$$

and, therefore, since $L^\infty_\mu(\mathbb{R}) = L^\infty(\mathbb{R})$, with equality in norms, we have

$$\lim_{p\to\infty} \|f\|_{L_\mu(p,1)} = \|f\|_{L_\mu(\infty,1)} = \|f\|_\infty\,.$$

Furthermore, by the monotonicity of the Lorentz norms with respect to the second index, $L^\infty_\mu(\mathbb{R}) = L_\mu(\infty, q)$ for all $1 \le q \le \infty$.

We denote the coefficients of f with respect to $\{p_{m,\alpha}\}$ in this setting by

$$d_m(\alpha) = \int_{\mathbb{R}} f(x)\, p_{m,\alpha}(x)\, W_\alpha(x)^2\, dx\,, \quad m = 0, 1, \dots, \tag{4.2}$$

indicating this correspondence by $f(x) \sim \sum_m d_m(\alpha)\, p_{m,\alpha}(x)$, and noting that for f in $L^2_\mu(\mathbb{R})$ the Parseval–Plancherel formula

$$\|f\|^2_{L^2_\mu} = \int_{\mathbb{R}} |f(x)|^2\, W_\alpha(x)^2\, dx = \sum_{m=0}^\infty |d_m(\alpha)|^2 = \|\{d_m(\alpha)\}\|^2_{\ell^2} \tag{4.3}$$

holds.

Now, more can be said about the Freud coefficients. Indeed, by (4.1),

$$\begin{aligned} \big|d_m(\alpha)\big| &\le \int_{\mathbb{R}} |f(x)|\, \big|\mathcal{F}_{m,\alpha}(x)\big|\, W_\alpha(x)\, dx \\ &\lesssim_\alpha m^{-1/\gamma} \Big(\int_{\mathbb{R}} W_\alpha(x)\, dx \Big) \|f\|_\infty\,, \end{aligned} \tag{4.4}$$

and, consequently, along the lines of (2.9) it follows that

$$\|\{d_{m,\alpha}\}\|_{\ell^{\gamma,\infty}} \lesssim_\alpha \|f\|_{L_\mu(\infty,1)} . \tag{4.5}$$

And, the estimate (4.5) assumes various useful forms. Indeed, (4.4) implies that $|d_m(\alpha)| \lesssim_\alpha \|f\|_{L^\infty_\mu}$, and, consequently,

$$\|\{d_m(\alpha\}\|_{\ell^\infty} \lesssim_\alpha \|f\|_{L^\infty_\mu} ,$$

which interpolated with the Parseval–Plancherel formula yields

$$\|\{d_m(\alpha\}\|_{\ell^p} \lesssim_{\alpha,p} \|f\|_{L^p_\mu} , \quad 2 < p < \infty . \tag{4.6}$$

Concerning the Hausdorff–Young inequality we then have:

THEOREM 4.1. *Given $1 < \alpha < 3$, let γ be defined by (2.1). Then, if f has the expansion $f(x) \sim \sum_m d_m(\alpha)\, p_{m,\alpha}(x)$, where the coefficients are defined as in (4.2) above, let T denote the linear map that assigns the sequence $\{d_m(\alpha)\}$ to f. Then, if $1 \le s \le \infty$, and p, q verify*

$$2 < p < \infty , \quad \textit{and,} \quad \Big(\frac{2}{\gamma} - 1\Big)\frac{1}{p} + \frac{1}{q} = \frac{1}{\gamma} , \tag{4.7}$$

it follows that

$$\|\{d_m(\alpha)\}\|_{\ell^{q,s}} \lesssim_{\alpha,p,s} \|f\|_{L_\mu(p,s)} , \tag{4.8}$$

and, in particular,

$$\|\{d_m(\alpha)\}\|_{\ell^{q,\infty}} \lesssim_{\alpha,p} \|f\|_{L^p_\mu} . \tag{4.9}$$

Moreover, if A, B are Young's functions such that $B(t)/t^2$ increases, $B(t)/t^\gamma$ decreases, and

$$\int_t^\infty \big(B(s)/s^\gamma\big)\, ds/s \lesssim B(t)/t^\gamma ,$$

T maps $L^A(\mathbb{R})$ continuously into the Orlicz sequence space ℓ^B provided that A, B verify

$$B^{-1}(t) = t^{1/\gamma}\, A^{-1}(t^{(\gamma-2)/\gamma}), \quad t > 0 . \tag{4.10}$$

PROOF. Note that by (4.5), T maps continuously $L_\mu(\infty,1)$ into the sequence space $\ell^{\gamma,\infty}$. Also, by (4.3) and the monotonicity of the Lorentz norms with respect to the second index, we have

$$\|\{d_m(\alpha)\}\|_{\ell^{2,\infty}} \lesssim \|\{d_m(\alpha)\}\|_{\ell^2} \lesssim \|f\|_{L^2_\mu} \lesssim \|f\|_{L_\mu(2,1)} ,$$

and T maps $L_\mu(2,1)$ continuously into $\ell^{2,\infty}$.

We are thus in the right framework to interpolate for Lorentz spaces, and, consequently, by [3, Corollary to Theorem 10, p. 293], T maps the Lorentz space $L(p,s)$ continuously into the Lorentz sequence space $\ell^{q,s}$, $1 \le s \le \infty$, where, for $0 < \theta < 1$,

$$\frac{1}{p} = \frac{\theta}{2}, \quad \text{and,} \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{\gamma}\,.$$

Now, upon eliminating θ, from the above relations it follows that

$$\Big(\frac{2}{\gamma} - 1\Big)\frac{1}{p} + \frac{1}{q} = \frac{1}{\gamma}\,,$$

which gives (4.7) above.

Moreover, on account of the monotonicity of the Lorentz norms with respect to the second index, since for p, q verifying (4.7) we have $q < p$, setting $s = p$ in (4.8), it follows that

$$\|\{d_m(\alpha)\}\|_{\ell^{q,\infty}} \lesssim \|\{d_m(\alpha)\}\|_{\ell^{q,p}} \lesssim_p \|f\|_{L_\mu(p,p)} \lesssim_p \|f\|_{L^p_\mu},$$

(4.9) holds, and T is of weak-type (p,q) .

Turning now to the Orlicz spaces, observe that the equation of the line passing through $(0, 1/\gamma)$ and $(1/2, 1/2)$ is given by

$$y = \Big(1 - \frac{2}{\gamma}\Big)\, x + \frac{1}{\gamma}\,,$$

and, consequently, (4.10) follows by interpolation [18, Theorem 2.8, p. 184]. □

As for the maximal coefficients of f, $D_m(\alpha)$, in analogy to (2.7), they are defined by

$$D_m(\alpha) = \sup_{\beta > 0} \Big| \int_{-\beta}^{\beta} f(x)\, p_{m,\alpha}(x)\, W_\alpha(x)^2\, dx \Big|, \quad m = 0, 1, 2, \dots . \tag{4.11}$$

Note that since $q < p$ in Theorem 4.1, the Christ–Kiselev maximal inequality does not apply in this context. We then have for the maximal coefficients:

THEOREM 4.2. *Let* $1 < \alpha < 3$ *and* γ *be as in* (2.1) *above, and let* T *be the sublinear mapping that assigns to* f *the sequence of its maximal Freud coefficients given by* (4.11). *Then, if* f *is a continuous function and* $2 < p < \infty$, *it follows that*

$$\|\{D_{m,\alpha}\}\|_{\ell^{p+1}} \lesssim_{\alpha,p} \|f\|_{L^p_\mu}\,. \tag{4.12}$$

PROOF. Note that it suffices to prove the assertion for functions f supported on $\mathbb{R}^+$. Suppose first that f is a nonnegative continuous function on $\mathbb{R}^+$ that belongs to $L^p_\mu(\mathbb{R})$. Then, on account of [4, Theorem 2.1], for $p > 2$ we have

$$\left(\int_0^\xi f(x)\, p_{m,\alpha}(x)\, W_\alpha(x)^2\, dx\right)^{p+1} \tag{4.13}$$
$$= (p+1)\int_0^\xi f(x)\, p_{m,\alpha}(x)\left(\int_0^x f(s) p_{m,\alpha}(s) W_\alpha(s)^2\, ds\right)^p W_\alpha(x)^2\, dx\,.$$

Now, by (4.1), the integral on the right–hand side above is dominated by

$$c_{\alpha,p}\int_0^\infty f(x)\, e^{|x|^\alpha/2}\left|\int_0^x f(s) p_{m,\alpha}(s) W_\alpha(s)^2 ds\right|^p W_\alpha(x)^2\, dx\,,$$

and, consequently, taking sup over ξ, it follows that

$$D_m(\alpha)^{p+1} \lesssim_{\alpha,p} \int_0^\infty f(x)\, e^{|x|^\alpha/2}\left|\int_0^x f(s) p_{m,\alpha}(s) W_\alpha(s)^2 ds\right|^p W_\alpha(x)^2\, dx\,.$$

Summing over m we then get that $\|\{D_m(\alpha)\}\|^{p+1}_{\ell^{p+1}}$ is bounded by

$$c_{\alpha,p}\int_0^\infty e^{|x|^\alpha/2}\, f(x)\sum_{m=0}^\infty\left|\int_0^x f(s) p_{m,\alpha}(s) W_\alpha(s)^2\, ds\right|^p W_\alpha(x)^2\, dx\,,$$

and so, on account of (4.6), it follows that

$$\|\{D_{m,\alpha}\}\|^{p+1}_{\ell^{p+1}} \lesssim_{\alpha,p} \int_0^\infty e^{|x|^\alpha/2}\, f(x)\left(\int_0^x f(s) W_\alpha(s)^2\, ds\right)^p W_\alpha(x)^2\, dx$$
$$\lesssim_{\alpha,p} \|f\|^p_{L^p_\mu}\int_0^\infty e^{|x|^\alpha/2}\, f(x) W_\alpha(x)^2\, dx\,.$$

Now, by Hölder's inequality with indices $1/p + 1/q = 1$, where $q < 2$ since $p > 2$, the above expression does not exceed

$$\|\{D_m(\alpha)\}\|^{p+1}_{\ell^{p+1}} \lesssim_{\alpha,p} \|f\|^p_{L^p_\mu}\|f\|_{L^p_\mu}\left(\int_0^\infty e^{|x|^{q\alpha}/2} W_\alpha^2(x)\, dx\right)^{1/q}, \tag{4.14}$$

and, consequently,

$$\|\{D_m(\alpha)\}\|_{\ell^{p+1}} \lesssim_{\alpha,p} \|f\|_{L^p_\mu}. \tag{4.15}$$

Now, given an arbitrary continuous function $f \in L^p_\mu(\mathbb{R})$, let $f_+(x) = \max(f(x),0)$, and $f_-(x) = f_+(x) - f(x)$. Then, $f_-(x), f_+(x)$ are nonnegative and continuous, and if $D^+_m(\alpha), D^-_m(\alpha)$ denote the maximal Freud coefficients of $f_+(x)$ and $f_-(x)$, respectively, then $D_m(\alpha) \le D^+_m(\alpha) + D^-_m(\alpha)$, and, consequently, by (4.15),

$$\|\{D_m(\alpha)\}\|_{\ell^{p+1}} \le \|\{D_m(\alpha)^+\}\|_{\ell^{p+1}} + \|\{D_m(\alpha)^-\}\|_{\ell^{p+1}}$$
$$\lesssim_{\alpha,p} \big(\|f^+\|_{L^p_\mu} + \|f^-\|_{L^p_\mu}\big) \lesssim_{\alpha,p} \|f\|_{L^p_\mu}\,,$$

(4.12) holds, and the proof is finished. □

And, for arbitrary functions in L^p_μ we have:

THEOREM 4.3. *Let $1 < \alpha < 3$ and γ be defined as in* (2.1) *above, and let T be the sublinear mapping that assigns to f the sequence of its maximal Freud coefficients given by* (4.11)*. Then, given $2 < p < \infty$, let $\varepsilon > 0$ be such that $p - \varepsilon > 2$. Then, it follows that*

$$\|\{D_m(\alpha)\}\|_{\ell^{q,s}} \lesssim_{\alpha,p} \|f\|_{L_\mu(p,s)}\,, \tag{4.16}$$

where $1 \le s \le \infty$, and

$$\frac{1}{q} = \frac{1}{p}\Big(\frac{p-\varepsilon}{p-\varepsilon+1} + \frac{\varepsilon}{\gamma}\Big)\,. \tag{4.17}$$

Furthermore, T is of weak-type (p,q) whenever p, q verify (4.17).

PROOF. Let χ_E denote the characteristic function of a measurable set E with $\mu(E) < \infty$. Then, on account of [4, Theorem 2.1], for $p > 2$ we have that (4.13) holds with χ_E in place of the continuous function f there, and, consequently, by (4.15), with $\{D_m(\alpha)\}$ the sequence of maximal coefficients of χ_E, it follows that

$$\|\{D_m(\alpha)\}\|_{\ell^{p-\varepsilon+1}} \lesssim_{\alpha,p} \|\chi_E\|_{L_\mu(p-\varepsilon,1)}\,,$$

and the mapping T is of restricted type $(p-\varepsilon, p-\varepsilon+1)$.

Also, when $f \in L^\infty(\mathbb{R}) = L^\infty_\mu(\mathbb{R})$, with $\{D_m(\alpha)\}$ the sequence of maximal coefficients of f,

$$D_m(\alpha) \lesssim_\alpha m^{-1/\gamma} \int_{\mathbb{R}} |f(x)|\, W_\alpha(x)\, dx \lesssim_\alpha m^{-1/\gamma} \|f\|_\infty\,,$$

and so, as the argument leading to (2.9) shows,

$$\|\{D_m(\alpha)\}\|_{\ell^{\gamma,\infty}} \lesssim_\alpha \|f\|_{L_\mu(\infty,1)}.$$

Thus, we are in the framework to apply interpolation in Lorentz spaces, and obtain that for $0 < \theta < 1$, and

$$\frac{1}{r} = \frac{\theta}{p-\varepsilon}\,, \quad \frac{1}{q} = \frac{\theta}{p-\varepsilon+1} + \frac{1-\theta}{\gamma}\,.$$

it follows that

$$\|\{D_m(\alpha)\}\|_{\ell^{q,s}} \lesssim_{\alpha,p} \|f\|_{L_\mu(r,s)}\,.$$

Now, the choice $\theta = (p-\varepsilon)/p$ gives that $1-\theta = \varepsilon/p$, and that $r = p$, and so

$$\frac{1}{q} = \frac{1}{p}\Big(\frac{p-\varepsilon}{p-\varepsilon+1} + \frac{\varepsilon}{\gamma}\Big)\,,$$

(4.17) holds, and (4.16) has been established.

Finally, since $q < p$ whenever (4.17) holds, T is of weak-type (p,q) on account of the monotonicity of the Lorentz spaces with respect to the second index. □

Because of the cumbersome expressions involved, the Orlicz spaces version of Theorem 4.3 is left for the reader to verify.

We close the note with two remarks. The first concerns the assumption that $1 < \alpha < 3$ throughout this note. When this is the case, γ as given by (2.1) is positive, and, therefore, the Freud coefficients of integrable functions tend to 0 as $m^{-1/\gamma}$ when $m \to \infty$. This allows us to establish the (unweighted) sharpened Hausdorff–Young inequality in this case.

Now, resting on [12, Theorem 13.2, p. 360], Ditzian showed that for $\alpha > 1$,

$$\max_{x\in\mathbb{R}} \big|\mathcal{F}_{m,\alpha}(x)\big| = \max_{x\in\mathbb{R}} \big|p_{m,\alpha}(x)\big|\, W_\alpha(x) \approx m^{1/6((\alpha-3)/\alpha)}\,, \quad m = 1,2,\dots,$$

thus hinting at the possible growth of the Freud coefficients when $\alpha > 3$, [7, (2.1), p. 583]. In this case, as noted above, Ditzian's (weighted) analogue to the Hausdorff–Young inequality holds for all $\alpha > 1$.

Which brings us to the second remark. Unlike the unweighted case above, where the coefficients are given by (0.2), the Freud coefficients in Ditzian's case are given by (4.2), and, as established in [7, Theorem 2.2, p. 583], they are bounded by

$$|d_0(\alpha)| \lesssim_\alpha \|fW_\alpha\|_1\,, \quad |d_m(\alpha)| \lesssim_\alpha m^{(1-\frac{3}{\alpha})\frac{1}{6}} \|fW_\alpha\|_1\,, \quad m = 1,2,\dots.$$

Then, along the lines of the proof of Theorems 2.1 and 2.2, the reader should have no difficulty in showing:

Theorem 4.4. *Suppose that f has the expansion*

$$f(x) \sim \sum_{m=0}^{\infty} d_m(\alpha)\, p_{m,\alpha}(x),$$

where the coefficients are defined as in (4.2) *above. Then, with $1 < \alpha < 3$, if $1 \le s \le \infty$, and p, q verify*

$$1 < p < 2\,, \quad \text{and,} \quad \frac{4\alpha-3}{3\alpha}\cdot\frac{1}{p}+\frac{1}{q}=\frac{14\alpha-6}{12\alpha}\,,$$

we have

$$\|\{d_m(\alpha)\}\|_{\ell^{q,s}} \lesssim_{\alpha,p,s} \|fW_\alpha\|_{p,s}\,,$$

and, in particular,

$$\|\{d_m(\alpha)\}\|_{\ell^{q}} \lesssim_{\alpha,p} \|fW_a\|_{p}\,.$$

Furthermore, let $(14-\alpha)/12\alpha < p < 2$, and suppose that q is such that

$$\frac{1}{p}+\frac{4\alpha-3}{3\alpha}\cdot\frac{1}{q}=\frac{14\alpha-6}{12\alpha}\,.$$

Then, given $\{d_m\} \in \ell(p,s)$, there is f such that $fW_\alpha \in L(q,s)$, $d_m(\alpha) = d_m$, and

$$\|fW_\alpha\|_{q,s} \lesssim_{\alpha,p,s} \|\{d_m(\alpha)\}\|_{\ell^{q,s}}\,,$$

and, in particular, $\|fW_\alpha\|_q \lesssim_{\alpha,p} \|\{d_m(\alpha)\}\|_{\ell^p}$.

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