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MAJORIZED MULTIVALUED MAPS, EQUILIBRIUM AND COINCIDENCE POINTS IN THE TOPOLOGICAL VECTOR SPACE SETTING

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Abstract. In this paper we begin by presenting an equilibrium result for abstract economies for majorized type maps defined on Hausdorff topological vector spaces. The ideas here motivate new results for maximal and coincidence points for collectively multivalued maps.

1. Introduction

In this paper we present existence theory for maximal type and coincidence elements for multivalued maps. Also we present an equilibrium theory for maps (constraints, preferences) majorized by upper semicontinuous maps with convex compact values. These general maps include as a special case the majorized maps in the literature [10,11,14–18]. In addition our theory is considered in the topological vector space setting which has not been considered in its full generality in the literature.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f \colon X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p \colon \Gamma \to X$ is called a Vietoris map (written $p \colon \Gamma \Rightarrow X$) if the following two conditions are satisfied:

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(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi\colon X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map; and

(ii) $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [7]. An upper semicontinuous map $\phi\colon X \to Y$ with compact values is said to be admissible (and we write $\phi \in \mathrm{Ad}(X,Y)$) provided there exists a selected pair (p,q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi\colon X \to CK(Y)$ is said to Kakutani (and we write $\phi \in \mathrm{Kak}(X,Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

We also discuss the following classes of maps in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in DKT(Z,W)$ [4] if W is convex and there exists a map $S\colon Z \to W$ with $\mathrm{co}(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$. We say $G \in HLPY(Z,W)$ [8,10] if W is convex and there exists a map $S\colon Z \to W$ with $\mathrm{co}(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\mathrm{int}\, S^{-1}(w) : w \in W\}$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F\colon X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \big\{\, Z : \mathrm{Fix}\, F \neq \emptyset \text{ for all } F \in \mathcal{X}(Z,Z)\big\}$$

where $\mathrm{Fix}\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;

(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii) $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1,2,\dots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

We say $F \in PK(X,Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

For a subset K of a topological space X, we denote by $\mathrm{Cov}_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $\mathrm{Cov}(K) = \mathrm{Cov}_X(K)$). Given two maps $F,G\colon X \to 2^Y$ and $\alpha \in \mathrm{Cov}(Y)$, F and G are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0\colon K \to Y$ extends to a continuous function $f\colon X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in \mathrm{Cov}(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f\colon X \to Y$ such that $f|_K$ is α-close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha\colon K \to V$ such that

(i) π_α and $i : K \to V$ are α-close;

(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES$(compact).

THEOREM 1.1 [2,12]. *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

REMARK 1.2. Other variations of Theorem 1.1 can be found in [1,13].

We also present another version (see [9]). A subset K of a Hausdorff topological vector space E is said to be convexly totally bounded (c.t.b. for brevity) if for every neighborhood V of 0 there exists a finite set $\{x_i : i \in I\} \subseteq E$ (I finite) and a finite family of convex sets $\{C_i : i \in I\}$ with $C_i \subseteq V$ for each $i \in I$ and $K \subseteq \bigcup_{i\in I}(x_i + C_i)$.

THEOREM 1.3. *Let X be a convex subset of a Hausdorff topological vector space E and $\Phi\colon X \to CK(X)$ a upper semicontinuous compact map. If $\overline{\Phi(X)}$ is a c.t.b. subset of X then Φ has a fixed point.*

We now list two well known results from the literature [16,17].

THEOREM 1.4. *Let X and Y be two topological spaces and A an open subset of X. Suppose $F_1\colon X \to 2^Y$, $F_2\colon A \to 2^Y$ are upper semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the map $F\colon X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & x \notin A, \\ F_2(x), & x \in A \end{cases}$$

is upper semicontinuous.

THEOREM 1.5. *Let X and Y be topological spaces. If $F, G\colon X \to 2^Y$ have compact values and are upper semicontinuous then $F \cap G$ is also upper semicontinuous.*

It is of interest to note that many books on multivalued maps focus on existence theory in Banach spaces (or more generally Fréchet spaces) for Kak maps or for monotone maps (e.e. J.P. Aubin and H. Frankowska, Set–Valued Analysis, Chapter 3) whereas in this paper we focus on Kak, Ad and DKT maximal type maps in Hausdorff topological vector spaces and we also note monotonicity plays no role in our theory.

2. Abstract economies and maximal elements

Let I be the set of agents and we describe the abstract economy as $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ where A_i, $B_i\colon X \equiv \prod_{i\in I} X_i \to E_i$ are constraint correspondences, $P_i\colon X \to E_i$ is a preference correspondence and X_i is a choice (or strategy) set which is a subset of a Hausdorff topological vector space E_i. We are interested in finding an equilibrium point for Γ i.e. a point $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$; here x_i denotes the projection of x on X_i.

We begin by discussing maximal type maps motivated from the literature (see [10,11,14–18]). Let Z and W be sets in a Hausdorff topological vector space with Z paracompact and W convex and compact. Suppose $H\colon Z \to W$ and for each $x \in Z$ assume there exists a map $A_x\colon Z \to W$ and an open set O_x containing x with $H(z) \subseteq A_x(z)$ for every $z \in O_x$, $A_x\colon O_x \to W$ is upper semicontinuous with convex compact values. We claim there exists a (compact) map $\Psi\colon Z \to W$ with $H(z) \subseteq \Psi(z)$ for $z \in Z$ and $\Psi\colon Z \to W$ is upper semicontinuous with convex compact values. To see this note $\{O_x\}_{x\in Z}$ is an open covering of Z and since Z is paracompact there exists [5,6] a locally finite open covering $\{V_x\}_{x\in Z}$ of Z with $x \in V_x$ and $V_x \subseteq O_x$ for $x \in Z$, and for each $x \in Z$ let

$$Q_x(z) = \begin{cases} A_x(z), & z \in V_x \\ W, & z \in Z\backslash V_x. \end{cases}$$

Now Theorem 1.4 guarantees that $Q_x\colon Z \to W$ is upper semicontinuous with convex compact values. Next note $H(z) \subseteq Q_x(z)$ for every $z \in Z$ since if $z \in V_x$ then since $V_x \subseteq U_x$ and $H(w) \subseteq A_x(w)$ for $w \in O_x$ we have $H(z) \subseteq Q_x(z)$ whereas if $z \in Z\backslash V_x$ then it is immediate since $Q_x(z) = W$. Now define $\Psi\colon Z \to W$ by

$$\Psi(z) = \bigcap_{x\in Z} Q_x(z) \quad \text{for } z \in Z.$$

Note $\Psi\colon Z \to W$ has convex compact values with $H(w) \subseteq \Psi(w)$ for $w \in Z$ since $H(z) \subseteq Q_x(z)$ for every $z \in Z$ (for each $x \in X$). It remains to show $\Psi\colon Z \to W$ is upper semicontinuous. Let $u \in Z$. There exists an open neighbourhood N_u of u such that $\{x \in Z : N_u \cap V_x \neq \emptyset\} = \{x_1, \dots, x_{n_u}\}$ (a finite set). Note if $x \notin \{x_1, \dots, x_{n_u}\}$ then $\emptyset = V_x \cap N_u$ so $Q_x(z) = W$ for $z \in N_u$ and so we have

$$\Psi(z) = \bigcap_{x \in Z} Q_x(z) = \bigcap_{j=1}^{n_u} Q_{x_j}(z) \quad \text{for } z \in N_u.$$

Now for $j \in \{1, \dots, n_u\}$ note $Q_{x_j} \colon Z \to W$ is upper semicontinuous (so $Q^{\star}_{x_j} \colon N_u \to W$, the restriction of Q_{x_j} to N_u, is upper semicontinuous) so Theorem 1.5 guarantees that $\Psi : N_u \to W$ is upper semicontinuous (at u). Since N_u is open we have that $\Psi \colon Z \to W$ is upper semicontinuous (at u).

Majorized maps were considered in [18] and put in a more abstract setting in for example in [17]. In the above description we put majorized type maps in the natural setting of our paper and indeed as one can see below in Theorem 2.3, Theorem 2.5, Theorem 2.6 and Theorem 2.10 we in fact discuss more general maps which incude these maximal element type maps.

THEOREM 2.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy with $\{X_i\}_{i \in I}$ a family of nonempty convex compact sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ assume the following conditions hold*:

$$\text{(2.1)} \qquad \text{cl } B_i (\equiv \overline{B_i}) \colon X \equiv \prod_{i \in I} X_i \to CK(X_i) \ \ \textit{is upper semicontinuous},$$

$$\text{(2.2)} \qquad U_i = \big\{ x \in X : A_i(x) \cap P_i(x) \neq \emptyset \big\} \ \ \textit{is paracompact and open in } X,$$

and

$$\text{(2.3)} \qquad X \textit{ is a Schauder admissible subset of } E \equiv \prod_{i \in I} E_i.$$

For each $i \in I$ suppose $A_i \cap P_i \colon U_i \to X_i$ and for each $x \in U_i$ assume there exists a map $A_{i,x} \colon U_i \to X_i$ and an open set $O_{i,x}$ (in U_i) containing x with $(A_i \cap P_i)(z) \subseteq A_{i,x}(z)$ for every $z \in O_{i,x}$ and $A_{i,x} : O_{i,x} \to X_i$ is upper semicontinuous with convex compact values. Also suppose either

$$\text{(2.4a)} \qquad z_i \notin A_{i,x}(z) \ \ \textit{for all } \ z \in O_{i,x}, \forall x \in U_i, \ \forall i \in I$$

or

$$\text{(2.4b)} \qquad \textit{there exists } j_0 \in I \textit{ with } z_{j_0} \notin A_{j_0,x}(z) \textit{ for all } z \in O_{j_0,x}, \ \forall x \in U_{j_0}$$

occurs (here z_i is the projection of z on X_i). Then there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ for each $i \in I$ and if (2.4a) holds we have $A_i(x) \cap P_i(x)) = \emptyset$ for each $i \in I$ whereas if (2.4b) holds we have $A_{j_0}(x) \cap P_{j_0}(x) = \emptyset$.

Proof. If $U_i = \emptyset$ for all $i \in I$ then from Theorem 1.1 (applied to $F = \prod_{i\in I} \overline{B_i}$) there exists a $y \in X$ with $y \in F(y)$ i.e. $y_i \in \overline{B_i}(y)$ for each $i \in I$. Now since $U_i = \emptyset$ for all $i \in I$ then by definition we have $A_i(x) \cap P_i(x)) = \emptyset$ for all $i \in I$.

As a result we assume for the remainder of the proof that there exists a $i_0 \in I$ with $U_{i_0} \neq \emptyset$. We will assume that $U_i \neq \emptyset$ for each $i \in I$ (we will also remark on the situation that $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I \backslash J$ at each step below). From the discussion before Theorem 2.1 (with $Z = U_i$, $W = X_i$, $H = A_i \cap P_i$ and $A_x = A_{i,x}$) there exists a map $\Psi_i \colon U_i \to X_i$ with $(A_i \cap P_i)(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i \colon U_i \to X_i$ is upper semicontinuous with convex compact values: here $\{O_{i,x}\}_{x\in U_i}$ is an open covering of U_i so there exists a locally finite open covering $\{V_{i,x}\}_{x\in U_i}$ of U_i (recall U_i is paracompact) with $x \in V_{i,x}$ and $V_{i,x} \subseteq O_{i,x}$ for $x \in U_i$, and for each $x \in U_i$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x} \\ X_i, & z \in U_i \backslash V_{i,x} \end{cases}$$

and $\Psi_i \colon U_i \to X_i$ is

$$\Psi_i(z) = \bigcap_{x\in U_i} Q_{i,x}(z) \quad \text{for } z \in U_i.$$

Note $\overline{B_i}|_{U_i} \colon U_i \to CK(X_i)$ is upper semicontinuous so from Theorem 1.5 we have that $\Psi_i \cap \overline{B_i} \colon U_i \to X$ is upper semicontinuous. Let $F_i \colon X \to X_i$ be given by

$$F_i(x) = \begin{cases} \overline{B_i}(x), & x \in U_i \\ (\Psi_i \cap \overline{B_i})(x), & x \in X \backslash U_i \end{cases}$$

so Theorem 1.4 guarantees that $F_i \colon X \to X_i$ is upper semicontinuous with nonempty convex and compact values (note for $x \in U_i$ that $(\Psi_i \cap \overline{B_i})(x) \subseteq \overline{B_i}(x)$). Note we also remark that if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I \backslash J$ then choose F_i as above if $i \in J$ whereas choose $F_i = \overline{B_i}$ if $i \in I \backslash J$.

Let $F \colon X \to CK(X)$ be given by

$$F(x) = \prod_{i\in I} F_i(x) \quad \text{for } x \in X.$$

Note $F \in \mathrm{Kak}(X, X)$ (notice $F_i(x) \subseteq \overline{B_i}(x)$ for $x \in X$ and $i \in I$) so Theorem 1.1 guarantees a $y \in X$ with $y \in F(y)$ i.e. $y_i \in F_i(y)$ for each $i \in I$. Thus $y_i \in \overline{B_i}(y)$ for each $i \in I$ since if $y \notin U_i$ we have $F_i(y) = \overline{B_i}(y)$ whereas if $y \in U_i$ we have $F_i(y) = (\Psi_i \cap \overline{B_i})(y) \subseteq \overline{B_i}(y)$ (we have a similar result if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I \backslash J$).

First suppose (2.4a) occurs. Fix $i \in I$. We claim $y \notin U_i$. If not then $y \in U_i$ so $y_i \in (\Psi_i \cap \overline{B_i})(y) \subseteq \Psi_i(y)$ i.e. $y_i \in \Psi_i(y)$. Now $y \in V_{i,x}$ for some $x \in U_i$ (since $\{V_{i,x}\}_{x \in U_i}$ is a locally finite open covering of U_i with $x \in V_{i,x}$ and $V_{i,x} \subseteq O_{i,x}$ for $x \in U_i$) and note $Q_{i,x}(y) = A_{i,x}(y)$ so $\Psi_i(y) \subseteq A_{i,x}(y)$. Now since $z_i \notin A_{i,x}(z)$ for all $z \in O_{i,x}$ and $V_{i,x} \subseteq O_{i,x}$ then $y_i \notin \Psi_i(y)$, a contradiction. Thus $y \notin U_i$. We can do this argument for all $i \in I$ so the result in the statement of Theorem 2.1 holds (note if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I \backslash J$ then note if $i \in I \backslash J$ we have $U_i = \emptyset$ so $y \notin U_i$ whereas if $i \in J$ then the argument above gives $y \notin U_i$, so in both cases we have $y \notin U_i$).

Next suppose (2.4b) occurs. Suppose $y \in U_{j_0}$. Then $y_{j_0} \in (\Psi_{j_0} \cap \overline{B_{j_0}})(y) \subseteq \Psi_{j_0}(y)$ i.e. $y_{j_0} \in \Psi_{j_0}(y)$. Also since $y \in V_{j_0,x^\star}$ for some $x^\star \in U_{j_0}$ then $Q_{j_0,x^\star}(y) = A_{j_0,x^\star}(y)$ so $\Psi_{j_0}(y) \subseteq A_{j_0,x^\star}(y)$. Now since $z_{j_0} \notin A_{j_0,x^\star}(z)$ for all $z \in O_{j_0,x^\star}$ and $V_{j_0,x^\star} \subseteq O_{j_0,x^\star}$ then $y_{j_0} \notin \Psi_{j_0}(y)$, a contradiction. Thus $y \notin U_{j_0}$ so the result in the statement of Theorem 2.1 holds (note if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I \backslash J$ then note if $j_0 \in I \backslash J$ we have $U_{j_0} = \emptyset$ so $y \notin U_{j_0}$ whereas if $j_0 \in J$ then the argument above gives $y \notin U_{j_0}$, so in both cases we have $y \notin U_{j_0}$). $\square$

REMARK 2.2. In Theorem 2.1 we could replace (2.3) with other conditions in [13] or alternatively with either (i) X is a c.t.b. subset of E, or (ii) $\overline{F(X)}$ is a c.t.b. subset of E (here F is as described in the proof of Theorem 2.1). Here we use Theorem 1.3 instead of Theorem 1.1 in the proof of Theorem 2.1.

Looking at the proof of Theorem 2.1 we see immediately that there is a more general result (the proof can be extracted from the proof of Theorem 2.1).

THEOREM 2.3. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy with $\{X_i\}_{i \in I}$ a family of nonempty convex compact sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ assume* (2.1), (2.2) (*the paracompactness of U_i is not needed here*) *and* (2.3) *hold and assume there exists a map $\Psi_i \colon U_i \to X_i$ with $(A_i \cap P_i)(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i \colon U_i \to X_i$ is upper semicontinuous with convex compact values. Also assume either*

$$\text{(2.5a)} \qquad y_i \notin \Psi_i(y) \ \ \textit{for all } y \in U_i, \ \forall i \in I$$

or

$$\text{(2.5b)} \qquad \textit{there exists a } j_0 \in I \textit{ with } y_{j_0} \notin \Psi_{j_0}(y) \textit{ for } y \in U_{j_0}$$

occurs. Then there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ for each $i \in I$ and if (2.5a) *holds we have $A_i(x) \cap P_i(x)) = \emptyset$ for each $i \in I$ whereas if* (2.5b) *holds we have $A_{j_0}(x) \cap P_{j_0}(x) = \emptyset$.*

Note special cases of Theorem 2.1 and Theorem 2.3 immediately guarantee maximal element type results.

THEOREM 2.4. *Let $\{X_i\}_{i\in I}$ be a family of nonempty convex compact sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ suppose $F_i \colon X \equiv \prod_{i\in I} X_i \to X_i$ and assume*

$$(2.6) \qquad U_i = \big\{ x \in X : F_i(x) \neq \emptyset \big\} \text{ is paracompact and open in } X.$$

For each $i \in I$ and for each $x \in U_i$ assume there exists a map $A_{i,x} \colon U_i \to X_i$ and an open set $O_{i,x}$ (in U_i) containing x with $F_i(z) \subseteq A_{i,x}(z)$ for every $z \in O_{i,x}$ and $A_{i,x} \colon O_{i,x} \to X_i$ is upper semicontinuous with convex compact values. Also suppose either

$$(2.7a) \qquad z_i \notin A_{i,x}(z) \text{ for all } z \in O_{i,x},\ \forall x \in U_i,\ \forall i \in I$$

or

$$(2.7b) \quad \text{there exists } j_0 \in I \text{ with } z_{j_0} \notin A_{j_0,x}(z) \text{ for all } z \in O_{j_0,x},\ \forall x \in U_{j_0}$$

occurs. Finally assume (2.3) holds. Then there exists a $x \in X$ with $F_i(x) = \emptyset$ for each $i \in I$ if (2.7a) holds whereas there exists a $x \in X$ with $F_{j_0}(x) = \emptyset$ if (2.7b) holds.

PROOF. The result follows immediately from Theorem 2.1 with $A_i(x) = B_i(x) = X_i$ for each $x \in X$ and $P_i(x) = F_i(x)$ for $x \in X$. □

THEOREM 2.5. *Let $\{X_i\}_{i\in I}$ be a family of nonempty convex compact sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ suppose $F_i \colon X \equiv \prod_{i\in I} X_i \to X_i$ and assume (2.3), (2.6) (the paracompactness of U_i is not needed here) hold and also suppose there exists a map $\Psi_i \colon U_i \to X_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i \colon U_i \to X_i$ is upper semicontinuous with convex compact values. Also assume either*

$$(2.8a) \qquad y_i \notin \Psi_i(y) \quad \text{for all } y \in U_i,\ \forall i \in I$$

or

$$(2.8b) \qquad \text{there exists a } j_0 \in I \text{ with } y_{j_0} \notin \Psi_{j_0}(y), \text{ for } y \in U_{j_0}$$

occurs. Then there exists a $x \in X$ with $F_i(x) = \emptyset$ for each $i \in I$ if (2.8a) holds whereas there exists a $x \in X$ with $F_{j_0}(x) = \emptyset$ if (2.8b) holds.

PROOF. The result follows immediately from Theorem 2.3 with $A_i(x) = B_i(x) = X_i$ for each $x \in X$ and $P_i(x) = F_i(x)$ for $x \in X$. □

Before we consider collectively coincidence type results we now present examples where Theorem 2.1, Theorem 2.3, Theorem 2.4 and Theorem 2.5

can be applied immediately i.e. we present examples where (2.2) (or alternatively (2.6)) and (2.3) hold. It is enough here to consider Theorem 2.4 and Theorem 2.5. Suppose X is metrizable and for $i \in I$ suppose $F_i\colon X \to X_i$ is lower semicontinuous (for example if F_i has open lower sections, i.e. $F_i^{-1}(y)$ is open in X for every $y \in X_i$, then $F_i\colon X \to X_i$ is lower semicontinuous [18]) then $U_i = \{x \in X : F_i(x) \neq \emptyset\} = \{x \in X : F_i(x) \cap X_i \neq \emptyset\}$ is open in X and U_i is paracompact i.e. (2.6) holds. If for $i \in I$ we assume E_i is a Hausdorff locally convex topological vector space and X is metrizable then from Dugundji's externsion theorem X is an AR (absolute retract) so (2.3) holds. In fact there are many examples of X's in (2.3) in the literature; see [7,12,13].

Next we use the idea in Theorem 2.3 (the basic idea in Theorem 2.1) to establish some new general collectively coincidence type results (our results improve those in [14,15]; we note the condition $F_i\colon X \to Y_i$ (empty values in $X \backslash U_i$) is upper semicontinuous was understood to mean $F_i\colon U_i \to Y_i$ is upper semicontinuous in [14, Theorems 2.5, 2.7, 2.14, 2.16]). We begin when the index set is finite and then remark about the general index set case.

THEOREM 2.6. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$ suppose $H_i\colon X \equiv \prod_{i=1}^{N} X_i \to Y_i$, $U_i = \{x \in X : H_i(x) \neq \emptyset\}$ is open and assume there exists a map $\Psi_i\colon U_i \to Y_i$ with $H_i(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i\colon U_i \to Y_i$ is upper semicontinuous with convex compact values. In addition assume*

$$(2.9)\qquad X \text{ is a Schauder admissible subset of } E \equiv \prod_{i=1}^{N} E_i.$$

For each $j \in \{1,\dots,N\}$ suppose $G_j\colon Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in \mathrm{Ad}(Y, X_j)$. Also assume either

$$(2.10a)\qquad \begin{cases} y_j \notin \Psi_j(w) \text{ for all } (w,y) \in U_j \times Y \text{ with} \\ w_i \in G_i(y) \text{ for all } i \in \{1,\dots,N\}, \forall j \in \{1,\dots,N_0\} \end{cases}$$

or

$$(2.10b)\qquad \begin{cases} \text{there exists } j_0 \in \{1,\dots,N_0\} \text{ with } y_{j_0} \notin \Psi_{j_0}(w) \text{ for all} \\ (w,y) \in U_{j_0} \times Y \text{ with } w_i \in G_i(y) \text{ for all } i \in \{1,\dots,N\} \end{cases}$$

occurs. Then there exists a $x \in X$ and a $y \in Y$ with $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$ and $H_j(x) = \emptyset$ for $j \in \{1,\dots,N_0\}$ if (2.10a) *occurs whereas there exists a $x \in X$ and a $y \in Y$ with $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$ and $H_{j_0}(x) = \emptyset$ if* (2.10b) *occurs.*

PROOF. Let $i \in \{1,\dots,N_0\}$ and define a map $F_i\colon X \to Y_i$ by

$$F_i(x) = \begin{cases} \Psi_i(x), & x \in U_i = \{x \in X : H_i(x) \neq \emptyset\} \\ Y_i, & x \in X \backslash U_i. \end{cases}$$

Note F_i has nonempty convex compact values and $F_i\colon X \to Y_i$ is upper semicontinuous from Theorem 1.4 i.e. $F_i \in \mathrm{Kak}(X, Y_i)$ (again we note if $U_{i_0} = \emptyset$ for some i_0 then $F_{i_0}(x) = Y_{i_0}$). Let $F\colon X \to Y$ be given by

$$F(x) = \prod_{j=1}^{N_0} F_j(x) \quad \text{for } x \in X$$

and note $F \in \mathrm{Kak}(X, Y)$. Let $G\colon Y \to X$ be given by

$$G(y) = \prod_{i=1}^{N} G_i(y) \quad \text{for } y \in Y,$$

and note $G \in \mathrm{Ad}(Y, X)$. Finally note $G\,F \in \mathrm{Ad}(X, X)$ is a compact map (recall Y is compact) so Theorem 1.1 guarantees a $x \in X$ with $x \in G\,F(x)$. Let $y \in F(x)$ with $x \in G(y)$. Now $x \in G(y)$ implies $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$. Also $y \in F(x)$ implies $y_j \in F_j(x)$ for $j \in \{1,\dots,N_0\}$.

Suppose (2.10a) occurs. Now for each $j \in \{1,\dots,N_0\}$ we claim $H_j(x) = \emptyset$. Suppose not i.e. suppose there exists a $j_0 \in \{1,\dots,N_0\}$ with $H_{j_0}(x) \neq \emptyset$. Then $x \in U_{j_0}$ so $y_{j_0} \in F_{j_0}(x) = \Psi_{j_0}(x)$, which contradicts (2.10a). Thus our claim is true so the result in the statement of Theorem 2.6 holds.

Suppose (2.10b) occurs. We claim $H_{j_0}(x) = \emptyset$ since if $H_{j_0}(x) \neq \emptyset$ then $x \in U_{j_0}$ so $y_{j_0} \in F_{j_0}(x) = \Psi_{j_0}(x)$, which contradicts (2.10b). The result in the statement of Theorem 2.6 holds. □

REMARK 2.7. (i) One could also consider the map FG instead of GF in the proof of Theorem 2.6 if one rephrases the statement of Theorem 2.6.

(ii) Note $\{1,\dots,N_0\}$ could be replaced by J (an index set) in Theorem 2.6.

(iii) For each $i \in \{1,\dots,N_0\}$ suppose in addition U_i is paracompact and for each $x \in U_i$ assume there exists a map $A_{i,x}\colon U_i \to Y_i$ and an open set $O_{i,x}$ (in U_i) containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in O_{i,x}$ and $A_{i,x}\colon O_{i,x} \to Y_i$ is upper semicontinuous with convex compact values. Also assume either

$$\text{(2.11a)} \quad \begin{cases} y_j \notin A_{j,x}(w) \text{ for all } (w,y) \in O_{j,x} \times Y \text{ with} \\ w_i \in G_i(y) \text{ for all } i \in \{1,\dots,N\},\ \forall x \in U_j,\ \forall j \in \{1,\dots,N_0\} \end{cases}$$

or

(2.11b)

$$\begin{cases} \text{there exists } j_0 \in \{1, \dots, N_0\} \text{ with } y_{j_0} \notin A_{j_0,x}(w) \text{ for all} \\ (w, y) \in O_{j_0,x} \times Y \text{ with } w_i \in G_i(y) \text{ for all } i \in \{1, \dots, N\}, \ \forall x \in U_{j_0} \end{cases}$$

occurs. From the discussion before Theorem 2.1 for each $i \in \{1, \dots, N_0\}$ there exists a map $\Psi_i \colon U_i \to Y_i$ with $H_i(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i \colon U_i \to Y_i$ is upper semicontinuous with convex compact values: here $\{O_{i,x}\}_{x \in U_i}$ is an open covering of U_i so there exists a locally finite open covering $\{V_{i,x}\}_{x \in U_i}$ of U_i (recall U_i is paracompact) with $x \in V_{i,x}$ and $V_{i,x} \subseteq O_{i,x}$ for $x \in X$, and for each $x \in U_i$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x}, \\ Y_i, & z \in U_i \backslash V_{i,x} \end{cases}$$

and $\Psi_i \colon U_i \to Y_i$ is

$$\Psi_i(z) = \bigcap_{x \in U_i} Q_{i,x}(z) \quad \text{for } z \in U_i.$$

Now from the proof of Theorem 2.6 we have $x_i \in G_i(y)$ for $i \in \{1, \dots, N\}$ and $y_j \in F_j(x)$ for $j \in \{1, \dots, N_0\}$. Assume there exists a $j_0 \in \{1, \dots, N_0\}$ with $H_{j_0}(x) \neq \emptyset$. Then $x \in U_{j_0}$ so since $\{V_{j_0,z}\}_{z \in U_{j_0}}$ is a covering of U_{j_0} there exists a $z^\star \in U_{j_0}$ with $x \in V_{j_0,z^\star}$ so

$$\Psi_{j_0}(x) = \bigcap_{z \in U_{j_0}} Q_{j_0,z}(x) \subseteq Q_{j_0,z^\star}(x) = A_{j_0,z^\star}(x)$$

and so $y_{j_0} \in F_{j_0}(x) = \Psi_{j_0}(x) \subseteq A_{j_0,z^\star}(x)$ which contradicts (2.11a) (with $j = j_0$) and (2.11b). Thus $H_{j_0}(x) = \emptyset$.

(iv) To get a contradiction in the proof of Theorem 2.6 one only needs the statement "there exists a $x \in X$ with $x \in GF(x)$" to be false, so one could list other conditions to guarantee the contradiction.

REMARK 2.8. Note Theorem 2.6 improves [15, Theorem 2.9]. Indeed part of an assumption (see (2.11a)) was inadvertently omitted in [15, Theorem 2.9] (but in fact it is a condition mentioned in Remark 2.7(iv)).

REMARK 2.9. We can replace $\{1, \dots, N\}$ and $\{1, \dots, N_0\}$ with I and J index sets and obtain an analogue of Theorem 2.6 if we replace $G_j \in \mathrm{Ad}(Y, X_j)$ with $G_j \in \mathrm{Kak}(Y, X_j)$ in the statement; here $Y \equiv \prod_{i \in J} Y_j$. We just need to note that $F \equiv \prod_{i \in J} F_i \in \mathrm{Kak}(X, Y)$ and $G \equiv \prod_{i \in I} G_i \in \mathrm{Kak}(Y, X)$ so $FG \in \mathrm{Ad}(X, X)$.

THEOREM 2.10. *Let* $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ *be families of convex sets each in a Hausdorff topological vector space and* $\{Y_i\}_{i=1}^{N_0}$ *is also a family of compact sets. For each* $i \in \{1, \ldots, N_0\}$ *suppose* $H_i \colon X \equiv \prod_{i=1}^N X_i \to Y_i$, $U_i = \{x \in X : H_i(x) \neq \emptyset\}$ *is open and assume there exists a map* $\Psi_i \colon U_i \to Y_i$ *with* $H_i(z) \subseteq \Psi_i(z)$ *for* $z \in U_i$ *and* $\Psi_i \colon U_i \to Y_i$ *is upper semicontinuous with convex compact values. For each* $j \in \{1, \ldots, N\}$ *suppose* $G_j \colon Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ *and* $G_j \in DKT(Y, X_j)$. *Also assume either* (2.10a) *or* (2.10b) *occurs. Then there exists a* $x \in X$ *and a* $y \in Y$ *with* $x_i \in G_i(y)$ *for* $i \in \{1, \ldots, N\}$ *and* $H_j(x) = \emptyset$ *for* $j \in \{1, \ldots, N_0\}$ *if* (2.10a) *occurs whereas there exists a* $x \in X$ *and a* $y \in Y$ *with* $x_i \in G_i(y)$ *for* $i \in \{1, \ldots, N\}$ *and* $H_{j_0}(x) = \emptyset$ *if* (2.10b) *occurs.*

PROOF. Let $i \in \{1, \ldots, N_0\}$ and let F_i and F be as in Theorem 2.6 and note $F \in \mathrm{Kak}(X, Y)$. Let $i \in \{1, \ldots, N\}$ and from [3,4] there exists a continuous (single valued) selection $g_i \colon Y \to X_i$ of G_i with $g_i(y) \in G_i(y)$ for $y \in Y$ and there exists a finite set R_i of X_i with $g_i(Y) \subseteq \mathrm{co}(R_i) \equiv Q_i$. Let $Q = \prod_{i=1}^N Q_i$ ($\subseteq X$) and note Q is compact. Let

$$g(y) = \prod_{i=1}^{N} g_i(y), \quad \text{for } y \in Y,$$

and note $g \colon Y \to Q$ is continuous. Let $F^\star$ denote the restriction of F to Q and note $F^\star \in \mathrm{Kak}(Q, Y)$. Now $g\, F^\star \in \mathrm{Ad}(Q, Q)$ (note $F^\star \in \mathrm{Ad}(Q, Y)$ and $g \in \mathrm{Ad}(Y, Q)$) and Q is a compact convex set in a finite dimensional subspace of $E = \prod_{i=1}^N E_i$, so Theorem 1.1 guarantees a $x \in Q$ with $x \in g\,(F^\star(x))$. Now let $y \in F^\star(x)$ with $x = g(y)$. Note $x_i = g_i(y) \in G_i(y)$ for $i \in \{1, \ldots, N\}$ and also $y_j \in F_j(x)$ for $j \in \{1, \ldots, N_0\}$. The conclusion of the theorem is completed the same way as in Theorem 2.6. □

REMARK 2.11. (i) In Theorem 2.10 note $G_j \in DKT(Y, X_j)$ could be replaced by $G_j \in HLPY(Y, X_j)$ since one can deduce immediately the existence of a continuous (single valued) selection $g_i \colon Y \to X_i$ of G_i.

(ii) We can replace $\{1, \ldots, N\}$ and $\{1, \ldots, N_0\}$ with I and J index sets and obtain an analogue of Theorem 2.10 if $Q \equiv \prod_{i \in I} Q_i$ is a Schauder admissible subset of $E \equiv \prod_{i \in I} E_i$.

(iii) To get a contradiction in the proof of Theorem 2.10 one only needs the statement "there exists a $x \in X$ with $x \in gF(x)$" to be false, so one could list other conditions to guarantee the contradiction.

(iv) Note one could also consider the map Fg instead of gF in the proof of Theorem 2.10 if one rephrases the statement of Theorem 2.10.

REMARK 2.12. (i) For each $i \in \{1, \ldots, N_0\}$ suppose in addition U_i is paracompact and for each $x \in U_i$ assume there exists a map $A_{i,x} \colon U_i \to Y_i$

and an open set $O_{i,x}$ (in U_i) containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in O_{i,x}$ and $A_{i,x} : O_{i,x} \to Y_i$ is upper semicontinuous with convex compact values. Also assume either (2.11a) or (2.11b) occurs. Now as in Remark 2.7 there exists a map $\Psi_i \colon U_i \to Y_i$ with $H_i(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i \colon U_i \to Y_i$ is upper semicontinuous with convex compact values and we have the conclusion as in Theorem 2.10.

(ii) Note Theorem 2.10 improves Theorem 2.10 in [15] and [14, Theorem 2.16, Theorem 2.18] (note part of the assumption in [14, Theorems 2.16, 2.18], and [15, Theorem 2.10] was inadvertently omitted (but in fact it is a condition mentioned in Remark 2.11(iii))).

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