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NON-TRIVIAL r-WISE INTERSECTING FAMILIES

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Abstract. A k-uniform family $\mathcal{F} \subset \binom{[n]}{k}$ is called non-trivial r-wise intersecting if $F_1 \cap F_2 \cap \dots \cap F_r \neq \emptyset$ for every $F_1, F_2, \dots, F_r \in \mathcal{F}$ and $\cap \mathcal{F} = \emptyset$. O'Neill and Verstraëte determined the maximum size of a non-trivial r-wise intersecting family for n sufficiently large. Actually, the Hilton–Milner–Frankl Theorem implies O'Neill–Verstraëte's result for $n \geq r(k-r+2)$. In the present paper, we show that the same result holds for a certain range when n is close to $2k$.

1. Introduction

Let $[n] = \{1, 2, \dots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. For $0 \leq k \leq n$ let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. Subsets of $2^{[n]}$ are called *families*. If $\mathcal{F} \subset 2^{[n]}$ satisfies $\mathcal{F} \subset \binom{[n]}{k}$, it is called k*-uniform*.

For integers r, t where $r \geq 2$, $t \geq 1$ a family $\mathcal{F} \subset 2^{[n]}$ is called r*-wise* t*-intersecting* if $|F_1 \cap \dots \cap F_r| \geq t$ for all $F_1, \dots, F_r \in \mathcal{F}$. In case of $r = 2$ the term t*-intersecting* and in case of $t = 1$ the terms r*-wise intersecting* and *intersecting* are used.

For a family $\mathcal{F} \subset 2^{[n]}$ one defines its *dual* $\mathcal{F}^c = \left\{ [n] \setminus F : F \in \mathcal{F} \right\}$. Note that the r-wise t-intersecting property of $\mathcal{F}$ is equivalent to $|G_1 \cup \dots \cup G_r| \leq n - t$ for all $G_1, \dots, G_r \in \mathcal{F}^c$. Families satisfying the latter property are called r*-wise* t*-union*.

Let us recall the Erdős–Ko–Rado Theorem, one of the cornerstones of extremal set theory.

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Theorem 1.1 [5]. *Suppose that* $\mathcal{F} \subset 2^{[n]}$ *is intersecting. Then*

$$|\mathcal{F}| \le 2^{n-1}. \tag{1.1}$$

Moreover, if $\mathcal{F} \subset \binom{[n]}{k}$ *and* $n \ge 2k$ *then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}. \tag{1.2}$$

Most readers could guess that equality in (1.1) and (1.2) can be achieved by *stars*, families in which a fixed element of $[n]$ is contained in all members. However, for (1.1) and in the special case $n = 2k$ also for (1.2) there are many other families attaining equality. Hilton and Milner [14] proved a strong stability result for (1.2), $n > 2k$.

Theorem 1.2 [14]. *If* $n > 2k$ *and* $\mathcal{F} \subset \binom{[n]}{k}$ *is intersecting and is not a star, then*

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1.3}$$

Note that an r-wise intersecting family is always r'-wise intersecting if $r > r' \ge 2$.

Definition 1.3 (Brace–Daykin Families). Let $r \ge 2$, $n > r$.
(a) $\mathcal{A}(n,r) = \left\{ A \subset [n] : |A \cap [r+1]| \ge r \right\}$,
(b) $\mathcal{B}(n,r) = \left\{ B \subset [n] : |B \cap [n-r,n]| \le 1 \right\}$.

It is easy to see that $\mathcal{A}(n,r)$ is r-wise intersecting, $\mathcal{B}(n,r)$ is r-wise union and $\mathcal{A}(n,r)^c$ is isomorphic to $\mathcal{B}(n,r)$.

For a family $\mathcal{F} \subset 2^{[n]}$ define $\bigcup \mathcal{F} = \bigcup_{F \in \mathcal{F}} F$. If $\bigcup \mathcal{F} = [n]$ then $\mathcal{F}$ is called *covering*. One of the early gems of extremal set theory is the following.

Theorem 1.4 (Brace–Daykin Theorem [2]). *Suppose that* $r \ge 3$, $\mathcal{B} \subset 2^{[n]}$ *is* r*-wise union and covering. Then*

$$|\mathcal{B}| \le |\mathcal{B}(n,r)| = (r+2)\, 2^{n-r-1} \tag{1.4}$$

with equality holding if and only if $\mathcal{B}$ *is isomorphic to* $\mathcal{B}(n,r)$.

For the proof of our main results we need a stronger result. To state it let us make a definition. A subset $H \subset [n]$ is called a *hole* for the family $\mathcal{F} \subset 2^{[n]}$ if $|F \cap H| \le 1$ for all $F \in \mathcal{F}$. Note that $\mathcal{B}(n,r)$ has a hole of size $r+1$, namely $[n-r,n]$. On the other hand possessing a hole of size $r+1$ guarantees the r-wise union property.

THEOREM 1.5 [12]. *Let $r \geq 3$, $n \geq r+2$ and $\mathcal{F} \subset 2^{[n]}$. Suppose that $\mathcal{F}$ is r-wise union, covering and it possesses no hole of size $r+1$. Then*

$$|\mathcal{F}| \leq (r+6)\, 2^{n-r-2}. \tag{1.5}$$

Even that (1.5) is only a slight improvement on (1.4), it will be sufficient for our proofs. Actually, (1.5) is best possible.

EXAMPLE 1.6. For $n \geq r+2 \geq 5$ define

$$\mathcal{D}(n,r) = \left\{ D \subset [n] : |D \cap [r+2]| \leq 1 \text{ or } D \cap [r+2] \in \binom{\{r, r+1, r+2\}}{2} \right\}.$$

Note that $[r]$, $[r-1] \cup \{r+1\}$ and $[r-1] \cup \{r+2\}$ are holes of size r in $\mathcal{D}(n,r)$. They permit to show that $\mathcal{D}(n,r)$ is r-wise union.

The extension of the uniform part, (1.2) of the Erdős–Ko–Rado Theorem to r-wise intersecting families was one of the first results of the first author. Let us state it for r-wise union families.

THEOREM 1.7 [6,10]. *Let n, k, r be positive integers, $k \leq n \leq rk$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise union. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k}. \tag{1.6}$$

Moreover, in case of equality $\mathcal{F}$ is isomorphic to $\binom{[n-1]}{k}$.

We should note that for $n \leq 2k$, (1.6) is a consequence of (1.2). However for $n > 2k$ it is no longer the case.

For $\mathcal{F} \subset 2^{[n]}$ and $1 \leq k \leq n$, define

$$\mathcal{F}^{(k)} := \left\{ F \in \mathcal{F} : |F| = k \right\}.$$

There are two natural constructions for relatively large covering k-uniform families that are r-wise union. The first is $\mathcal{B}^{(k)}(n,r)$. The second is from [8]

$$\mathcal{H}(n,k,r) = \left\{ H \in \binom{[n]}{k} : [r-1] \subset H,\ H \cap [r, k+1] \neq \emptyset \right\} \cup \binom{[k+1]}{k}.$$

Note that $\mathcal{H}(n,k,r)$ is r-wise intersecting. Therefore the actual example that is needed is its dual $\mathcal{H}(n,k,r)^c$, which is r-wise union.

Let us note that both $\mathcal{B}(n,r)$ and $\mathcal{H}(n,k,r)^c$ are s-wise $(r+1-s)$-union for all s, $2 \leq s < r$. This is no coincidence.

CLAIM 1.8. *Suppose that $\mathcal{F} \subset 2^{[n]}$ is r-wise union and covering, $r \geq 3$. Then $\mathcal{F}$ is s-wise $(r+1-s)$-union for all s, $2 \leq s < r$.*

PROOF. Suppose the contrary and fix $F_1, \dots, F_s \in \mathcal{F}$ such that

$$|F_1 \cup \dots \cup F_s| \geq n - (r - s).$$

Then $Y := [n] \setminus (F_1 \cup \dots \cup F_s)$ satisfies $|Y| \leq r - s$. Since $\mathcal{F}$ is covering we may choose $F_{s+1}, \dots, F_r \in \mathcal{F}$ satisfying $Y \subset F_{s+1} \cup \dots \cup F_r$. Then $F_1 \cup \dots \cup F_r = [n]$, a contradiction. $\square$

THEOREM 1.9 (O'Neill, Verstraëte [18]). *Let $k \leq n - r$, $n > n_0((n-k))$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise union and covering. Then*

$$|\mathcal{F}| \leq \max\left\{ |\mathcal{B}^{(k)}(n,r)|, |\mathcal{H}(n,k,r)| \right\}. \tag{1.7}$$

Let us recall two results concerning 2-wise t-intersecting families for $t \geq 2$.

THEOREM 1.10 (Exact Erdős–Ko–Rado Theorem [5], [7], [20]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is (2-wise) t-intersecting, $n \geq (k - t + 1)(t + 1)$. Then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}. \tag{1.8}$$

Note that Erdős, Ko, Rado proved (1.8) for $n > n_0(k,t)$. The exact bound $(k - t + 1)(t + 1)$ was proved in [7] for $t \geq 15$ and later by Wilson [20] for all $t \geq 2$.

Let us state the corresponding stability result.

THEOREM 1.11 (Hilton–Milner–Frankl Theorem [8], [7], [1]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is $(r-1)$-intersecting and non-trivial, $n \geq r(k - r + 2)$. Then*

$$|\mathcal{F}| \leq \max\left\{ |\mathcal{A}^{(k)}(n,r)|, |\mathcal{H}(n,k,r)| \right\}. \tag{1.9}$$

In view of Claim 1.8 one can deduce Theorem 1.9 from Theorem 1.11. In the dual version, that is, for r-wise intersecting families in $\binom{[n]}{k}$ the condition on (n,k,r) is

$$k \leq \frac{n}{r} + r - 2. \tag{1.10}$$

Being linear in n, this bound is rather strong however compared with the restriction $k \leq \frac{r-1}{r} n$ of Theorem 1.7 the gap is still very large. The aim of the present paper is to show that (1.7) holds for a certain range close to $\frac{n}{2}$. The proof is completely different from the above results. It relies on Theorem 1.5.

Theorem 1.12. *Let* $\varepsilon > 0$, $n \geq \frac{4}{\varepsilon^2} + 7$, $\left(\frac{1}{2} + \varepsilon\right) n \leq k \leq \frac{3n}{5} - 3$ *and let* $\mathcal{F} \subset \binom{[n]}{k}$ *be a* 3*-wise union covering family. Then*

$$|\mathcal{F}| \leq |\mathcal{B}^{(k)}(n, 3)|.$$

Theorem 1.13. *Let* $\varepsilon > 0$, $n \geq \frac{4}{\varepsilon^2} + 8$, $\left(\frac{1}{2} + \varepsilon\right) n \leq k \leq 0.65n - 4$ *and let* $\mathcal{F} \subset \binom{[n]}{k}$ *be a* 4*-wise union covering family. Then*

$$|\mathcal{F}| \leq |\mathcal{B}^{(k)}(n, 4)|.$$

By the same method, we can also show that if $\mathcal{F} \subset \binom{[n]}{k}$ is 5-wise union and covering then $|\mathcal{F}| \leq |\mathcal{B}^{(k)}(n, 5)|$ for $n \geq \frac{3.2}{\varepsilon^2} + 8$, $\left(\frac{1}{2} + \varepsilon\right) n \leq k \leq 0.675n - 4$. Moreover, if $\mathcal{F} \subset \binom{[n]}{k}$ is 6-wise union and covering then $|\mathcal{F}| \leq |\mathcal{B}^{(k)}(n, 6)|$ for $n \geq \frac{3.3}{\varepsilon^2} + 9$, $\left(\frac{1}{2} + \varepsilon\right) n \leq k \leq 0.65n - 4$.

For $r \geq 11$, we prove the following theorem.

Theorem 1.14. *Let* $\mathcal{F} \subset \binom{[n]}{k}$ *be* r*-wise union and covering. If* $(\frac{1}{2} + \varepsilon)n \leq k < \left(\frac{1}{2} + \frac{1}{4(r+5)}\right) n - r$, $\varepsilon > 0$, $r \geq 11$ *and* $n \geq 2\log(r + 10)/\varepsilon^2$, *then*

$$|\mathcal{F}| \leq |\mathcal{B}^{(k)}(n, r)|. \tag{1.11}$$

For $\mathcal{F} \subset \binom{[n]}{k}$ and $0 \leq \ell < k$, define the *ℓth shadow* $\partial^{(\ell)}\mathcal{F}$ as

$$\partial^{(\ell)}\mathcal{F} = \left\{ E \in \binom{[n]}{\ell} : \text{there exists } F \in \mathcal{F} \text{ such that } E \subset F \right\}.$$

The celebrated Kruskal–Katona Theorem [16,17] gives the best possible lower bounds on $|\partial^{(\ell)}\mathcal{F}|$ for given size of $\mathcal{F}$.

For every positive integer m, one can write m in k-cascade form uniquely:

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$$

with $a_k > a_{k-1} > \cdots > a_s \geq 1$. We need the following version of the Kruskal–Katona Theorem (see [9] for a short proof).

Theorem 1.15 (The Kruskal–Katona Theorem [16,17]). *If* $\mathcal{F} \subset \binom{[n]}{k}$, $|\mathcal{F}| = m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$, *then*

$$|\partial^{(\ell)}\mathcal{F}| \geq \binom{a_k}{\ell} + \binom{a_{k-1}}{\ell - 1} + \cdots + \binom{a_s}{\ell - k + s}. \tag{1.12}$$

Let us recall the following version of the Chernoff bound.

THEOREM 1.16 [4,15]. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i = 1$ with probability p and $X_i = 0$ with probability $1-p$, $i = 1, 2, \ldots, n$. Let $X = X_1 + X_2 + \cdots + X_n$ and $\lambda = np$. Then for $t \geq 0$*

$$\Pr(X \leq \mathbb{E}X - t) \leq \exp\Big(-\frac{t^2}{2\lambda}\Big). \tag{1.13}$$

The following inequality concerning the tails of sum of binomial coefficients is an easy consequence of the Chernoff bound.

LEMMA 1.17. *Let n be a positive integer and let $m = \big(\frac{1}{2} - \varepsilon\big)n$ with $\varepsilon > 0$. Then*

$$\sum_{0 \leq i \leq m} \binom{n}{i} \leq e^{-\varepsilon^2 n} 2^n. \tag{1.14}$$

PROOF. Consider independent random variables $X_1, X_2, \ldots, X_n$ with $X_i = 1$ with probability $\frac{1}{2}$ and $X_i = 0$ with probability $\frac{1}{2}$, $i = 1, 2, \ldots, n$. Let $X = X_1 + X_2 + \cdots + X_n$. Then it is easy to see that

$$\Pr(X \leq m) = \frac{\sum_{0 \leq i \leq m} \binom{n}{i}}{2^n}.$$

By applying (1.13) with $t = \varepsilon n$ and $\lambda = \frac{n}{2}$, (1.14) follows. $\square$

2. Non-trivial 3-wise union families

In this section, we prove Theorem 1.12. The following lemma gives a lower bound on $|\mathcal{B}^{(k)}(n,3)|$ in the k-cascade form, which admits the use of the Kruskal–Katona Theorem.

LEMMA 2.1. *For $n \geq \frac{5}{3}k + 4$,*

$$\binom{n-4}{k} + 4\binom{n-4}{k-1} > \binom{n-2}{k} + \binom{n-5}{k-1} + \binom{n-7}{k-2}. \tag{2.1}$$

PROOF. Note that

$$\binom{n-4}{k} + 4\binom{n-4}{k-1} = \binom{n-3}{k} + 3\binom{n-4}{k-1}$$
$$= \binom{n-3}{k} + \left(\binom{n-4}{k-1} + \binom{n-4}{k-2}\right) + 2\binom{n-4}{k-1} - \binom{n-4}{k-2}$$
$$= \binom{n-2}{k} + 2\binom{n-4}{k-1} - \binom{n-4}{k-2}$$

$$= \binom{n-2}{k} + \binom{n-5}{k-1} + \binom{n-5}{k-2} + \binom{n-4}{k-1} - \binom{n-4}{k-2}$$
$$= \binom{n-2}{k} + \binom{n-5}{k-1} + \binom{n-5}{k-2} + \binom{n-5}{k-1} - \binom{n-5}{k-3}.$$

To prove (2.1), it suffices to show that

$$\binom{n-5}{k-1} + \binom{n-5}{k-2} > \binom{n-7}{k-2} + \binom{n-5}{k-3}.$$

By expanding,

$$\binom{n-7}{k-1} + 2\binom{n-7}{k-2} + \binom{n-7}{k-3} + \binom{n-7}{k-2} + 2\binom{n-7}{k-3} + \binom{n-7}{k-4}$$
$$> \binom{n-7}{k-2} + \binom{n-7}{k-3} + 2\binom{n-7}{k-4} + \binom{n-7}{k-5}.$$

Equivalently,

$$\binom{n-7}{k-1} + 2\binom{n-7}{k-2} + 2\binom{n-7}{k-3} > \binom{n-7}{k-4} + \binom{n-7}{k-5}. \tag{2.2}$$

Set $n = ck + 4$. Using $c - 1 > \frac{1}{2}$, we have

$$\frac{\binom{n-7}{k-1}}{\binom{n-7}{k-3}} = \frac{(n-k-4)(n-k-5)}{(k-1)(k-2)} \geq (c-1)^2, \quad \frac{\binom{n-7}{k-2}}{\binom{n-7}{k-3}} = \frac{n-k-4}{k-2} \geq c-1,$$
$$\frac{\binom{n-7}{k-4}}{\binom{n-7}{k-3}} = \frac{k-3}{n-k-3} \leq \frac{1}{c-1}, \quad \frac{\binom{n-7}{k-5}}{\binom{n-7}{k-3}} = \frac{(k-3)(k-4)}{(n-k-2)(n-k-3)} \leq \frac{1}{(c-1)^2}.$$

Then it is sufficient to show that

$$(c-1)^2 + 2(c-1) + 2 > \frac{1}{c-1} + \frac{1}{(c-1)^2}. \tag{2.3}$$

Let

$$f(x) = (x-1)^2 + 2(x-1) + 2 - \frac{1}{x-1} - \frac{1}{(x-1)^2}.$$

Now by (2.3) it suffices to show that $f(x) > 0$ for $x \geq \frac{5}{3}$. Note that

$$f'(x) = 2(x-1) + 2 + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3} > 0 \quad \text{for } x > 1$$

and $f\left(\frac{5}{3}\right) = \frac{1}{36}$. Thus $f(x) > 0$ for $x \geq \frac{5}{3}$ and the lemma follows. □

Proof of Theorem 1.12. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise union covering family. Assume indirectly that $|\mathcal{F}| > |\mathcal{B}^{(k)}(n,3)| = \binom{n-4}{k} + 4\binom{n-4}{k-1}$. Define

$$\mathcal{F}_* = \left\{ G \subset [n] : \exists F \in \mathcal{F},\ G \subset F \right\}.$$

Obviously, $\mathcal{F}_* \subset 2^{[n]}$ is 3-wise union, $\bigcup \mathcal{F}_* = [n]$. Since $|\mathcal{F}_*^{(k)}| = |\mathcal{F}| > |\mathcal{B}^{(k)}(n,3)|$, $\mathcal{F}_*$ is not contained in a copy of $\mathcal{B}(n,3)$, i.e., it possesses no hole of size 4. Thus by (1.5) we infer that

$$|\mathcal{F}_*| \le \frac{9}{32} 2^n. \tag{2.4}$$

By (2.1), we have

$$|\mathcal{F}| > \binom{n-4}{k} + 4\binom{n-4}{k-1} > \binom{n-2}{k} + \binom{n-5}{k-1} + \binom{n-7}{k-2}.$$

Then by (1.12)

$$|\mathcal{F}_*^{(\ell)}| = |\partial^{(\ell)}\mathcal{F}| \ge \binom{n-2}{\ell} + \binom{n-5}{\ell-1} + \binom{n-7}{\ell-2}. \tag{2.5}$$

Summing (2.5) for $0 \le \ell \le k$ gives

(2.6)

$$\begin{aligned}|\mathcal{F}_*| &\ge \sum_{0\le \ell \le k} \left(\binom{n-2}{\ell} + \binom{n-5}{\ell-1} + \binom{n-7}{\ell-2} \right) = 2^{n-2} + 2^{n-5} + 2^{n-7} \\ &- \left(\sum_{k<j\le n-2} \binom{n-2}{j} + \sum_{k-1<j\le n-5} \binom{n-5}{j} + \sum_{k-2<j\le n-7} \binom{n-7}{j} \right) \\ &\ge \frac{9}{32} 2^n + 2^{n-7} - \left(\sum_{k<j\le n-2} \binom{n-2}{j} \right. \\ &\left. + \sum_{k-1<j\le n-5} \binom{n-5}{j} + \sum_{k-2<j\le n-7} \binom{n-7}{j} \right).\end{aligned}$$

Since $k \ge \left(\frac{1}{2} + \varepsilon\right) n$ implies $n-k-3 < \left(\frac{1}{2} - \varepsilon\right)(n-2)$, by (1.14) we infer that

$$\sum_{k<j\le n-2} \binom{n-2}{j} = \sum_{0\le j\le n-k-3} \binom{n-2}{j} \le e^{-\varepsilon^2 (n-2)} 2^{n-2}.$$

Similarly, we have

$$\sum_{k-1<j\le n-5}\binom{n-5}{j}\le e^{-\varepsilon^2(n-5)}\,2^{n-5},\qquad \sum_{k-2<j\le n-7}\binom{n-7}{j}\le e^{-\varepsilon^2(n-7)}\,2^{n-7}.$$

Since $n\ge \frac{4}{\varepsilon^2}+7>\frac{\log 37}{\varepsilon^2}+7$,

$$\sum_{k<j\le n-2}\binom{n-2}{j}+\sum_{k-1<j\le n-5}\binom{n-5}{j}+\sum_{k-2<j\le n-7}\binom{n-7}{j}$$
$$\le \Big(32e^{-\varepsilon^2(n-2)}+4e^{-\varepsilon^2(n-5)}+e^{-\varepsilon^2(n-7)}\Big)2^{n-7}\le 37e^{-\varepsilon^2(n-7)}\,2^{n-7}<2^{n-7}.$$

By (2.6) it follows that $|\mathcal{F}_*|>\frac{9}{32}2^n$, contradicting (2.4). Thus the theorem holds. □

3. Non-trivial 4-wise union families

By a similar argument as in Section 2, we prove Theorem 1.13.

LEMMA 3.1. *For $n\ge 1.53k+5$,*

$$\binom{n-5}{k}+5\binom{n-5}{k-1}>\binom{n-3}{k}+\binom{n-5}{k-1}+\binom{n-8}{k-2}. \tag{3.1}$$

PROOF. Using $\binom{n-3}{k}=\binom{n-5}{k}+2\binom{n-5}{k-1}+\binom{n-5}{k-2}$, we see that (3.1) is equivalent to

$$2\binom{n-5}{k-1}>\binom{n-5}{k-2}+\binom{n-8}{k-2}.$$

Equivalently,

$$2\binom{n-8}{k-1}+4\binom{n-8}{k-2}+3\binom{n-8}{k-3}>\binom{n-8}{k-4}+\binom{n-8}{k-5}.$$

Set $n=ck+5$. Then by $c-1>\frac{1}{2}$

$$\frac{\binom{n-8}{k-1}}{\binom{n-8}{k-3}}=\frac{(n-k-5)(n-k-6)}{(k-1)(k-2)}\ge (c-1)^2,\quad \frac{\binom{n-8}{k-2}}{\binom{n-8}{k-3}}=\frac{n-k-5}{k-2}\ge c-1,$$

$$\frac{\binom{n-8}{k-4}}{\binom{n-8}{k-3}}=\frac{k-3}{n-k-4}\le\frac{1}{c-1},\quad \frac{\binom{n-8}{k-5}}{\binom{n-8}{k-3}}=\frac{(k-3)(k-4)}{(n-k-3)(n-k-4)}\le\frac{1}{(c-1)^2}.$$

Then it is sufficient to show that

$$2(c-1)^2+4(c-1)+3>\frac{1}{c-1}+\frac{1}{(c-1)^2}. \tag{3.2}$$

Let

$$f(x)=2(x-1)^2+4(x-1)+3-\frac{1}{x-1}-\frac{1}{(x-1)^2}.$$

Now by (3.2), it suffices to show that $f(x)>0$ for $x\geq 1.53$. Note that

$$f'(x)=4(x-1)+4+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}>0 \quad \text{for } x>1$$

and $f(1.53)=0.235022$. Thus $f(x)>0$ for $x\geq 1.53$ and the lemma follows. □

Proof of Theorem 1.13. Assume indirectly that $|\mathcal{F}|>|\mathcal{B}^{(k)}(n,4)|=\binom{n-5}{k}+5\binom{n-5}{k-1}$. Define

$$\mathcal{F}_*=\left\{G\subset[n]:\exists F\in\mathcal{F},\ G\subset F\right\}.$$

Obviously, $\mathcal{F}_*\subset 2^{[n]}$ is 4-wise union, $\bigcup\mathcal{F}_*=[n]$. Since $|\mathcal{F}_*^{(k)}|=|\mathcal{F}|>|\mathcal{B}^{(k)}(n,4)|$, $\mathcal{F}_*$ is not contained in a copy of the Brace–Daykin family. Thus by (1.5) we infer that

$$|\mathcal{F}_*|\leq\frac{5}{32}2^n. \tag{3.3}$$

By (3.1) we have

$$|\mathcal{F}|>\binom{n-5}{k}+5\binom{n-5}{k-1}>\binom{n-3}{k}+\binom{n-5}{k-1}+\binom{n-8}{k-2}.$$

Then by (1.12)

$$|\mathcal{F}_*^{(\ell)}|=|\partial^{(\ell)}\mathcal{F}|\geq\binom{n-3}{\ell}+\binom{n-5}{\ell-1}+\binom{n-8}{\ell-2}. \tag{3.4}$$

Summing (3.4) for $0\leq\ell\leq k$ gives

$$|\mathcal{F}_*|\geq\sum_{0\leq\ell\leq k}\left(\binom{n-3}{\ell}+\binom{n-5}{\ell-1}+\binom{n-8}{\ell-2}\right)=2^{n-3}+2^{n-5}+2^{n-8} \tag{3.5}$$

$$-\left(\sum_{k<j\leq n-3}\binom{n-3}{j}+\sum_{k-1<j\leq n-5}\binom{n-5}{j}+\sum_{k-2<j\leq n-8}\binom{n-8}{j}\right)$$

$$\geq \frac{5}{32}2^n + 2^{n-8} - \Bigg(\sum_{k<j\leq n-3} \binom{n-3}{j} + \sum_{k-1<j\leq n-5} \binom{n-5}{j} + \sum_{k-2<j\leq n-8} \binom{n-8}{j} \Bigg).$$

Since $k \geq \left(\frac{1}{2}+\varepsilon\right)n$ implies $n-k-4 < \left(\frac{1}{2}-\varepsilon\right)(n-3)$, by (1.14) we infer that

$$\sum_{k<j\leq n-3} \binom{n-3}{j} = \sum_{0\leq j\leq n-k-4} \binom{n-3}{j} \leq e^{-\varepsilon^2(n-3)}\, 2^{n-3}.$$

Similarly, we have

$$\sum_{k-1<j\leq n-5} \binom{n-5}{j} \leq e^{-\varepsilon^2(n-5)}\, 2^{n-5}, \quad \sum_{k-2<j\leq n-8} \binom{n-8}{j} \leq e^{-\varepsilon^2(n-8)}\, 2^{n-8}.$$

Since $n \geq \frac{4}{\varepsilon^2} + 8 > \frac{\log 41}{\varepsilon^2} + 8$,

$$\sum_{k<j\leq n-3} \binom{n-3}{j} + \sum_{k-1<j\leq n-5} \binom{n-5}{j} + \sum_{k-2<j\leq n-8} \binom{n-8}{j}$$
$$\leq \left(32e^{-\varepsilon^2(n-3)} + 8e^{-\varepsilon^2(n-5)} + e^{-\varepsilon^2(n-8)}\right)2^{n-8} \leq 41\, e^{-\varepsilon^2(n-8)}\, 2^{n-8} < 2^{n-8}.$$

By (3.5) it follows that $|\mathcal{F}_*| > \frac{5}{32}2^n$, contradicting (3.3). Thus the theorem holds. $\square$

4. The general case

In the general case, we fail to find the proper k-cascade form lower bound on $|\mathcal{B}^{(k)}(n,r)|$. Instead of the Kruskal–Katona Theorem, we shall use Sperner's shadow bound [19] as follows: For $\mathcal{F} \subset \binom{[n]}{k}$ and $0 \leq \ell < k$,

$$\frac{|\partial^{(\ell)}\mathcal{F}|}{\binom{n}{\ell}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}. \tag{4.1}$$

Proof of Theorem 1.14. Assume indirectly that $|\mathcal{F}| > |\mathcal{B}^{(k)}(n,r)| = \binom{n-r-1}{k} + (r+1)\binom{n-r-1}{k-1}$. Define

$$\mathcal{F}_* = \left\{ G \subset [n] : \exists F \in \mathcal{F},\ G \subset F \right\}.$$

Obviously, $\mathcal{F}_* \subset 2^{[n]}$ is r-wise union, $\bigcup \mathcal{F}_* = [n]$. Since $|\mathcal{F}_*^{(k)}| = |\mathcal{F}| > |\mathcal{B}^{(k)}(n,r)|$, $\mathcal{F}_*$ is not contained in a copy of the Brace–Daykin family. Thus by (1.5) we infer that

$$|\mathcal{F}_*| \le \frac{r+6}{2^{r+2}} 2^n. \tag{4.2}$$

On the other hand we define $\alpha = \frac{|\mathcal{F}|}{\binom{n}{k}}$ and note

$$\alpha > \frac{\binom{n-r}{k} + r\binom{n-r-1}{k-1}}{\binom{n}{k}} = \Big(1 + \frac{rk}{n-r}\Big)\frac{\binom{n-r}{k}}{\binom{n}{k}} \ge \frac{n+(k-1)r}{n-r}\Big(\frac{n-k-r}{n-r}\Big)^r.$$

Let $k = \big(\frac{1}{2} - \beta\big)n - r = \big(\frac{1}{2} + \beta - \frac{r}{n}\big)n \ge \frac{n}{2} + \varepsilon$. Then $\varepsilon + \frac{r}{n} \le \beta \le \frac{1}{4(r+5)}$ and

$$\alpha > \frac{n+kr}{n}\Big(\frac{n-k-r}{n}\Big)^r \ge \frac{r+2}{2}\Big(\frac{1}{2} - \beta\Big)^r.$$

By (4.1), $|\mathcal{F}_*^{(\ell)}| = |\partial^{(\ell)}\mathcal{F}| \ge \alpha\binom{n}{\ell}$ for all $0 \le \ell \le k$. Then by (1.14) and $n - k = \big(\frac{1}{2} - \beta + \frac{r}{n}\big)n$ we have

$$\begin{aligned} |\mathcal{F}_*| &\ge \sum_{0\le\ell\le k} \alpha\binom{n}{\ell} = \alpha 2^n - \sum_{0\le\ell\le n-k-1} \alpha\binom{n}{\ell} \\ &> \alpha\big(1 - e^{-(\beta-\frac{r}{n})^2 n}\big)2^n \ge \frac{r+2}{2}\Big(\frac{1}{2} - \beta\Big)^r\big(1 - e^{-\varepsilon^2 n}\big)2^n. \end{aligned} \tag{4.3}$$

Comparing with (4.2) we get

$$\frac{r+6}{2(r+2)} > (1-2\beta)^r\big(1 - e^{-\varepsilon^2 n}\big) \ge (1 - 2r\beta)\big(1 - e^{-\varepsilon^2 n}\big). \tag{4.4}$$

Note that $n \ge 2\log(r+10)/\varepsilon^2$ and $r \ge 11$ imply

$$1 - e^{-\varepsilon^2 n} > \frac{(r+5)(r+6)}{(r+2)(r+10)}.$$

By $\beta \le \frac{1}{4(r+5)}$, we infer

$$(1-2r\beta)\big(1 - e^{-\varepsilon^2 n}\big) > \frac{r+10}{2(r+5)} \cdot \frac{(r+5)(r+6)}{(r+2)(r+10)} = \frac{r+6}{2(r+2)},$$

contradicting (4.4). □

5. Concluding remarks

We should mention that the bounds of Theorem 1.9 on $n_0(n-k)$ were greatly improved by Cao, Lv, Wang [3]. Our result shows that the same bounds hold for k close to $n/2$.

Theorem 1.7 suggests that one should put the bar even higher and try to determine the maximum size of k-uniform non-trivial r-wise intersecting families for the range $k < \frac{r-1}{r}n$.

Let us mention two related results.

THEOREM 5.1 [11,12]. *Let $\mathcal{F} \subset 2^{[n]}$ be r-wise t-union, $n \geq t$. If $r \geq 3$ and $t \leq 2^r - r - 1$, then*

$$|\mathcal{F}| \leq 2^{n-t}. \tag{5.1}$$

THEOREM 5.2 [13]. *Let m, p, r be positive integers, $r \geq 3$. Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_r \subset \binom{[m]}{p}$ are non-empty and cross-union. For $\frac{m}{r-1} \leq p \leq \frac{r-1}{r}m$,*

$$\sum_{1 \leq i \leq r} |\mathcal{F}_i| \leq r\binom{m-1}{p}. \tag{5.2}$$

Let us note for non-trivial r-wise union families $\mathcal{F} \subset \binom{[n]}{k}$ that under the assumption that $\mathcal{F}$ has a hole of size r one can use Theorem 5.2 to prove that $|\mathcal{F}| \leq |\mathcal{B}^{(k)}(n,r)|$. However the general case appears to be very difficult to handle.

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