Acta Math. Hungar., **169** (**1**) (2023), 238–251
https://doi.org/10.1007/s10474-023-01304-y
First published online March 1, 2023

CERTAIN PROPERTIES OF THE ENHANCED POWER GRAPH ASSOCIATED WITH A FINITE GROUP

J. KUMAR[1,*,†], X. MA[2], PARVEEN[1,‡] and S. SINGH[1]

[1]Department of Mathematics, Birla Institute of Technology and Science Pilani, Pilani-333031, India
e-mails: jitenderarora09@gmail.com, p.parveenkumar144@gmail.com, sidharth_0903@hotmail.com

[2]School of Science, Xi'an Shiyou University, Xi'an 710065, China
e-mail: xuanlma@mail.bnu.edu.cn

(Received August 31, 2022; accepted December 10, 2022)

Abstract. The enhanced power graph of a finite group G, denoted by $\mathcal{P}_E(G)$, is a simple undirected graph whose vertex set is G and two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in G$. In this article, we determine all finite groups such that the minimum degree and the vertex connectivity of $\mathcal{P}_E(G)$ are equal. Also, we classify all groups whose (proper) enhanced power graphs are strongly regular. Further, the vertex connectivity of the enhanced power graphs associated to some nilpotent groups is obtained. Finally, we obtain the upper and lower bounds of the Wiener index of $\mathcal{P}_E(G)$, where G is a nilpotent group. The finite nilpotent groups attaining these bounds are also characterized.

1. Introduction

The study of graphs related to various algebraic structures becomes important, because graphs of this type have valuable applications and are related to automata theory (see [16,17] and the books [14,15]). Certain graphs, viz. power graphs, commuting graphs, Cayley graphs etc., associated to groups have been studied by various researchers, see [5,13,26]. Segev [25,26], Segev and Seitz [27] used combinatorial parameters of certain commuting graphs to establish long standing conjectures in the theory of division algebras. A variant of commuting graphs on groups has played an important role

* Corresponding author.

† The first author wishes to acknowledge the support of Core Research Grant (CRG/2022/001142) funded by SERB.

‡ The third author gratefully acknowledge for providing financial support to CSIR (09/719(0110)/2019-EMR-I) government of India.

Key words and phrases: enhanced power graph, nilpotent group, vertex connectivity, minimum degree, Wiener index.

Mathematics Subject Classification: q05C25.

in classification of finite simple groups, see [2]. Hayat et al. [12] used commuting graphs associated with groups to establish some NSSD (non-singular with a singular deck) molecular graph.

In order to measure how much the power graph is close to the commuting graph of a group G, Aalipour et al. [1] introduced a new graph called *enhanced power graph.* The enhanced power graph of a group G is a simple undirected graph whose vertex set is G and two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in G$. Indeed, the enhanced power graph contains the power graph and is a spanning subgraph of the commuting graph. Aalipour et al. [1] characterized the finite group G, for which equality holds for either two of the three graphs viz. power graph, enhanced power graph and commuting graph of G. Further, the enhanced power graphs have received the considerable attention by various researchers. Bera et al. [3] characterized the abelian groups and the non abelian p-groups having dominatable enhanced power graphs. Dupont et al. [10] determined the rainbow connection number of enhanced power graph of a finite group G. Later, Dupont et al. [9] studied the graph theoretic properties of enhanced quotient graph of a finite group G. A complete description of finite groups with enhanced power graphs admitting a perfect code have been studied in [19]. Ma et al. [21] investigated the metric dimension of the enhanced power graph of a finite group. Hamzeh et al. [11] derived the automorphism groups of enhanced power graphs of finite groups. Zahirović et al. [29] proved that two finite abelian groups are isomorphic if their enhanced power graphs are isomorphic. Also, they supplied a characterization of finite nilpotent groups whose enhanced power graphs are perfect. Recently, Panda et al. [24] studied the graph-theoretic properties, viz. minimum degree, independence number, matching number, strong metric dimension and perfectness, of enhanced power graphs over finite abelian groups. Moreover, the enhanced power graphs associated to non-abelian groups such as semidihedral, dihedral, dicyclic, U_{6n}, V_{8n} etc., have been studies in [7,24]. Bera et al. [4] gave an upper bound for the vertex connectivity of enhanced power graph of any finite abelian group. Moreover, they classified the finite abelian groups whose proper enhanced power graphs are connected. The complement of the enhanced power graph has been studied in [18,23]. For a comprehensive list of results and open questions on enhanced power graphs of groups, we refer the reader to [20].

In this paper, we aim to enhance the investigation of the interplay between algebraic properties of the group G and its enhanced power graph $\mathcal{P}_E(G)$. This paper is arranged as follows. In Section 2, we provide the necessary background material and fix our notations used throughout the paper. In Section 3, we classify all finite groups such that the minimum degree is equal to the vertex connectivity of $\mathcal{P}_E(G)$. Section 4 comprises the classification of groups whose enhanced power graphs are (strongly) regular.

In Section 5, we obtain the vertex connectivity of $\mathcal{P}_E(G)$, where G belongs to a class of nilpotent groups. Finally, in Section 6, we study the Wiener index of $\mathcal{P}_E(G)$, where G is a nilpotent group.

2. Preliminaries

In this section, first we recall the graph theoretic notions from [28]. A *graph* Γ is a pair $\Gamma = (V, E)$, where $V(\Gamma)$ and $E(\Gamma)$ are the set of vertices and edges of Γ, respectively. Two distinct vertices u_1, u_2 are *adjacent*, denoted by $u_1 \sim u_2$ if there is an edge between u_1 and u_2. Otherwise, we write it as $u_1 \nsim u_2$. Let Γ be a graph. A *subgraph* Γ' of Γ is the graph such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. For $X \subseteq V(\Gamma)$ the subgraph of Γ induced by X is the graph $\Gamma(X)$ with vertex set X and two vertices of $\Gamma(X)$ are adjacent if and only if they are adjacent in Γ. A graph Γ is said to be *complete* if every two distinct vertices are adjacent. All the vertices which are adjacent to a vertex $v \in V(\Gamma)$ is called the *neighbours* of v. The *degree* $\deg(v)$ of a vertex v in a graph Γ, is the number of edges incident to v. The *minimum degree*, denoted by $\delta(\Gamma)$, is defined by $\delta(\Gamma) = \min\{\deg(v) : v \in V(\Gamma)\}$. A graph Γ is *k-regular* if the degree of every vertex in $V(\Gamma)$ is k. A graph Γ is said to be *strongly regular graph* with parameters (n, k, λ, μ) if it is k-regular graph on n vertices such that each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours. A *path* in a graph is the sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the next vertex of it. A graph Γ is *connected* if each pair of vertices has a path in Γ. Otherwise, Γ is *disconnected*. The *distance* between $u, v \in V(\Gamma)$, denoted by $d(u, v)$, is the number of edges in a shortest path connecting them. For a connected graph Γ, the Wiener index $W(\Gamma)$ is defined by

$$W(\Gamma) = \sum_{x \in V(\Gamma)} \sum_{y \in V(\Gamma)} \frac{d(x, y)}{2}.$$

The *diameter* of Γ is the maximum distance between the pair of vertices in Γ. A *vertex (or edge) cut-set* in a connected graph Γ is a set X of vertices (or edges) such that the remaining subgraph $\Gamma \setminus X$, by removing the set X, is either disconnected or has only one vertex. The cardinality of a smallest vertex (or edge) cut-set of Γ is called the *vertex (or edge) connectivity* of Γ and it is denoted by $\kappa(\Gamma)$ (or $\kappa'(\Gamma)$). For a connected graph Γ, it is well known that $\kappa(\Gamma) \leq \kappa'(\Gamma) \leq \delta(\Gamma)$. The *strong product* $\Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_r$ of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ is a graph such that

• the vertex set of $\Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_r$ is the Cartesian product $V(\Gamma_1) \times V(\Gamma_2) \times \cdots \times V(\Gamma_r)$; and

• distinct vertices $(u_1, u_2, \ldots, u_r)$ and $(v_1, v_2, \ldots, v_r)$ are adjacent in $\Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_r$ if and only if either $u_i = v_i$ or $u_i \sim v_i$ in Γ_i for each $i \in [r] = \{1, 2, \ldots, r\}$.

We refer the readers to [8] for basic definitions and results of group theory. A cyclic subgroup of a group G is called a *maximal cyclic subgroup* if it is not properly contained in any cyclic subgroup of G. If G is a cyclic group, then G is the only maximal cyclic subgroup of G. The set of all maximal cyclic subgroups of G is denoted by $\mathcal{M}(G)$. Let G be a group. The *order of an element* x in G is the cardinality of the subgroup generated by x and it is denoted by $o(x)$. The *exponent* of a group G is the least common multiple of the orders of all elements of G and it is denoted by $\exp(G)$. A group G is called a *torsion group* if every element of G is of finite order. The following result is useful for latter use.

THEOREM 2.1 [8]. *Let G be a finite group. Then the following statements are equivalent*:

(i) *G is a nilpotent group.*

(ii) *Every Sylow subgroup of G is normal.*

(iii) *G is the direct product of its Sylow subgroups.*

(iv) *For $x, y \in G$, x and y commute whenever $o(x)$ and $o(y)$ are relatively primes.*

All the groups considered in this paper are finite. We write $p, p_1, p_2, \ldots, p_r$ to be prime numbers such that $p_1 < p_2 < \cdots < p_r$ and P_i the unique Sylow p_i-subgroup of G for $i \in [r]$. In view of Theorem 2.1, for a nilpotent group G and $x \in G$, there exists a unique element $x_i \in P_i$ such that $x = x_1 x_2 \cdots x_r$, for $i \in [r]$. The *enhanced power graph* $\mathcal{P}_E(G)$ of a finite group G is a simple undirected graph with vertex set G and two vertices are adjacent if they belong to the same cyclic subgroup of G. For $X \subseteq G$, we denote by $\mathcal{P}_E(X)$ the subgraph induced by X. The following results will be useful in the sequel.

THEOREM 2.2 [3, Theorem 2.4]. *The enhanced power graph $\mathcal{P}_E(G)$ of the group G is complete if and only if G is cyclic.*

THEOREM 2.3 [24, Theorem 3.2]. *For a finite group G, the minimum degree $\delta(\mathcal{P}_E(G)) = m - 1$, where m is the order of a maximal cyclic subgroup of minimum possible order.*

LEMMA 2.4 [6, Lemma 2.11]. *Any maximal cyclic subgroup of a finite nilpotent group $G = P_1 P_2 \cdots P_r$ is of the form $M_1 M_2 \cdots M_r$, where M_i is a maximal cyclic subgroup of P_i, $(1 \leq i \leq r)$.*

COROLLARY 2.5. *Let $G = P_1 P_2 \cdots P_r$ be a nilpotent group and P_i is cyclic for some i. Then P_i is contained in every maximal cyclic subgroup of G.*

THEOREM 2.6 [22, Theorem 4.2]. *Let G be a non-trivial finite group. Then the proper power graph of G is strongly regular graph if and only if G is a p-group of order p^m for which $\exp(G) = p$ or p^m.*

3. Equality of the minimum degree and the vertex connectivity of $\mathcal{P}_E(G)$

It is well known that the diameter of $\mathcal{P}_E(G)$ is at most two. Consequently, $\kappa'(\mathcal{P}_E(G)) = \delta(\mathcal{P}_E(G))$ and so $\kappa(\mathcal{P}_E(G)) \leq \delta(\mathcal{P}_E(G))$. In this section, we classify the group G such that $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$. We begin with the following lemma.

LEMMA 3.1. *Let G be a non-cyclic group and $M \in \mathcal{M}(G)$. Then $\overline{M}$ is a cut-set of $\mathcal{P}_E(G)$, where $\overline{M}$ is the union of all sets of the form $M \cap \langle x \rangle$, for $x \in G \setminus M$.*

PROOF. Let $M = \langle a \rangle$ and $M' = \langle b \rangle$ be two maximal cyclic subgroups of G. Then we claim that there is no path between a and b in $\mathcal{P}_E(G \setminus \overline{M})$. If possible, suppose that there is a path $a \sim x_1 \sim x_2 \sim \cdots \sim x_k \sim b$ from a to b in $\mathcal{P}_E(G \setminus \overline{M})$. Then $x_1 \in M$. Otherwise, $\langle a, x_1 \rangle$ is a cyclic subgroup which is not contained in M, which is impossible. We may now suppose that $x_1, x_2, \ldots, x_{r-1} \in M$ and $x_r \notin M$ for some $r \in [k] \setminus \{1\}$. Note that such r exists because $x_k \sim b$ and if $x_r \in M$ for each $r \in [k]$, then $x_k \in \overline{M}$ which is impossible. Now if $x_{r-1} \in M$ then by using a similar argument, we obtain $x_{r-1} \in \overline{M}$. It follows that no such path exists and so $\overline{M}$ is a cut-set. □

THEOREM 3.2. *For the group G, $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$ if and only if one of the following holds*:

(i) *G is a cyclic group.*

(ii) *G is non-cyclic and contains a maximal cyclic subgroup of order* 2.

PROOF. First suppose that $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$. If G is cyclic then we have nothing to prove. If possible, let G be non-cyclic group and it does not have a maximal cyclic subgroup of order 2. By Theorem 2.3, $\delta(\mathcal{P}_E(G)) = |M| - 1$, where $M \in \mathcal{M}(G)$ such that $|M|$ is minimum. By Lemma 3.1, $\overline{M}$ is a cut-set. Note that every generator of M does not belong to $\overline{M}$. Consequently, we get

$$\kappa(\mathcal{P}_E(G)) \leq |\overline{M}| < |M| - 1 = \delta(\mathcal{P}_E(G)),$$

which is impossible. Thus, G must have a maximal cyclic subgroup of order 2.

To prove the converse part, suppose that G is cyclic. Then $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G)) = n - 1$ (by Theorem 2.2). If G is non-cyclic and has a maximal cyclic subgroup M of order 2, then by Lemma 3.1, $\overline{M} = \{e\}$ is a cut-set. It

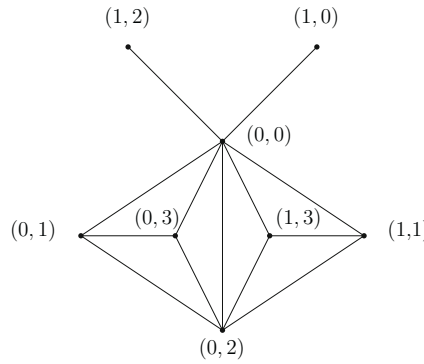

Fig. 1: $\mathcal{P}_E(\mathbb{Z}_2 \times \mathbb{Z}_4)$

follows that $\kappa(\mathcal{P}_E(G)) = 1$. By Theorem 2.3, $\delta(\mathcal{P}_E(G)) = |M| - 1 = 1$ and so $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$. □

EXAMPLE 3.3. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then G has two maximal cyclic subgroups $\langle(1,2)\rangle$ and $\langle(1,0)\rangle$ of order 2. By Fig. 1, observe that the minimum degree is 1 and $\{(0,0)\}$ is the smallest cut-set of $\mathcal{P}_E(G)$. Thus, $\kappa(\mathcal{P}_E(G)) = \delta(\mathcal{P}_E(G)) = 1$.

4. Regularity of $\mathcal{P}_E(G)$

The identity element of the group G is adjacent to all the other elements of G in $\mathcal{P}_E(G)$. Thus, $\mathcal{P}_E(G)$ is regular if and only if G is a finite cyclic group (cf. Theorem 2.2). The proper enhanced power graph $\mathcal{P}_E^*(G)$ is the subgraph of $\mathcal{P}_E(G)$ induced by $G \setminus \{e\}$. In this section, we classify the group G such that $\mathcal{P}_E^*(G)$ is (strongly) regular.

THEOREM 4.1. *Let G be a finite group. Then $\mathcal{P}_E^*(G)$ is regular if and only if one of the following holds*:

(i) *G is a cyclic group.*

(ii) *$|M_i| = |M_j|$ and $M_i \cap M_j = \{e\}$, where $M_i, M_j \in \mathcal{M}(G)$.*

PROOF. Suppose that $\mathcal{P}_E^*(G)$ is regular. If G is cyclic then there is nothing to prove. We may now suppose that G is a non-cyclic group. Assume that $M_i = \langle x\rangle$ and $M_j = \langle y\rangle$ are two maximal cyclic subgroups of G. Since $|M_i| - 2 = \deg(x) = \deg(y) = |M_j| - 2$, we deduce that $|M_i| = |M_j|$. Moreover, if $M_i \cap M_j \neq \{e\}$ for some i, j $(i \neq j)$ then for $a \neq e \in M_i \cap M_j$, we obtain $\deg(a) \geq |M_i \cup M_j| - 2 \neq |M_i| - 2 = \deg(x)$. Consequently, $\mathcal{P}_E^*(G)$ is not regular; a contradiction.

Conversely, suppose that G is a cyclic group then by Theorem 2.2, $\mathcal{P}_E^*(G)$ is complete and so is regular. We may now suppose that G is non-cyclic.

If G satisfies condition (ii) then note that every element of $G \setminus \{e\}$ lies in exactly one maximal cyclic subgroup of G, consequently for each $x \in G \setminus \{e\}$, we have $\deg(x) = |M_i| - 2$, where M_i is the maximal cyclic subgroup of G containing x. Hence, $\mathcal{P}_E^*(G)$ is regular. □

REMARK 4.2. Notice that there are several groups which satisfy the condition (ii) of the Theorem 4.1. For instance, an elementary abelian p-group, a non-abelian group $G = \langle x, y, z;\ x^p = y^p = z^p = e,\ yz = zyx,\ xy = yx, xz = zx\rangle$, etc. where p is an odd prime.

Clearly, a strongly regular graph is always regular. However, the converse need not be true. We show that the converse is also true for $\mathcal{P}_E(G)$ in the following theorem.

THEOREM 4.3. *Let G be a finite group. Then $\mathcal{P}_E^*(G)$ is regular if and only if $\mathcal{P}_E^*(G)$ is strongly regular.*

PROOF. To prove the result, it is sufficient to show that if $\mathcal{P}_E^*(G)$ is regular then $\mathcal{P}_E^*(G)$ is strongly regular. Suppose that $\mathcal{P}_E^*(G)$ is regular then G must satisfy one of the conditions given in Theorem 4.1. If G is cyclic then being a complete graph, $\mathcal{P}_E^*(G)$ is strongly regular. If G satisfies condition (ii), then by the proof of Theorem 4.1, for each $x \in V(\mathcal{P}_E^*(G))$ we obtain $\deg(x) = m - 2$, where m is the order of a maximal cyclic subgroup containing x. For $m = 2$, $\mathcal{P}_E^*(G)$ is a null graph and so is strongly regular. If $m \geq 3$, then observe that in $\mathcal{P}_E^*(G)$, each pair of adjacent vertices has exactly $m - 3$ common neighbours and each pair of non-adjacent vertices has no common neighbour. Hence, $\mathcal{P}_E^*(G)$ is strongly regular with parameters $(n, m-2, m-3, 0)$. □

In view of [1, Theorem 28] and Theorem 2.6, we have the following corollary.

COROLLARY 4.4. *If G is a non-cyclic p-group then $\mathcal{P}_E^*(G)$ is regular if and only if the exponent of G is p.*

THEOREM 4.5. *Let G be a non-cyclic nilpotent group. Then $\mathcal{P}_E^*(G)$ is regular if and only if G is a p-group with exponent p.*

PROOF. Let G be a non-cyclic nilpotent group of order $n = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$. To prove our result it is sufficient to prove that if $\mathcal{P}_E^*(G)$ is regular then G is a p-group. If possible, let $r \geq 2$. Since G is a non-cyclic group there exists a non-cyclic Sylow subgroup P_i. Consequently, P_i has at least two maximal cyclic subgroups, namely: M_i and M_i'. Consider the maximal cyclic subgroups $M = M_1 M_2 \cdots M_i \cdots M_r$ and $M' = M_1 M_2 \cdots M_i' \cdots M_r$ of G, here M_j is a maximal cyclic subgroup of P_j for $j \in [r] \setminus \{i\}$. By Lemma 2.4, we obtain that M and M' are maximal cyclic subgroups of G such that $M \cap M' \neq \{e\}$; a contradiction of Theorem 4.1. Thus, $r = 1$ and so G is p-group. □

COROLLARY 4.6. *Let G be a finite non-cyclic abelian group. Then $\mathcal{P}_E^*(G)$ is regular if and only if G is an elementary abelian p-group.*

Based on the results obtained in this section, we posed the following conjecture which we are not able to prove.

CONJECTURE. *Let G be a finite non-cyclic group. If $\mathcal{P}_E^*(G)$ is regular then G is a p-group with exponent p.*

5. The vertex connectivity of $\mathcal{P}_E(G)$

In this section, we investigate the vertex connectivity of the enhanced power graph of some nilpotent groups. Recall that if G and H are two torsion groups then $\mathcal{P}_E(G \times H) \cong \mathcal{P}_E(G) \boxtimes \mathcal{P}_E(H)$ if and only if $\gcd(o(g), o(h)) = 1$ for all $g \in G$ and $h \in H$ (see [29, Lemma 2.1]). Let $G = P_1P_2 \cdots P_r$ be a nilpotent group. For our purpose, first we show that the enhanced power graph of a finite nilpotent group is isomorphic to the strong product of the enhanced power graph of its Sylow subgroups (see Theorem 5.4). Using this and ascertaining a minimum cut-set, we obtain the vertex connectivity of $\mathcal{P}_E(G)$, where G is a nilpotent group such that each of its Sylow subgroups is cyclic except P_k for some $k \in [r]$.

Let $G = P_1P_2 \cdots P_r$ be a nilpotent group. For $x = x_1x_2 \cdots x_r \in G$, where $x_i \in P_i$, define $\tau_x = \{j \in [r] : x_j \neq e\}$. Note that if $\langle x \rangle \in \mathcal{M}(G)$ then $\tau_x = [r]$.

LEMMA 5.1. *Let $H = \langle x \rangle$ and $x = \prod_{i \in \tau_x} x_i$. Then $\langle x_i \rangle \subseteq \langle x \rangle$ for all $i \in \tau_x$.*

PROOF. Consider $i_0 \in \tau_x$ and $l = \prod_{i \in [r] \setminus \{i_0\}} o(x_i)$. Then $x^l = x_{i_0}^l$ (cf. Theorem 2.1). Since $\gcd(l, o(x_{i_0})) = 1$, we have $\langle x^l \rangle = \langle x_{i_0}^l \rangle = \langle x_{i_0} \rangle$ and so $x_{i_0} \in \langle x^l \rangle$. Hence, $\langle x_{i_0} \rangle \subseteq \langle x \rangle$. □

LEMMA 5.2. *Let G be a nilpotent group. Then $\langle x \rangle = \big\langle \prod_{i \in \tau_x} x_i \big\rangle = \prod_{i \in \tau_x} \langle x_i \rangle$, where $\langle x_i \rangle \langle x_j \rangle = \big\{ ab : a \in \langle x_i \rangle \text{ and } b \in \langle x_j \rangle \big\}$.*

PROOF. Clearly, $\big\langle \prod_{i \in \tau_x} x_i \big\rangle \subseteq \prod_{i \in \tau_x} \langle x_i \rangle$. If $a \in \prod_{i \in \tau_x} \langle x_i \rangle$. Then $a = \prod_{i \in \tau_x} a_i$ such that $a_i \in \langle x_i \rangle$. Thus, $a_i = x_i^{k_i}$ for some $k_i \in \mathbb{N}$. By Lemma 5.1, $a_i = x^{\lambda_i k_i}$ for some $\lambda_i \in \mathbb{N}$ and so $a = x^{\sum_{i \in \tau_x} \lambda_i k_i}$. Consequently, we get $a \in \langle x \rangle$. Thus, the result holds. □

LEMMA 5.3. *Let G be a nilpotent group such that $x = \prod_{i=1}^r x_i$ and $y = \prod_{i=1}^r y_i$. Then $x \sim y$ in $\mathcal{P}_E(G)$ if and only if $x_i \sim y_i$ in $\mathcal{P}_E(P_i)$ whenever $x_i \neq y_i$.*

PROOF. First suppose that $x \sim y$ in $\mathcal{P}_E(G)$. Then there exists $z \in G$ such that $x, y \in \langle z \rangle$. We may now suppose that $x_i \neq y_i$ for some i. By

Lemma 5.1, $x_i \in \langle x\rangle \subseteq \langle z\rangle$. Similarly, $y_i \in \langle z\rangle$. Thus, $\langle x_i, y_i\rangle \subseteq \langle z\rangle$ follows that $\langle x_i, y_i\rangle$ is a cyclic subgroup of P_i. Thus, $x_i \sim y_i$ in $\mathcal{P}_E(P_i)$. Conversely, suppose that $x_i \sim y_i$ in $\mathcal{P}_E(P_i)$ for $x_i \neq y_i$. Consider $K = \{j \in [r] : x_j \sim y_j \text{ in } \mathcal{P}_E(P_j)\}$. Consequently, for $i \in K$, we have $\langle x_i, y_i\rangle \subseteq \langle z_i\rangle$ for some $z_i \in P_i$. Choose $z = \prod_{i\in K} z_i \cdot \prod_{i\in[r]\setminus K} x_i$. Thus by Lemma 5.2, $\langle z\rangle = \prod_{i\in K}\langle z_i\rangle \cdot \prod_{i\in[r]\setminus K}\langle x_i\rangle$. Consequently, $x = \prod_{i\in[r]} x_i \in \langle z\rangle$ and $y = \prod_{i\in[r]} y_i \in \langle z\rangle$. Hence, $x \sim y$ in $\mathcal{P}_E(G)$. □

THEOREM 5.4. *Let G be a nilpotent group. Then*

$$\mathcal{P}_E(G) \cong \mathcal{P}_E(P_1) \boxtimes \mathcal{P}_E(P_2) \boxtimes \cdots \boxtimes \mathcal{P}_E(P_r)$$

where P_i is the Sylow p_i-subgroup of G.

PROOF. Let $x = x_1x_2\cdots x_r \in G$. Then define $\psi\colon V(\mathcal{P}_E(G)) \to V(\mathcal{P}_E(P_1) \boxtimes \mathcal{P}_E(P_2) \boxtimes \cdots \boxtimes \mathcal{P}_E(P_r))$ such that $x \longmapsto (x_1, x_2, \ldots, x_r)$, where $x_i \in P_i$. In view of Lemma 5.3, note that ψ is a graph isomorphism. □

LEMMA 5.5. *Let G be a non-cyclic group and $T = \bigcap_{M\in\mathcal{M}(G)} M$. Then T is contained in every cut-set of $\mathcal{P}_E(G)$.*

PROOF. Let $x \in T$ and $y(\neq x) \in G$. Since $y \in M$ for some $M \in \mathcal{M}(G)$ and $x \in T$, we have $x \in M$. Consequently, $x \sim y$. It follows that x is adjacent to every vertex of $\mathcal{P}_E(G)$. Thus, x must belongs to every cut-set of $\mathcal{P}_E(G)$ and so is T. □

THEOREM 5.6. *Let $G = P_1P_2\cdots P_r$ be a non-cyclic nilpotent group of order $n = p_1^{\lambda_1}p_2^{\lambda_2}...p_r^{\lambda_r}$ with $r \geq 2$. Suppose that each Sylow subgroup P_i of G is cyclic except P_k for some $k \in [r]$.*

(i) *If P_k is not a generalized quaternion group, then the set $Q = P_1P_2\cdots P_{k-1}P_{k+1}\cdots P_r$ is the only minimum cut-set of $\mathcal{P}_E(G)$ and hence $\kappa(\mathcal{P}_E(G)) = \frac{n}{p_k^{\lambda_k}}$.*

(ii) *If P_k is a generalized quaternion group, then the set*

$$Q' = Z(Q_{2^\alpha})P_2\cdots P_r$$

is the only minimum cut-set of $\mathcal{P}_E(G)$ and hence $\kappa(\mathcal{P}_E(G)) = \frac{n}{2^{\lambda_1-1}}$.

PROOF. (i) First suppose that P_k is not a generalized quaternion group. By Corollary 2.5, Q is contained in every maximal cyclic subgroup of G. By Lemma 5.5, Q is contained in every cut-set of $\mathcal{P}_E(G)$. Now, to prove our result we first prove the following claim.

CLAIM. *Let T_i be a cut-set of $\mathcal{P}_E(P_i)$. Then $T = P_1\cdots P_{i-1}T_iP_{i+1}\cdots P_r$ is a cut-set of $\mathcal{P}_E(G)$.*

PROOF OF THE CLAIM. Let T_i be a cut-set of $\mathcal{P}_E(P_i)$ and let $a, b \in P_i$ such that there exist no path between a and b in $\mathcal{P}_E(P_i \setminus T_i)$. It follows that,

for the isomorphism ψ defined in the proof of Theorem 5.4, there is no path between $\psi(a)$ and $\psi(b)$ in the subgraph induced by $V(\boxtimes_{i=1}^{r}\mathcal{P}_E(P_i)) \setminus \psi(T)$. Consequently, there is no path between a and b in $\mathcal{P}_E(G \setminus T)$. Hence, T is a cut-set of $\mathcal{P}_E(G)$. $\square$

Now by [4, Theorem 1], $\kappa(\mathcal{P}_E(P_k)) = 1$ and $\{e\}$ is the only cut-set of $\mathcal{P}_E(P_k)$. Thus, above claim follows that the set Q is the only minimum cut-set of $\mathcal{P}_E(G)$. Hence, $\kappa(\mathcal{P}_E(G)) = \frac{n}{p_k^{\lambda_k}}$.

(ii) Now suppose that $P_k = Q_{2^\alpha}$ is a generalized quaternion group. Note that the center $Z(Q_{2^\alpha})$ of Q_{2^α} is contained in every maximal cyclic subgroup of Q_{2^α}. Consequently, Q' is contained in every maximal cyclic subgroup of G [cf. Lemma 2.4]. Thus, by Lemma 5.5, Q' is contained in every cut-set of $\mathcal{P}_E(G)$. By claim, Q' is a cut-set of $\mathcal{P}_E(G)$. Hence, Q' is the only minimum cut-set of $\mathcal{P}_E(G)$ and so $\kappa(\mathcal{P}_E(G)) = \frac{n}{2^{\lambda_1-1}}$. $\square$

6. The Wiener index of $\mathcal{P}_E(G)$

In this section, we study the Wiener index of $\mathcal{P}_E(G)$, where G is a finite nilpotent group. We obtain a lower bound and an upper bound of $W(\mathcal{P}_E(G))$. We also characterize the finite nilpotent groups attaining these bounds. Define

- $S_{0,i} = \{(x,x) : x \in P_i\}$.
- $S_{1,i} = \{(x,y) : x \sim y \text{ in } \mathcal{P}_E(P_i)\}$.
- $S_{2,i} = \{(x,y) : x \nsim y \text{ in } \mathcal{P}_E(P_i)\}$ such that $|S_{2,i}| = m_i$.

and

- $S_0 = \{(x,x) : x \in G\}$.
- $S_1 = \{(x,y) : x \sim y \text{ in } \mathcal{P}_E(G)\}$.
- $S_2 = \{(x,y) : x \nsim y \text{ in } \mathcal{P}_E(G)\}$.

Then by the definition of Wiener index, we have

$$W(\mathcal{P}_E(G)) = \frac{|S_1| + 2|S_2|}{2}.$$

Now we obtain the Wiener index of $\mathcal{P}_E(G)$, where G is a nilpotent group.

THEOREM 6.1. *Let G be a nilpotent group of order $n = p_1^{\lambda_1}p_2^{\lambda_2}\cdots p_r^{\lambda_r}$. Then*

$$W(\mathcal{P}_E(G)) = \frac{2n^2 - n - \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i)}{2}.$$

PROOF. Let $G = P_1P_2\cdots P_r$ be a nilpotent group such that $|P_i| = p_i^{\lambda_i}$ and let $x = x_1x_2\cdots x_r$, $y = y_1y_2\cdots y_r \in G$. By Theorem 5.4, note that

$$S_1 = \big\{(x,y) : \text{either } x_i = y_i \text{ or } x_i \sim y_i \text{ in } \mathcal{P}_E(P_i)\big\} \setminus S_0\,.$$

Let $m_i = |S_{2,i}|$. Then

$$|S_1| = \prod_{i=1}^{r}(|S_{1,i}| + |S_{0,i}|) - n$$

$$= \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i - p_i^{\lambda_i} + p_i^{\lambda_i}) - n = \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i) - n$$

and

$$|S_2| = n^2 - |S_0| - |S_1| = n^2 - \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i).$$

Hence,

$$W(\mathcal{P}_E(G)) = \frac{2n^2 - n - \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i)}{2}. \quad \square$$

COROLLARY 6.2. *Let G, G' be nilpotent groups such that $|G| = |G'| = p_1^{\lambda_1}p_2^{\lambda_2}\cdots p_r^{\lambda_r}$. If $m_i \leq m_i'$ for all $i \in [r]$, then $W(\mathcal{P}_E(G)) \leq W(\mathcal{P}_E(G')$.*

LEMMA 6.3. *Let G be a p-group. Then $|S_2| \leq (o(G) - p)(o(G) - 1)$.*

PROOF. Let $x \neq e \in G$. Since G is a p-group, we have $o(x) \geq p$. Thus, x is adjacent to at least $p-1$ vertices in $\mathcal{P}_E(G)$. It follows that x is not adjacent to at most $o(G) - p + 1$ elements in $\mathcal{P}_E(G)$. Since x is at distance 0 from itself, it implies that the number of elements at distance two from x is at most $o(G) - p$. Note that the identity element is adjacent to all other vertices in $\mathcal{P}_E(G)$. Thus, for $S_2 = \{(x, y) : x \nsim y \text{ in } \mathcal{P}_E(G)\}$, we have $|S_2| \leq (o(G) - p)(o(G) - 1)$. $\square$

In view of Theorem 6.1 and Lemma 6.3, we have the following corollary.

COROLLARY 6.4. *Let G be a p-group. Then*

$$W(\mathcal{P}_E(G)) \leq \frac{(o(G) - 1)(2o(G) - p)}{2}.$$

For the nilpotent group G, now we give a sharp lower bound and an upper bound of $W(\mathcal{P}_E(G))$ (independent from m_i) in the following theorem.

THEOREM 6.5. *Let G be a nilpotent group of order $n = p_1^{\lambda_1}p_2^{\lambda_2}\cdots p_r^{\lambda_r}$. Then*

(i)

$$\frac{n(n-1)}{2} \leq W(\mathcal{P}_E(G)) \leq \frac{2n^2 - n - \prod_{i=1}^{r}(p_i^{\lambda_i+1} + p_i^{\lambda_i} - p_i)}{2}.$$

(ii) $W(\mathcal{P}_E(G))$ *attains its lower bound if and only if* G *is cyclic.*

(iii) $W(\mathcal{P}_E(G))$ *attains its upper bound if and only if* $|M| = p_1p_2\cdots p_r$ *for every* $M \in \mathcal{M}(G)$.

PROOF. (i)-(ii). From Lemma 6.3, it follows that $m_i \leq (p_i^{\lambda_i} - p_i)(p_i^{\lambda_i} - 1)$ for all $i \in [r]$. Consequently, by Theorem 6.1 and Corollary 6.2, we get $W(\mathcal{P}_E(G)) \leq \frac{2n^2 - n - \prod_{i=1}^{r}(p_i^{\lambda_i+1} + p_i^{\lambda_i} - p_i)}{2}$. Notice that $W(\mathcal{P}_E(G))$ is smallest if and only if $\mathcal{P}_E(G)$ is complete if and only if G is cyclic (cf. Theorem 2.2). Since the Wiener index of the complete graph on n vertices is $\frac{n(n-1)}{2}$, we obtain $\frac{n(n-1)}{2} \leq W(\mathcal{P}_E(G))$.

(iii) By Theorem 6.1, observe that $W(\mathcal{P}_E(G))$ is maximum if and only if m_i is maximum for all $i \in [r]$. First, we prove that m_i is maximum if and only if $|M'| = p_i$ for every $M' \in \mathcal{M}(P_i)$.

For simplicity, we write $p_i = p$ and $\lambda_i = \lambda$ so that $m_i \leq (p^\lambda - p)(p^\lambda - 1)$. Now let $|M'| = p$ for every $M' \in \mathcal{M}(P_i)$. Then for any non-identity element $x \in P_i$, $o(x) = p$. Since x is a generator of a maximal cyclic subgroup H of P_i note that $x \in H$ only. It follows that x is adjacent to $p-1$ vertices of $\mathcal{P}_E(P_i)$. Consequently, x is at distance 2 from $p^\lambda - p$ vertices of $\mathcal{P}_E(P_i)$. Since x is an arbitrary non-identity element of P_i, we have $m_i = (p^\lambda - 1)(p^\lambda - p)$. Thus m_i is maximum. Conversely, suppose that the m_i is maximum. On contrary, suppose $|M'| = p^\alpha$ for some $\alpha \geq 2$ and $M' \in \mathcal{M}(P_i)$. Further, assume that $x \in P_i$. Clearly, $o(x) \geq p$. If $x \in M'$ then x is adjacent to at least $p^\alpha - 1$ vertices of $\mathcal{P}_E(P_i)$ and so at most $p^\lambda - p^\alpha$ vertices are at distance 2 from x in $\mathcal{P}_E(P_i)$. Similarly, if $x \in P_i \setminus M'$ then there are at most $p^\lambda - p$ elements at distance 2 from x in $\mathcal{P}_E(P_i)$. Consequently, we get

$$m_i \leq (p^\alpha - 1)(p^\lambda - p^\alpha) + (p^\lambda - p)(p^\lambda - p^\alpha) < (p^\lambda - 1)(p^\lambda - p);$$

a contradiction. Hence, m_i is maximum if and only if $|M'| = p_i$ for all $M' \in \mathcal{M}(P_i)$.

Thus by Lemma 2.4, we get $W(\mathcal{P}_E(G))$ is maximum if and only if $|M| = p_1p_2\cdots p_r$ for every $M \in \mathcal{M}(G)$. □

Note that the given upper bound is tight and it is attained by the group $G = \mathbb{Z}_{p_1}^{\lambda_1} \times \mathbb{Z}_{p_2}^{\lambda_2} \times \cdots \times \mathbb{Z}_{p_r}^{\lambda_r}$. Moreover, in this case, the graph $\mathcal{P}_E(G)$ has minimum number of edges.

References

[1] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish and F. Shaveisi, On the structure of the power graph and the enhanced power graph of a group, *Electron. J. Combin.*, **24** (2017), Paper No. 3.16, 18 pp.

[2] M. Aschbacher, *Finite Group Theory*, Cambridge University Press (Cambridge, 2000).

[3] S. Bera and A. K. Bhuniya, On enhanced power graphs of finite groups, *J. Algebra Appl.*, **17** (2018), 1850146.

[4] S. Bera, H. K. Dey and S. K. Mukherjee, On the connectivity of enhanced power graphs of finite groups, *Graphs Combin.*, **37** (2021), 591–603.
[5] I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum*, **78** (2009), 410–426.
[6] S. Chattopadhyay, K. L. Patra and B. K. Sahoo, Minimal cut-sets in the power graphs of certain finite non-cyclic groups, *Comm. Algebra*, **49** (2021), 1195–1211.
[7] S. Dalal and J. Kumar, On enhanced power graphs of certain groups, *Discrete Math. Algorithms Appl.*, **13** (2021), 2050099.
[8] D. S. Dummit and R. M. Foote, *Abstract Algebra*, Prentice Hall, Inc. (Englewood Cliffs, NJ, 1991).
[9] L. A. Dupont, D. G. Mendoza and M. Rodríguez, The enhanced quotient graph of the quotient of a finite group, arXiv:1707.01127 (2017).
[10] L. A. Dupont, D. G. Mendoza and M. Rodríguez, The rainbow connection number of enhanced power graph, arXiv:1708.07598 (2017).
[11] A. Hamzeh and A. R. Ashrafi, Automorphism groups of supergraphs of the power graph of a finite group, *European J. Combin.*, **60** (2017), 82–88.
[12] U. Hayat, M. Umer, I. Gutman, B. Davvaz and A. Nolla de Celis, A novel method to construct NSSD molecular graphs, *Open Math.*, **17** (2019), 1526–1537.
[13] A. V. Kelarev, On undirected Cayley graphs, *Australas. J. Combin.*, **25** (2002), 73–78.
[14] A. V. Kelarev, *Ring Constructions and Applications*, World Scientific Publishing Co., Inc. (River Edge, NJ, 2002).
[15] A. Kelarev, *Graph Algebras and Automata*, Marcel Dekker, Inc. (New York, 2003).
[16] A. V. Kelarev, Labelled Cayley graphs and minimal automata, *Australas. J. Combin.*, **30** (2004), 95–101.
[17] A. Kelarev, J. Ryan, and J. Yearwood, Cayley graphs as classifiers for data mining: the influence of asymmetries, *Discrete Math.*, **309** (2009), 5360–5369.
[18] X. Ma, A. Doostabadi and K. Wang, Notes on the diameter of the complement of the power graph of a finite group, arXiv:2112.13499v2 (2021).
[19] X. Ma, R. Fu, X. Lu, M. Guo and Z. Zhao, Perfect codes in power graphs of finite groups, *Open Math.*, **15** (2017), 1440–1449.
[20] X. Ma, A. Kelarev, Y. Lin and K. Wang, A survey on enhanced power graphs of finite groups, *Electron. J. Graph Theory Appl.*, **10** (2022), 89–111.
[21] X. Ma and Y. She, The metric dimension of the enhanced power graph of a finite group, *J. Algebra Appl.*, **19** (2020), 2050020, 14 pp.
[22] A. R. Moghaddamfar, S. Rahbariyan and W. J. Shi, Certain properties of the power graph associated with a finite group, *J. Algebra Appl.*, **13** (2014), 1450040.
[23] Parveen and J. Kumar, The complement of enhanced power graph of a finite group, arXiv:2207.04641 (2022).
[24] R. Prasad Panda, S. Dalal and J. Kumar, On the enhanced power graph of a finite group, *Comm. Algebra*, **49** (2021), 1697–1716.
[25] Y. Segev, On finite homomorphic images of the multiplicative group of a division algebra, *Ann. of Math. (2)*, **149** (1999), 219–251.
[26] Y. Segev, The commuting graph of minimal nonsolvable groups, *Geom. Dedicata*, **88** (2001), 55–66.
[27] Y. Segev and G. M. Seitz, Anisotropic groups of type A_n and the commuting graph of finite simple groups, *Pacific J. Math.*, **202** (2002), 125–225.
[28] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Inc. (Upper Saddle River, NJ, 1996).
[29] S. Zahirović, I. Bošnjak and R. Madarász, A study of enhanced power graphs of finite groups, *J. Algebra Appl.*, **19** (2020), 2050062.