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HINDMAN'S THEOREM AND CHOICE

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Abstract. In **ZF** (i.e. the Zermelo–Fraenkel set theory without the Axiom of Choice (**AC**)), we investigate the set-theoretic strength of a generalized version of Hindman's theorem and of certain weaker forms of this theorem, which were introduced by Fernández-Bretón [8], with respect to their interrelation with several weak choice principles. In this direction, we determine the status of (this general version of) Hindman's theorem (and of weaker forms) in certain permutation models of $\mathbf{ZFA} + \neg\mathbf{AC}$ and transfer the results to **ZF**, strengthen some results of [8] and settle a related open problem from Howard and Rubin [10]; thus filling the gap in information in both [8] and [10].

1. Introduction

A cornerstone of the Ramsey theory of numbers is undoubtedly the celebrated *Hindman's theorem* which states the following: "For every finite colouring of the natural numbers there exists an infinite set X such that all finite sums of distinct elements of X have the same colour". In fact, the latter statement was originally a conjecture of R. Graham and B. Rothschild that was proved to be true, in 1974, by Hindman [9] who used a long and involved combinatorial argument. Although Hindman does not state exactly his assumptions, it is implicit that he is assuming the **ZFC** axioms. In terms of analyzing the minimal set of assumptions that he employs, Hindman's original proof uses relatively little, certainly much less than even **ZF**; as shown by Blass, Hirst and Simpson [2], Hindman's proof goes through in the subsystem $\mathbf{ACA}_0^+$ of second order arithmetic. A different proof that requires stronger assumptions, but still well within the scope of **ZF**, is Baumgartner's [1]. However, the standard by now **ZFC**-proof (which is considerably easier than Hindman's original proof) utilizes ultrafilters and

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is due to Galvin, whose argument was based on a result of Glazer, namely "There exists a free idempotent ultrafilter on ω", which is not provable in **ZF**; see [10, Feferman's model $\mathcal{M}2$, Blass' model $\mathcal{M}15$]—in $\mathcal{M}2$, every ultrafilter on ω is principal and in $\mathcal{M}15$ every ultrafilter on any infinite set is principal.[1] But even Galvin's proof of Hindman's theorem can be carried out in **ZF** (in spite of the apparent need for the **AC** encapsulated in the use of a free (idempotent) ultrafilter on ω) by using Shoenfield's absoluteness theorem. In fact, it was Comfort [4] who used Shoenfield's theorem (actually, a theorem stronger than the latter result) to establish that Hindman's theorem is provable in **ZF**, i.e. provable without appeal to any form of choice; see [4, Subsection 4.2]. A different but more explicit (and probably more illuminating) argument was suggested to us by the referee and is as follows: given $c : \omega \to k$, consider $\mathbf{L}[c]$ (i.e. the constructible universe relativized to c) and obtain within $\mathbf{L}[c]$ a free idempotent ultrafilter on ω, $\mathcal{U}$ say. As $\mathcal{U}$ is an ultrafilter on ω and c is a finite colouring of ω, there exists $i \in k$ such that $U = c^{-1}(\{i\}) \in \mathcal{U}$. Following Galvin's proof, construct an infinite (in fact, denumerable) $X \subseteq U$ in $\mathbf{L}[c]$ such that $\mathrm{FS}(X)$, the set of all finite sums of distinct elements of X, is a subset of U in $\mathbf{L}[c]$, i.e. $\mathrm{FS}(X)$ is c-monochromatic (in colour i) in $\mathbf{L}[c]$. All of this happens in $\mathbf{L}[c]$ but of course X still exists and satisfies that $\mathrm{FS}(X)$ is c-monochromatic in the "real world".

Now, Hindman's theorem is equivalent to the statement: "For every finite colouring of $[\omega]^{<\omega}$ (the set of finite subsets of ω), there exists an infinite, disjointed $Y \subseteq [\omega]^{<\omega}$ such that all unions of finitely many elements of Y have the same colour". Fernández-Bretón [8], continuing the research initiated by Brot, Cao and Fernández-Bretón [3], considered a natural generalization of the above statement by replacing ω with any infinite set X and studied the implication relations between the resulting statement and various weak choice principles. In particular, the author [8] formulated the following proposition, which he referred to as Hindman's theorem and denoted by **HT**: "For every infinite set X and for every colouring $c : [X]^{<\omega} \to 2$, there exists an infinite, pairwise disjoint family $Y \subseteq [X]^{<\omega}$ such that the set

$$\mathrm{FU}(Y) = \left\{ \bigcup_{y \in F} y : F \in [Y]^{<\omega} \setminus \{\emptyset\} \right\}$$

is c-monochromatic". (In [8, Proposition 2], it was shown—within **ZF**—that **HT** is equivalent to $\mathbf{HT}(k)$ for any integer $k \geq 2$, where $\mathbf{HT}(k)$ is the statement resulting from **HT** by replacing "2" with "k".) Based on a result of

[1] For Glazer's result and Galvin's proof, see, for example, Jech [13, Theorem 29.1, Lemma 29.2]. For a thorough study on free idempotent ultrafilters on ω, as well as on the Ellis–Numakura Lemma, in set theory without the full power of **AC**, the reader is referred to the fairly recent works of Di Nasso and Tachtsis [7] and Tachtsis [21].

[3] (specifically, [3, Theorem 3.2]), the author gave an exact characterization of **HT** as a weak choice form, namely as "For every infinite set X, $[X]^{<\omega}$ is Dedekind-infinite"; see [8, Proposition 4].

Fernández-Bretón [8] also considered a class of weaker Boolean forms of **HT**, denoted by $\mathbf{HT}_n(k)$ ($n, k \in \omega \setminus \{0\}$), and investigated the set-theoretic strength of $\mathbf{HT}_2(k)$ with regard to its placement among several weak choice principles. (Complete definitions shall be given in the forthcoming Section 2.)

The purpose of this paper is to continue the research initiated in [8] and [3] on this intriguing topic by providing new information on the status of **HT**, $\mathbf{HT}_2(k)$ and $\mathbf{HT}_3(k)$ in certain (relatively recent) permutation models of $\mathbf{ZFA} + \neg\mathbf{AC}$ and by strengthening some results of [8] as well as *resolving a related open problem* from Howard and Rubin [10]; in particular, we show (in Theorem 7) that there is a model of **ZF** in which **HT** is true but the axiom of multiple choice for denumerable families of denumerable sets and the axiom of countable choice for non-empty finite sets are both false (the corresponding problem from [10] concerned the statement "For every infinite set X, $\wp(X)$ is Dedekind-infinite" rather than **HT**, which, nonetheless, is weaker than **HT** in **ZF**). The latter result thus fills the gap in information in [8], [10] and, moreover, it properly strengthens a result of [8].

2. Terminology and known results

Notation 1. 1. **ZF** is the Zermelo–Fraenkel set theory minus the **AC**.

2. **ZFA** is **ZF** with the Axiom of Extensionality weakened to allow the existence of atoms.

3. **ZFC** is **ZF** + **AC**.

4. ω denotes as usual the set of natural numbers.

5. Let X be any set. $[X]^{<\omega}$ denotes the set of finite subsets of X and, for every $n \in \omega \setminus \{0\}$, $[X]^n$ denotes the set of all n-element subsets of X.

Definition 1. A set X is called:

1. *finite* if there exists a bijection $f : X \to n$ for some $n \in \omega$. Otherwise, X is called *infinite*.

2. *denumerable* if there is a bijection $f : \omega \to X$.

3. *countable* if it is finite or denumerable.

4. *Dedekind-finite* if there is no injection $f : \omega \to X$. Otherwise, X is called *Dedekind-infinite*.

5. *Ia-finite* if X cannot be expressed as a disjoint union of two infinite subsets (i.e. if every subset of X is either finite or co-finite in X).[2]

6. *amorphous* if X is infinite and Ia-finite.

[2] This notion of finiteness was first formulated by A. Lévy [17].

7. *cuf* if it can be expressed as a countable union of finite sets.

Definition 2. Let $(P, \leq)$ be a partially ordered set (poset).

(a) The binary relation $<$ on P defined by $x < y$ if $x \leq y$ and $x \neq y$, is called a *strict partial order* on P; hence, a strict partial order $<$ on P is an irreflexive ($\forall x \in P$, $(x, x) \notin <$) and transitive binary relation on P. The ordered pair $(P, <)$ is called a *strict poset*.

(b) $(P, \leq)$ is called a *lattice* if, in addition, every pair of elements of P has a supremum and an infimum. The corresponding strict poset $(P, <)$ is called a *strict lattice*.

(c) A set $A \subseteq P$ is called a *chain* in P if, for every $x, y \in A$, $x \leq y$ or $y \leq x$ (i.e. if any two elements of A are comparable with respect to $\leq$). A set $A \subseteq P$ is called an *anti-chain* in P if, for every two distinct $x, y \in A$, $x \nleq y$ and $y \nleq x$ (i.e. if any two elements of A are incomparable with respect to $\leq$).

Below, we list the weak choice principles we shall refer to in this paper. To the best of our knowledge, the principle $\mathbf{DC}_{l,fp}$ was introduced by Da Silva [5] and, as already mentioned, **HT** and $\mathbf{HT}_n(k)$ $(n, k \in \omega \setminus \{0\})$ were introduced by Fernández-Bretón [8].

Definition 3 (**AC** and weak choice principles).

1. *Axiom of Choice* (**AC** and Form 1 in [10]): Every family of non-empty sets has a choice function.
2. *Axiom of Countable Choice* ($\mathbf{AC}^\omega$ and Form 8 in [10]): Every denumerable family of non-empty sets has a choice function.
3. *Axiom of Countable Multiple Choice* ($\mathbf{MC}^\omega$ and Form 126 in [10]): Every denumerable family of non-empty sets has a multiple choice function (i.e. a function which chooses a non-empty, finite subset from each element of the given family).
4. $\mathbf{MC}^\omega_\omega$ (Form 350 in [10]): Every denumerable family of denumerable sets has a multiple choice function.
5. $\mathbf{AC}^\omega_{\mathbf{fin}}$ (Form 10 in [10]): Every denumerable family of non-empty finite sets has a choice function.[3]
6. $\mathbf{DF} = \mathbf{F}$ (Form 9 in [10]): Every Dedekind-finite set is finite.
7. *Principle of Dependent Choices* (**DC** and Form 43 in [10]): Let R be a binary relation on a non-empty set A such that $(\forall x \in A)(\exists y \in A)(x\ R\ y)$. Then there is a sequence $(x_n)_{n\in\omega}$ of elements of A such that $x_n\ R\ x_{n+1}$ for all $n \in \omega$.
8. $\mathbf{DC}_{l,fp}$: **DC** restricted to strict lattices for which every element has only finitely many predecessors.

[3] In **ZF**, $\mathbf{AC}^\omega_{\mathbf{fin}}$ is equivalent to the statement "The union of a countable family of finite sets is countable"; see [10].

9. *Hindman's theorem* (**HT**): For every infinite set X and for every colouring $c : [X]^{<\omega} \to 2$, there exists an infinite, pairwise disjoint family $Y \subseteq [X]^{<\omega}$ such that the set

$$\mathrm{FU}(Y) = \left\{ \bigcup_{y \in F} y : F \in [Y]^{<\omega} \setminus \{\emptyset\} \right\}$$

is c-monochromatic.

10. Given a set X, a family $Y \subseteq [X]^{<\omega}$, and an $n \in \omega \setminus \{0\}$, we let

$$\mathrm{FS}_{\leq n}(Y) = \{F_1 \triangle \cdots \triangle F_t : t \leq n \text{ and } F_1, \ldots, F_t \in Y\},$$

where $\triangle$ denotes the operation of symmetric difference.

Let $n, k \in \omega \setminus \{0\}$. $\mathbf{HT}_n(k)$: For every infinite set X and every colouring $c : [X]^{<\omega} \to k$, there exists an infinite $Y \subseteq [X]^{<\omega}$ such that $\mathrm{FS}_{\leq n}(Y)$ is c-monochromatic.

11. *Ramsey's theorem* (**RT** and Form 17 in [10]): For every infinite set X and for every colouring $c : [X]^2 \to 2$, there exists an infinite $Y \subseteq X$ such that $[Y]^2$ is c-monochromatic.

12. *Chain/Anti-chain Principle* (**CAC** and Form 217 in [10]): Every infinite poset has either an infinite chain or an infinite anti-chain.

13. **LW** (Form 91 in [10]): Every linearly ordered set can be well ordered.[4]

14. Form 82 of [10]: For every infinite set X, $\wp(X)$ is Dedekind-infinite.[5]

Next, we list some of the most representative known results in this area.

THEOREM 1. 1. (**ZF**) *Any infinite, well-orderable set X satisfies the conclusion of* **HT**, *and thus of $\mathbf{HT}_n(k)$ for all $n, k \in \omega \setminus 1$.*

2. *Let $k \in \omega \setminus 1$ and $n \in \omega \setminus 4$. Then* **HT** *is equivalent to $\mathbf{HT}_n(k)$.*

3. *Let $k \in \omega \setminus 1$. Then,* $\mathbf{HT} \Rightarrow \mathbf{HT}_3(k) \Rightarrow \mathbf{HT}_2(k)$.

4. $\mathbf{HT}_2(k)$ *does not imply* **HT** *in* **ZF**, *for any integer $k \geq 2$.*

5. *The following are equivalent*:

(a) **HT**.

(b) *For every infinite set X, $[X]^{<\omega}$ is Dedekind-infinite.*

(c) *Every infinite set has an infinite cuf subset.*

(d) $\mathbf{DC}_{l,fp}$.

Thus, **HT** *(equivalently, $\mathbf{DC}_{l,fp}$) implies that there are no amorphous sets. Moreover, the latter implication is not reversible in* **ZF**.

6. $\mathbf{DF} = \mathbf{F}$ *is equivalent to* $\mathbf{HT} \wedge \mathbf{AC}^{\omega}_{\mathbf{fin}}$.

[4] **LW** is equivalent to **AC** in **ZF**, but it is not equivalent to **AC** in **ZFA** (see [12, Theorems 9.1, 9.2].

[5] Note that Form 82 is equivalent to the statement: "For every infinite set X, there is a surjection $f : X \to \omega$".

7. *Form 82 is weaker than* **HT** *in* **ZF**. *In particular, the Basic Cohen Model (Model* $\mathcal{M}1$ *in* [10]) *satisfies* $82 \wedge \neg\mathbf{HT}_2(2)$ (*and thus satisfies* $82 \wedge \neg\mathbf{HT}$).

8. *For any integer* $k \geq 2$, **RT** *implies* $\mathbf{HT}_2(k)$ *and the implication is not reversible in* **ZF**.

9. *In each of the Basic Fraenkel Model (Model* $\mathcal{N}1$ *of* [10]) *and the Mostowski Linearly Ordered Model (Model* $\mathcal{N}3$ *of* [10]), $(\forall k \geq 2)(\mathbf{HT}_2(k))$ *is true but* $\mathbf{HT}_3(2)$ (*and thus* $\mathbf{HT}_3(k)$ *for any integer* $k \geq 2$) *is false.*

In Theorem 1, parts (1)–(4), the equivalence between (a) and (b) in part (5), part (6), the fact of (7) that $\mathbf{HT}_2(2)$ is false in the Basic Cohen Model $\mathcal{M}1$ and part (8), were established by Fernández-Bretón [8] (see [8, Propositions 4, 6, Corollaries 25, 29, Theorems 28, 30]). In fact, in [8], it is mentioned that (2) and "(a) $\Longleftrightarrow$ (b)" of (5) follow from [3, Theorem 3.2]. The equivalence between (c) and (d) of Theorem 1(5) was shown by Tachtsis [19, Theorem 4], and the equivalence between (b) and (c) is straightforward. Furthermore, the assertion (in part (5) of Theorem 1) that "There are no amorphous sets" (Form 64 in [10]) does not imply **HT** in **ZF** follows from the fact that the former principle is true in the Basic Cohen Model $\mathcal{M}1$ (see [10]), while (as mentioned above) $\mathbf{HT}_2(2)$, and thus **HT**, is false in $\mathcal{M}1$. Part (9) of Theorem 1 was established in [8, Theorem 28] and [3, Propositions 4.2, 4.17].

Let us also note here that the validity of Form 82 in $\mathcal{M}1$ readily follows from Jech [12, Lemma 5.25] (this was also observed by Truss [23, Theorem 5]), and that the failure of **HT** in $\mathcal{M}1$ is derived from part (6) of Theorem 1 and the fact that $\mathcal{M}1$ satisfies the Countable Union Theorem (i.e. "The union of a countable family of countable sets is countable"), and thus satisfies $\mathbf{AC}^{\omega}_{\mathbf{fin}}$, and $\neg(\mathbf{DF} = \mathbf{F})$ (see [10]).

Last but not least, let us mention that Keremedis and Tachtsis [15] have given some topological characterizations of the principle "For every infinite set X, $[X]^{<\omega}$ is Dedekind-infinite", and thus of **HT**. To name one, the statement "*For every infinite set* X*, the Tychonoff product* 2^X *(where* $2 = \{0,1\}$ *is equipped with the discrete topology) has a denumerable, disjointed family of non-empty, open sets*" is equivalent to the above principle, and thus to **HT**; see [15, Lemma 1, Theorem 3]. Further topological equivalents of **HT** can be found in Keremedis and Wajch [16].

3. Terminology for permutation models and a transfer theorem of Pincus

For the reader's convenience, we provide a brief account of the construction of permutation models of **ZFA**; a detailed account can be found in Jech [12, Chapter 4].

One starts with a model M of $\mathbf{ZFA} + \mathbf{AC}$ which has A as its set of atoms. Let G be a group of permutations of A and also let $\mathcal{F}$ be a filter on the lattice of subgroups of G which satisfies the following two properties:

$$(\forall a \in A)(\exists H \in \mathcal{F})(\forall \phi \in H)(\phi(a) = a)$$

and $\mathcal{F}$ is closed under conjugation, i.e.

$$(\forall \phi \in G)(\forall H \in \mathcal{F})(\phi H \phi^{-1} \in \mathcal{F}).$$

Such a filter $\mathcal{F}$ of subgroups of G is called a *normal filter* on G. Every permutation of A extends uniquely to an $\in$-automorphism of M by $\in$-induction, and for any $\phi \in G$, we identify ϕ with its (unique) extension. If $x \in M$ and H is a subgroup of G, then $\mathrm{fix}_H(x)$ denotes the (pointwise stabilizer) subgroup $\{\phi \in H : \forall y \in x(\phi(y) = y)\}$ of H and $\mathrm{Sym}_H(x)$ denotes the (stabilizer) subgroup $\{\phi \in H : \phi(x) = x\}$ of H.

An element x of M is called *$\mathcal{F}$-symmetric* (or *symmetric* when no confusion arises) if $\mathrm{Sym}_G(x) \in \mathcal{F}$ and it is called *hereditarily $\mathcal{F}$-symmetric* (or *hereditarily symmetric*) if x and all elements of $\mathrm{TC}(x)$ (the transitive closure of x) are $\mathcal{F}$-symmetric.

Let $\mathcal{N}$ be the class which consists of all hereditarily $\mathcal{F}$-symmetric elements of M. Then $\mathcal{N}$ is a model of $\mathbf{ZFA}$ and $A \in \mathcal{N}$ (see Jech [12, Theorem 4.1, p. 46]); it is called the *permutation model* (or the *Fraenkel–Mostowski model*) determined by M, G and $\mathcal{F}$.

Definition 4. For any set X, let $\wp^{\alpha}(X)$, where α ranges over ordinal numbers, be defined as follows:

$$\wp^0(X) = X, \quad \wp^{\alpha+1}(X) = \wp^{\alpha}(X) \cup \wp(\wp^{\alpha}(X)),$$

$$\wp^{\alpha}(X) = \bigcup_{\beta<\alpha} \wp^{\beta}(X) \quad (\alpha \text{ limit}).$$

In the subsequent Definitions 5 and 6(2), the notation $\mathbf{x}$ stands for a tuple $(x_1, x_2, \dots, x_n)$ of variables. In Definition 6(2), the variables of $\mathbf{y} = (y_1, y_2, \dots, y_n)$ are assumed disjoint from those of $\mathbf{x}$. $\exists\mathbf{x}$ ($\forall\mathbf{x}$) stands for $\exists x_1 \exists x_2 \cdots \exists x_n$ ($\forall x_1 \forall x_2 \cdots \forall x_n$). $\bigcup \mathbf{x}$ stands for $x_1 \cup x_2 \cup \cdots \cup x_n$.

Definition 5 [14]. Let C be a class and also let $\Phi(\mathbf{x})$ be a formula in the language of set theory with atoms. $\Phi^C(\mathbf{x})$ is Φ with quantifiers restricted to C. Similarly, if $\sigma(\mathbf{x})$ is a term, then $\sigma^C(\mathbf{x})$ is defined by the same formula that defines σ but with its quantifiers restricted to C.

$\Phi(\mathbf{x})$ is *boundable* if for some ordinal γ, $\mathbf{ZFA} \vdash \Phi(\mathbf{x}) \leftrightarrow \Phi^{\wp^{\gamma}(\bigcup \mathbf{x})}(\mathbf{x})$. Similarly, the term $\sigma(x)$ is boundable if for some ordinal γ, $\mathbf{ZFA} \vdash \sigma(\mathbf{x}) = \sigma^{\wp^{\gamma}(\bigcup \mathbf{x})}(\mathbf{x})$.

A *statement* is boundable if it is the existential closure of a boundable formula.

Definition 6 [18]. 1. Let x be a set. We define

$$|x|_- = \sup \left\{ \kappa : \kappa \text{ is a well-ordered cardinal such that } \kappa \le |x| \right\}.$$

$|x|_-$ is called the *injective cardinality* of x.

2. A formula $\Phi(\mathbf{y})$ is *injectively boundable* if it is a conjunction of $\Phi_i(\mathbf{y})$:

$$\Phi_i(\mathbf{y}) = \forall \mathbf{x}\Big(\Big(\Big|\bigcup \mathbf{x}\Big|_- \le \sigma_i(\mathbf{y}) \wedge \bigcup \mathbf{x} \cap \mathrm{TC}\Big(\bigcup \mathbf{y}\Big) = \emptyset\Big) \to \Psi_i(\mathbf{x}, \mathbf{y})\Big),$$

where $\sigma_i(\mathbf{y})$ and $\Psi_i(\mathbf{x}, \mathbf{y})$ are boundable.

A *statement* is injectively boundable if it is the existential closure of an injectively boundable formula.

The following fact was noted in [18, p. 722].

Fact 1. *Boundable formulae and statements are* (*up to equivalence*) *injectively boundable.*

Theorem 2 [18, Theorem 3A3]. *Let* Φ *be a conjunction of injectively boundable statements which hold in a permutation model* V_0. *There is a* **ZF**-*model* $V \supset V_0$ *with the same ordinals and cofinalities where* Φ *holds.*

4. Main results

We start by proving that $\mathbf{MC}^\omega$ implies **HT** and that the implication is not reversible in **ZF** (cf. Proposition 1 and Theorem 3). This *strengthens* [8, Theorem 18] that **HT** is weaker than $\mathbf{AC}^\omega$ in **ZF**; recall that $\mathbf{MC}^\omega$ is weaker than $\mathbf{AC}^\omega$ in **ZFA** (see, for example [10, Second Fraenkel Model $\mathcal{N}2$]).

Proposition 1. $\mathbf{MC}^\omega$ *implies* **HT** (*and thus, by Theorem* 1[(2),(3)], *implies* $\mathbf{HT}_n(k)$ *for all* $n, k \in \omega \setminus \{0\}$).

Proof. Assume $\mathbf{MC}^\omega$. Let X be an infinite set and also let

$$\mathcal{U} = \left\{ [X]^n : n \in \omega \setminus \{0\} \right\}.$$

Clearly, $\mathcal{U}$ is denumerable. By $\mathbf{MC}^\omega$, let F be a multiple choice function for $\mathcal{U}$. It is reasonably clear that

$$\mathcal{V} = \left\{ \bigcup F([X]^n) : n \in \omega \setminus \{0\} \right\}$$

is a denumerable subset of $[X]^{<\omega}$. Hence, $[X]^{<\omega}$ is Dedekind-infinite, i.e. **HT** is true. □

Note that since $\mathbf{DF} = \mathbf{F}$ implies **HT** and $\mathbf{DF} = \mathbf{F}$ does not imply $\mathbf{MC}^\omega$ in **ZF** (see [10, Model $\mathcal{M}6$]), it follows that **HT** is weaker than $\mathbf{MC}^\omega$ in **ZF**.

However, since (as already mentioned in Section 1) we consider it *important* to provide *new* information about the status of **HT** in certain models of $\mathbf{ZFA}+\neg\mathbf{AC}$, we show next that **HT** is true in a permutation model constructed in [6], which does not satisfy any of $\mathbf{DF}=\mathbf{F}$ and $\mathbf{MC}^{\omega}$, and for which the status of **HT** was *unknown until now.* In Theorem 4, we will observe that the above **ZFA**-independence result can be transferred to **ZF**.

The permutation model from [6]. We start with a model M of $\mathbf{ZFA}+\mathbf{AC}$ with a set A of atoms such that A has a denumerable partition $\{A_i : i\in\omega\}$ into denumerable sets, and for each $i\in\omega$, A_i has a denumerable partition

$$P_i=\left\{\, A_{i,j} : j\in\omega\setminus\{0\}\right\}$$

into finite sets such that, for every $j\in\omega\setminus\{0\}$, $|A_{i,j}|=j$. Let

$$G=\left\{\phi\in \mathrm{Sym}(A) : (\forall i\in\omega)(\phi(A_i)=A_i) \text{ and } |\{a\in A : \phi(a)\neq a\}|<\aleph_0\right\},$$

where $\mathrm{Sym}(A)$ is the group of *all* permutations of A. Let

$$\mathbf{P}_i=\{\phi(P_i) : \phi\in G\}$$

and also let

$$\mathbf{P}=\bigcup\{\mathbf{P}_i : i\in\omega\}.$$

Let $\mathcal{F}$ be the filter of subgroups of G generated by the subgroups $\mathrm{fix}_G(E)$, $E\in[\mathbf{P}]^{<\omega}$; $\mathcal{F}$ is a normal filter on G.

Definition 7. Let M, G and $\mathcal{F}$ be given as above. We let $\mathcal{N}$ be the permutation model determined by M, G and $\mathcal{F}$.

If $x\in\mathcal{N}$, then $\mathrm{Sym}_G(x)\in\mathcal{F}$, and thus (by definition of $\mathcal{F}$) there exists $E\in[\mathbf{P}]^{<\omega}$ such that $\mathrm{fix}_G(E)\subseteq \mathrm{Sym}_G(x)$. Under these circumstances, we call E a *support* of x.

A few observations are in order:

1. For every $i\in\omega$, every $Q\in\mathbf{P}_i$ is a partition of A_i into sets of different cardinalities. Thus, for any $\phi\in G$, ϕ fixes Q if and only if ϕ fixes Q pointwise. Moreover, since every $\phi\in G$ moves only finitely many elements of A,

$$(\forall Q\in\mathbf{P}_i)(\exists j_Q\in\omega)(Q\supseteq\{A_{i,j} : j>j_Q\}). \tag{1}$$

2. The set A is (in $\mathcal{N}$) the denumerable union of the cuf sets A_i, $i\in\omega$ (since, for every $i\in\omega$, $P_i=\{A_{i,j} : j\in\omega\setminus\{0\}\}$ is a denumerable partition of A_i in $\mathcal{N}$—$\{P_i\}$ is a support of every element of P_i—comprising finite sets), which fails to be cuf in $\mathcal{N}$.

3. For every $i\in\omega$, A_i is not well orderable in $\mathcal{N}$ since no $E\in[\mathbf{P}]^{<\omega}$ is a support of a well ordering on A_i (and thus $\mathbf{AC}^{\omega}_{\mathrm{fin}}$ is false in $\mathcal{N}$). If not,

then for some $i \in \omega$, there exists $E \in [\mathbf{P}]^{<\omega}$ which is a support of every element of A_i. Then, necessarily, $E \cap \mathbf{P}_i \neq \emptyset$. Otherwise, pick two distinct $a, b \in A_i$ and consider the transposition $\phi = (a, b)$. Then $\phi \in \mathrm{fix}_G(E) \setminus \mathrm{fix}_G(A_i)$, which is a contradiction. So, $E \cap \mathbf{P}_i \neq \emptyset$. Since E is finite, equation (1) yields the existence of a $j_0 \in \omega$ such that every member of $E \cap \mathbf{P}_i$ contains $\{A_{i,j} : j > j_0\}$. Let $j > j_0$ and also let a, b be two distinct elements of $A_{i,j}$ (and note that $|A_{i,j}| = j > j_0 \geq 1$). Let $\eta = (a, b)$. Clearly $\eta \in \mathrm{fix}_G(E)$. However, η does not fix A_i pointwise, which is a contradiction.

Moreover, in much the same way as the previous argument, one shows that, for every $i \in \omega$, A_i has no infinite well orderable subsets in $\mathcal{N}$, and thus is Dedekind-finite in $\mathcal{N}$.

THEOREM 3. *Let $\mathcal{N}$ be the permutation model of Definition 7. Then,*

$$\mathcal{N} \models \mathbf{HT} \wedge \neg\mathbf{MC}^{\omega} \wedge \neg(\mathbf{DF} = \mathbf{F}).$$

PROOF. Via standard Fraenkel–Mostowski techniques, it can be shown that the family $\mathcal{A} = \{A_i : i \in \omega\}$, which is denumerable in $\mathcal{N}$ since any permutation of A in G fixes $\mathcal{A}$ pointwise, has no multiple choice function in $\mathcal{N}$; hence $\mathbf{MC}^{\omega}$ is false in $\mathcal{N}$—we leave the details to the reader. Furthermore, as shown in the paragraph preceding this theorem, for every $i \in \omega$, A_i is Dedekind-finite in $\mathcal{N}$. Thus, $\mathbf{DF} = \mathbf{F}$ is false in $\mathcal{N}$.[6]

We will now prove that

$$\mathcal{N} \models \text{“Every infinite set has an infinite cuf subset”}.$$

By Theorem 1(5), this will yield $\mathbf{HT}$ is true in $\mathcal{N}$. First, we establish the following lemma.

LEMMA 1. *For every $i \in \omega$, $\mathbf{P}_i$ is cuf in $\mathcal{N}$.*[7]

PROOF. Fix $i \in \omega$. For every $F \in [P_i]^{<\omega} \setminus \{\emptyset\}$, we let

$$Z_F = \Big\{\phi(P_i) : \phi \in \mathrm{Sym}\big(\bigcup F\big)\Big\},$$

where we have identified the group $\mathrm{Sym}(\bigcup F)$ of all permutations of $\bigcup F$ with the subgroup of G comprising all $\phi \in G$ such that $\phi \restriction (A \setminus \bigcup F) = \mathrm{id}$ (i.e. the identity mapping). Clearly, Z_F is finite for all $F \in [P_i]^{<\omega} \setminus \{\emptyset\}$. Let

$$\mathcal{Z}_i = \big\{ Z_F : F \in [P_i]^{<\omega} \setminus \{\emptyset\} \big\}.$$

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[6] We also note that, in [6, Theorem 3.3], it was shown that the statement “The union of a denumerable family of denumerable sets is cuf” is true in $\mathcal{N}$.

[7] Recall that $\mathbf{P}_i = \{\phi(P_i) : \phi \in G\}$, i.e. $\mathbf{P}_i$ is the G-orbit of the partition $P_i = \{A_{i,j} : j \in \omega \setminus \{0\}\}$ of A_i into the j-element sets $A_{i,j}$. Also, using the features of the construction of $\mathcal{N}$, it is not hard to verify that, for every $i \in \omega$, $\mathbf{P}_i$ is *not* a denumerable union of denumerable sets in $\mathcal{N}$.

The collection $\mathcal{Z}_i$ is in $\mathcal{N}$ and is denumerable in $\mathcal{N}$. Indeed, it is easy to see that $\{P_i\}$ is a support of every element of $\mathcal{Z}_i$ (and recall that, for every $\phi \in G$, if ϕ fixes an element Q of $\mathbf{P}_i$, then ϕ fixes Q pointwise). Thus, $\mathcal{Z}_i$ is well orderable in $\mathcal{N}$ and since it is denumerable in the ground model M (because $[P_i]^{<\omega}$ is denumerable in M), it follows that $\mathcal{Z}_i$ is denumerable in $\mathcal{N}$. We assert that

$$\mathbf{P}_i = \bigcup \mathcal{Z}_i.$$

Let $\phi \in G$. Since the set $\{a \in A : \phi(a) \neq a\}$ is finite, so is its subset $U_i = \{a \in A_i : \phi(a) \neq a\}$. Let F be minimal among the non-empty finite subsets of P_i such that $U_i \subseteq \bigcup F$. By definition of U_i, we have that ϕ fixes $A_i \setminus \bigcup F$ pointwise. Let $\eta \in G$ be defined by

$$\eta \restriction \bigcup F = \phi \restriction \bigcup F \quad \text{and} \quad \eta \restriction \Big(A \setminus \bigcup F\Big) = \text{id}.$$

Clearly, $\eta(P_i) \in Z_F$ and $\eta(P_i) = \phi(P_i)$. Hence, $\phi(P_i) \in Z_F \subseteq \bigcup \mathcal{Z}_i$, and therefore $\mathbf{P}_i = \bigcup \mathcal{Z}_i$ as asserted. Thus, $\mathbf{P}_i$ is cuf in the model $\mathcal{N}$, finishing the proof of the lemma. $\square$

Now, we are ready to prove that, in $\mathcal{N}$, every infinite set has an infinite cuf subset. Fix an infinite set X which is in $\mathcal{N}$. If X is well orderable in $\mathcal{N}$, or has an infinite subset which is well orderable in $\mathcal{N}$, then clearly X has an infinite cuf subset in $\mathcal{N}$. So we assume that no infinite subset of X (which is in $\mathcal{N}$) is well orderable in $\mathcal{N}$. Let E be a support of X. By our assumption on X, all but finitely many elements of X are not supported by E (recall the definition of support and see also Jech [12, Equation (4.2), p. 47]). Without loss of generality, we assume that, for every $x \in X$, E is not a support of x. Indeed, if the set $Y = \{x \in X : E \text{ is a support of } x\}$ is infinite, then $Y \in \mathcal{N}$ and Y is well orderable in $\mathcal{N}$ (since E is a support of every element of Y), which is contrary to our assumption on X. It follows that Y is finite and, without loss of generality, we assume $Y = \emptyset$.

Now, if, for every $x \in X$, there exists a support E_x of x such that

$$(\forall i \in \omega)\big[(E_x \cap \mathbf{P}_i \neq \emptyset) \Longleftrightarrow (E \cap \mathbf{P}_i \neq \emptyset)\big], \tag{2}$$

then, for every $x \in X$, $\mathrm{Orb}_{\mathrm{fix}_G(E)}(x)$ is finite. Indeed, let $x \in X$ and also let $E_x \in [\mathbf{P}]^{<\omega}$ be a support of x satisfying (2). Let $I = \{i \in \omega : E \cap \mathbf{P}_i \neq \emptyset\}$. By (2), $I = \{i \in \omega : E_x \cap \mathbf{P}_i \neq \emptyset\}$. Since E is finite, so is I. By equation (1) (at the beginning of this subsection), we conclude that, for every $i \in I$, there exists $j_i \in \omega$ such that, for every $Q \in (E \cup E_x) \cap \mathbf{P}_i$, $Q \supseteq \{A_{i,j} : j > j_i\}$. It follows that any $\phi \in \mathrm{fix}_G(E)$ fixes, for every $i \in I$, $\{A_{i,j} : j > j_i\}$ pointwise. This, together with the facts that $\bigcup\{A_{i,j} : i \in I, j \leq j_i\}$ is finite and E_x is a support of x, easily yields $\mathrm{Orb}_{\mathrm{fix}_G(E)}(x)$ is finite.

Since E is a support of X, we have

$$X = \bigcup \{\mathrm{Orb}_{\mathrm{fix}_G(E)}(x) : x \in X\}.$$

As the collection $\{\mathrm{Orb}_{\mathrm{fix}_G(E)}(x) : x \in X\}$ is well orderable in $\mathcal{N}$ (having E as a support of each of its elements) and, for every $x \in X$, $\mathrm{Orb}_{\mathrm{fix}_G(E)}(x)$ is finite, we conclude that X has an infinite cuf subset in $\mathcal{N}$.

Now, we assume that there exists $x \in X$ such that, for every support E_x of x, equation (2) fails for E_x. In view of the arguments of the previous paragraphs, we may choose such an x in X for which $\mathrm{Orb}_{\mathrm{fix}_G(E)}(x)$ is infinite. Let E_x be a support of x. Since E is not a support of x, we may assume that $E \subsetneq E_x$. This, together with the fact that (2) fails for E_x, yields the existence of an $i \in \omega$ such that $E_x \cap \mathbf{P}_i \neq \emptyset$ and $E \cap \mathbf{P}_i = \emptyset$. For simplicity's sake, and noting (in view of the forthcoming arguments) that, in **ZF**, a finite product of cuf sets is cuf, we assume that $E_x \cap \mathbf{P}_i = \{P_i\}$. Define

$$f = \left\{ \langle \phi(P_i), \phi(x) \rangle : \phi \in \mathrm{fix}_G(E_x \setminus \{P_i\}) \right\}.$$

Then, $f \in \mathcal{N}$ since $E_x \setminus \{P_i\}$ is a support of f.[8] Furthermore, f is a function. Indeed, let $\phi, \psi \in \mathrm{fix}_G(E_x \setminus \{P_i\})$ such that $\phi(P_i) = \psi(P_i)$. Then, $\phi^{-1}\psi \in \mathrm{fix}_G(E_x)$, and thus $\phi^{-1}\psi(x) = x$ since E_x is a support of x. Therefore, $\phi(x) = \psi(x)$, and so f is a function as required.

We also have that $\mathrm{dom}(f) = \mathbf{P}_i$. To see this, let $\phi \in G$. Let η be the element of G which is the identity on $A \setminus A_i$ and which agrees with ϕ on A_i. Clearly, $\eta \in \mathrm{fix}_G(E_x \setminus \{P_i\})$ and $\eta(P_i) = \phi(P_i)$. As $\eta(P_i) \in \mathrm{dom}(f)$, we conclude that $\phi(P_i) \in \mathrm{dom}(f)$. Therefore, $\mathrm{dom}(f) = \mathbf{P}_i$. By Lemma 1, it follows that $\mathrm{dom}(f)$ is a (infinite) cuf set in $\mathcal{N}$; in particular, $\mathrm{dom}(f) = \bigcup\{Z_F : F \in [P_i]^{<\omega} \setminus \{\emptyset\}\}$ (see the proof of Lemma 1).

It is clear that $\mathrm{ran}(f) = \mathrm{Orb}_{\mathrm{fix}_G(E_x \setminus \{P_i\})}(x)$. Moreover, as $E \subseteq E_x \setminus \{P_i\}$, $x \in X$ and E is a support of X, we deduce that $\mathrm{ran}(f) \subseteq X$. Since $\mathrm{Orb}_{\mathrm{fix}_G(E)}(x)$ is infinite, and using, if necessary, an argument similar to the one following equation (2), it is not hard to verify that $\mathrm{ran}(f)$ is an infinite subset of X.

By the proof of Lemma 1, we obtain the following:

$$\begin{aligned}\mathrm{ran}(f) = f[\mathbf{P}_i] &= f\Big[\bigcup\{Z_F : F \in [P_i]^{<\omega} \setminus \{\emptyset\}\}\Big] \\ &= \bigcup \left\{ f[Z_F] : F \in [P_i]^{<\omega} \setminus \{\emptyset\} \right\},\end{aligned}$$

[8] If $E_x \cap \mathbf{P}_i = \{Q_1, \ldots, Q_n\}$, then define $f = \{\langle \phi(\langle Q_1, \ldots, Q_n \rangle), \phi(x) \rangle : \phi \in \mathrm{fix}_G(E_x \setminus \mathbf{P}_i)\}$. The subsequent argument goes through under the obvious, minor changes, taking into account Lemma 1 and the fact that, in **ZF**, a finite product of cuf sets is cuf.

and since (by the proof of Lemma 1) $\{Z_F : F \in [P_i]^{<\omega} \setminus \{\emptyset\}\}$ is, in $\mathcal{N}$, a denumerable cover of $\mathbf{P}_i$ by the finite sets Z_F, and $\mathrm{ran}(f)$ is an infinite subset of X in $\mathcal{N}$, we conclude that $\mathrm{ran}(f)$ is an infinite cuf subset of X in $\mathcal{N}$. This completes the proof of the theorem. □

THEOREM 4. **HT** *is weaker than each of* $\mathbf{MC}^\omega$ *and* $\mathbf{DF} = \mathbf{F}$ *in* **ZF**.

PROOF. Consider the statement

$$\Phi = \mathbf{HT} \wedge \neg\mathbf{MC}^\omega \wedge \neg(\mathbf{DF} = \mathbf{F}).$$

In [8], it was shown that **HT** is an injectively boundable statement (see [8, Theorem 13 and Section 3.3]). Also, both $\neg\mathbf{MC}^\omega$ and $\neg(\mathbf{DF} = \mathbf{F})$ are injectively boundable since both are boundable (recall Fact 1 of Section 3). Thus, Φ is a conjunction of injectively boundable statements and since it has a permutation model (by Theorem 3), it follows from Theorem 2 (of Section 3) that Φ has a **ZF**-model. □

It was an *open problem, until now*, whether or not **HT** implies $\mathbf{MC}^\omega_\omega$ in **ZF**. Furthermore, in Howard and Rubin [10], it is *mentioned as unknown* whether or not there is either a model of **ZF**, or of **ZFA**, in which (the weaker than **HT**) Form 82 (i.e. "The power set of an infinite set is Dedekind-infinite") is true but $\mathbf{MC}^\omega_\omega$ and $\mathbf{AC}^\omega_{\mathbf{fin}}$ are both false. We *completely settle these open problems* by showing next (Cf. Theorems 5, 6 and 7) that:

(3) There is a model of **ZFA** in which $\mathbf{LW} \wedge \mathbf{HT} \wedge \neg\mathbf{MC}^\omega_\omega \wedge \neg\mathbf{AC}^\omega_{\mathbf{fin}}$ is true

and

(4) there is a model of **ZF** in which $\mathbf{HT} \wedge \neg\mathbf{MC}^\omega_\omega \wedge \neg\mathbf{AC}^\omega_{\mathbf{fin}}$ is true.

(Note that (4) strengthens the result of Theorem 4.) To establish (3), we will use a permutation model recently constructed by Howard and Tachtsis [11]. For (4), we will transfer the previous **ZFA**-result (that concerns only the conjunction given by (4), since, in **ZF**, **LW** is equivalent to the full **AC**) into **ZF** by using Pincus' Theorem 2 of Section 3.

The permutation model from [11]. We start with a model M of **ZFA** + **AC** with a denumerable set A of atoms which is written as a disjoint union $\bigcup\{A_n : n \in \omega\}$, where $|A_n| = \aleph_0$ for all $n \in \omega$.

For each $n \in \omega$, let $\mathrm{FSym}(A_n)$ be the group of all permutations of A_n which move only finitely many elements of A_n. Let G be the group of all permutations ϕ of A such that:

1. $\phi \restriction A_n \in \mathrm{FSym}(A_n)$ for all $n \in \omega$,
2. $\phi \restriction A_n = \mathrm{id}_{A_n}$ (the identity function on A_n) for all but finitely many $n \in \omega$.

It follows that, for every $\phi \in G$, $\phi(A_n) = A_n$ for all $n \in \omega$ and ϕ moves only finitely many elements of A. (Note that G is essentially the weak direct product of the groups $\mathrm{FSym}(A_n)$, $n \in \omega$.) Let $\mathcal{F}$ be the filter on the lattice of subgroups of G generated by $\{\mathrm{fix}_G(F) : (\exists S \in [\omega]^{<\omega})(F \subseteq \bigcup\{A_i : i \in S\})\}$; $\mathcal{F}$ is a normal filter on G.

Definition 8. Let $\mathcal{N}$ be the permutation model determined by M, G and $\mathcal{F}$.

By the definition of $\mathcal{F}$, it follows that, if $x \in \mathcal{N}$, then there is a finite $S \subset \omega$ such that $\mathrm{fix}_G(\bigcup\{A_i : i \in S\}) \subseteq \mathrm{Sym}_G(x)$. Under these circumstances, we call $\bigcup\{A_i : i \in S\}$ a *support* of x.

The following result about $\mathcal{N}$ was established in [11, Theorem 3.4 and Remark 3.5].

Theorem 5. *Let $\mathcal{N}$ be the permutation model of Definition* 8. *Then*

$$\mathcal{N} \models \mathbf{LW} \wedge \neg \mathbf{MC}_\omega^\omega \wedge \neg \mathbf{AC}_{\mathbf{fin}}^\omega.$$

We show next that **HT** is true in the model $\mathcal{N}$. Indeed, we have the following theorem.

Theorem 6. *Let $\mathcal{N}$ be the permutation model of Definition* 8. *Then*

$$\mathcal{N} \models \mathbf{HT}.$$

Proof. First, we prove the following lemma.

Lemma 2. *For every $x \in \mathcal{N}$, the G-orbit of x, $\mathrm{Orb}_G(x) = \{\phi(x) : \phi \in G\}$, is countable in $\mathcal{N}$. In particular, every element of $\mathcal{N}$ has a well orderable partition into countable sets in $\mathcal{N}$.*

Proof. Fix $x \in \mathcal{N}$. Let $E = \bigcup\{A_n : n \in S\}$, for some finite $S \subset \omega$, be a support of x. First, we observe that $\mathrm{Orb}_G(x)$ is well orderable in $\mathcal{N}$. Indeed, let $\phi \in G$. Then $\phi(E)$ is a support of $\phi(x)$. Since E is a (finite) union of A_n's and every element of G fixes A_n for all $n \in \omega$, it follows that $\phi(E) = E$. Therefore, E is a support of $\phi(x)$. Since ϕ was arbitrary, we conclude that E is a support of every element of $\mathrm{Orb}_G(x)$, i.e. $\mathrm{Orb}_G(x)$ is well orderable in $\mathcal{N}$.

Now, we assert that

$$\mathrm{Orb}_G(x) = \mathrm{Orb}_{\mathrm{fix}_G(A \setminus E)}(x). \tag{5}$$

Let $\phi \in G$. Let η be the permutation of A in G which agrees with ϕ on E and is the identity on $A \setminus E$. Since ϕ, η agree on E, it follows that $\eta^{-1}\phi \in \mathrm{fix}_G(E)$ and since E is a support of x, $\eta^{-1}\phi(x) = x$, or equivalently $\phi(x) = \eta(x)$. As $\eta(x) \in \mathrm{Orb}_{\mathrm{fix}_G(A \setminus E)}(x)$, it follows that $\phi(x) \in \mathrm{Orb}_{\mathrm{fix}_G(A \setminus E)}(x)$. Hence, $\mathrm{Orb}_G(x) \subseteq \mathrm{Orb}_{\mathrm{fix}_G(A \setminus E)}(x)$, and therefore (5) is true.

The group $\mathrm{fix}_G(A \setminus E)$ is clearly isomorphic to $\prod_{n\in S} \mathrm{FSym}(A_n)$. As, for every $n \in \omega$, A_n is denumerable and every element of $\mathrm{FSym}(A_n)$ moves only finitely many elements of A_n, it follows that $|\mathrm{FSym}(A_n)| = \aleph_0$ for all $n \in \omega$. This, together with the fact that S is finite, yields $\left|\prod_{n\in S} \mathrm{FSym}(A_n)\right| = \aleph_0$, and thus $|\mathrm{fix}_G(A \setminus E)| = \aleph_0$. Therefore, $\left|\mathrm{Orb}_{\mathrm{fix}_G(A\setminus E)}(x)\right| \le \aleph_0$ in M (the ground model), and hence, by (5), $|\mathrm{Orb}_G(x)| \le \aleph_0$ in M. Since $\mathrm{Orb}_G(x)$ is well orderable in $\mathcal{N}$ (as shown in the first paragraph of the proof), it follows that $|\mathrm{Orb}_G(x)| \le \aleph_0$ in $\mathcal{N}$ as required.

For the second assertion of the lemma, note that since E is a support of x, this yields

$$x = \bigcup\{\mathrm{Orb}_{\mathrm{fix}_G(E)}(y) : y \in x\}.$$

The family $\mathcal{Q} = \{\mathrm{Orb}_{\mathrm{fix}_G(E)}(y) : y \in x\}$ is a partition of x and it is well orderable in $\mathcal{N}$ since E is a support of every member of $\mathcal{Q}$. Furthermore, by the first assertion of the lemma, every member of $\mathcal{Q}$ is countable in $\mathcal{N}$. Therefore, x has, in $\mathcal{N}$, a well orderable partition into countable sets, finishing the proof of the lemma. □

To complete the proof of the theorem, fix an infinite set $x \in \mathcal{N}$. By Lemma 2, there exists, in $\mathcal{N}$, a well-orderable partition of x, $\mathcal{O}$ say, such that every member of $\mathcal{O}$ is countable in $\mathcal{N}$. If all elements of $\mathcal{O}$ are finite, then $\mathcal{O}$ is infinite (since x is infinite). It follows that $[x]^{<\omega}$ is Dedekind-infinite in $\mathcal{N}$. If for some $O \in \mathcal{O}$, O is infinite (and thus denumerable in $\mathcal{N}$), then $|[O]^{<\omega}| = \aleph_0$ in $\mathcal{N}$, and thus $[x]^{<\omega}$ is Dedekind-infinite in $\mathcal{N}$.

By the above arguments and Theorem 1(5), we conclude that **HT** is true in $\mathcal{N}$, finishing the proof of the theorem. □

By Theorems 5 and 6, we immediately obtain the following corollary.

Corollary 1. *In* **ZFA**, $\mathbf{LW} \wedge \mathbf{HT}$ *does not imply* $\mathbf{MC}_\omega^\omega \vee \mathbf{AC}_{\mathbf{fin}}^\omega$.

Theorem 7. *In* **ZF**, **HT** *does not imply* $\mathbf{MC}_\omega^\omega \vee \mathbf{AC}_{\mathbf{fin}}^\omega$.

Proof. Let

$$\Pi = \mathbf{HT} \wedge \neg\mathbf{MC}_\omega^\omega \wedge \neg\mathbf{AC}_{\mathbf{fin}}^\omega.$$

Since Π is a conjunction of injectively boundable statements and has (by Theorems 5 and 6) a permutation model, it follows, by Theorem 2, that it has a **ZF**-model. □

Fernández-Bretón [8] showed that, in **ZF**, $\mathbf{HT}_2(k)$ does not imply **RT**, for any integer $k \ge 2$ (see [8, Corollary 29]). This was proved by firstly establishing independence in the Second Fraenkel Model $\mathcal{N}2$ of [10] and then using Pincus' Theorem 2 to transfer the result into **ZF**. Moreover, **CAC** is known to be false in $\mathcal{N}2$ (see [10]), and thus so is **RT** since **RT** implies **CAC**; see, for example, [20, Theorem 1.6]). Therefore, the natural question that emerges is whether or not **CAC** in conjunction with $(\forall k \geq 2)(\mathbf{HT}_2(k))$ implies **RT** in **ZF**. We *settle this open problem* and *strengthen the above result* of [8] by showing next that

$$\text{(6)} \qquad \mathbf{CAC} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))] \not\Rightarrow (\mathbf{HT}_3(2) \vee \mathbf{RT}) \text{ in } \mathbf{ZF}.$$

We will also provide further new information by showing that

$$\text{(7)} \quad \mathbf{LW} \wedge \mathbf{CAC} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))] \not\Rightarrow (\mathbf{HT}_3(2) \vee \mathbf{RT}) \text{ in } \mathbf{ZFA}$$

(recall that **LW** is equivalent to **AC** in **ZF**, but is *not* equivalent to **AC** in **ZFA**). For (6), we will first prove independence in **ZFA**. Then, we will transfer the result to **ZF** using Theorem 2 (of Section 3). For the establishment of (6) and (7) in **ZFA**, we will employ a permutation model constructed by Tachtsis [20] (for the independence of **RT** from **CAC**).

The permutation model from [20]. We start with a model M of **ZFA** + **AC** with a set of atoms $A = \bigcup\{A_i : i \in \omega\}$ which is a denumerable disjoint union of pairs $A_i = \{a_i, b_i\}$, $i \in \omega$. Let G be the group of all permutations ϕ of A such that ϕ moves only finitely many atoms and, for every $i \in \omega$, there exists $k \in \omega$ such that $\phi(A_i) = A_k$. Let Γ be the (normal) filter of subgroups of G generated by $\{\operatorname{fix}_G(E) : E \in [A]^{<\omega}\}$.

DEFINITION 9. We define $\mathcal{N}$ to be the permutation model determined by M, G and Γ.

By the definition of Γ, it follows that, if $x \in \mathcal{N}$, then there exists a finite $S \subset \omega$ such that $\operatorname{fix}_G\big(\bigcup\{A_i : i \in S\}\big) \subseteq \operatorname{Sym}_G(x)$. Under these circumstances, we call $\bigcup\{A_i : i \in S\}$ a *support* of x.

Let us recall the following facts about $\mathcal{N}$, which were established in [20, Theorem 2.1].

FACT 2. 1. *The family* $\mathcal{A} = \{A_i : i \in \omega\}$ *is amorphous in* $\mathcal{N}$ *and has no partial choice function in* $\mathcal{N}$. *Thus,* A *is amorphous in* $\mathcal{N}$.[9]

2. **RT** *is false in* $\mathcal{N}$.

3. *Every element* x *of* $\mathcal{N}$ *is either well-orderable or has an infinite subset* y *with a partition into sets each of cardinality at most* 2, *indexed by a co-finite subset of* $\mathcal{A}$, *and thus indexed by an amorphous set. In the second case, it follows that* y *is an amorphous subset of* x.

4. *The union of a well-orderable family of well-orderable sets in* $\mathcal{N}$ *is well orderable in* $\mathcal{N}$ (*and thus* $\mathbf{MC}^{\omega}_{\omega}$ *and* $\mathbf{AC}^{\omega}_{\mathbf{fin}}$ *are both true in* $\mathcal{N}$).

5. **CAC** *is true in* $\mathcal{N}$.

[9] In [22, Theorem 5.1], we further showed that every amorphous set in $\mathcal{N}$ is bounded; in particular, if $x \in \mathcal{N}$ is amorphous and Π is a partition of x into infinitely many pieces in $\mathcal{N}$, then all but finitely many elements of Π have the same cardinality, which is less than or equal to 2.

For the proof of the forthcoming Theorem 8, we will need parts of the argument for Fact 2(3). Therefore, for the reader's convenience, we provide a sketch of the argument and label Fact 2(3) as Lemma 3 below.

LEMMA 3. *Let $\mathcal{N}$ be the permutation model of Definition* 9. *Then, every element x of $\mathcal{N}$ is either well-orderable or has an infinite subset y with a partition into sets each of cardinality at most* 2, *indexed by a co-finite subset of $\mathcal{A} = \{A_i : i \in \omega\}$, and thus indexed by an amorphous set. In the second case, it follows that y is an amorphous subset of x.*

PROOF. Let $x \in \mathcal{N}$ be non-well-orderable in $\mathcal{N}$. Let $E = \bigcup\{A_i : i \in K\}$ for some finite $K \subset \omega$, be a support of x. Since x is not well orderable in $\mathcal{N}$, there exists $z \in x$ which is not supported by E. Let $E_z = \bigcup\{A_i : i \in K'\}$ for some finite $K' \subset \omega$, be a support of z. Since E is not a support of z, it follows that $E_z \setminus E \neq \emptyset$. Without loss of generality, assume that $E \subsetneq E_z$ and also assume that $\{A_i : i \in K'\}$ has the fewest possible copies A_j outside $\{A_i : i \in K\}$. Let $i_0 \in K'$ such that $A_{i_0} \cap E = \emptyset$, where $A_{i_0} = \{a_{i_0}, b_{i_0}\}$.

We define the following set:

$$f = \left\{ \langle \phi(z), \phi(A_{i_0}) \rangle : \phi \in \operatorname{fix}_G(E_z \setminus A_{i_0}) \right\}.$$

Then, f has the following properties:

1. $f \in \mathcal{N}$ (since $E_z \setminus A_{i_0}$ is a support of f).

2. f is a function with $\operatorname{dom}(f) \subseteq x$ and $\operatorname{ran}(f) = \mathcal{A} \setminus \{A_i : i \in K', i \neq i_0\}$, where $\mathcal{A} = \{A_i : i \in \omega\}$. That f is a function follows from the fact that, if $w \in \mathcal{N}$ and E_1, E_2 are supports of w, then $E_1 \cap E_2$ is a support of w; see [20, Lemma 1].

3. For every $\phi \in \operatorname{fix}_G(E_z \setminus A_{i_0})$, we have:

$$f^{-1}(\{\phi(A_{i_0})\}) = \{\phi(z), \phi\rho(z)\},$$

where $\rho = (a_{i_0}, b_{i_0})$, i.e. ρ interchanges a_{i_0} and b_{i_0} and fixes all other atoms. It follows that the family

$$\begin{aligned} \mathcal{Y} &= \left\{ f^{-1}(\{w\}) : w \in \mathcal{A} \setminus \{A_i : i \in K', i \neq i_0\} \right\} \\ &= \left\{ \{\phi(z), \phi\rho(z)\} : \phi \in \operatorname{fix}_G(E_z \setminus A_{i_0}) \right\} \end{aligned}$$

is a partition of the set $y = \operatorname{dom}(f)$ which is amorphous in $\mathcal{N}$ since, in $\mathcal{N}$, $\mathcal{Y}$ is equipotent to $\mathcal{A} \setminus \{A_i : i \in K', i \neq i_0\}$ and the latter set is (by Fact 2(1)) amorphous in $\mathcal{N}$. Furthermore, y is amorphous in $\mathcal{N}$. This yields that either for all but finitely many $u \in \mathcal{Y}$, $|u| = 1$ or for all but finitely many $u \in Y$, $|u| = 2$.

The proof of the lemma is complete. $\square$

THEOREM 8. *Let $\mathcal{N}$ be the permutation model of Definition* 9. *Then,*

$$\mathcal{N} \models \mathbf{LW} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))] \wedge \neg\mathbf{HT}_3(2)$$

and thus

$$\mathcal{N} \models \mathbf{LW} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))] \wedge \neg\mathbf{HT}.$$

Proof. We start by proving **LW** is true in $\mathcal{N}$. Indeed, we have the following claim.

Claim. $\mathcal{N} \models \mathbf{LW}$.

Proof. Fix a linearly ordered set $(L, \leq)$ in $\mathcal{N}$. Let E be a support of $(L, \leq)$. We will show that $\mathrm{fix}_G(E) \subseteq \mathrm{fix}_G(L)$, and thus $\mathrm{fix}_G(L) \in \Gamma$; this will yield L is well orderable in $\mathcal{N}$ (see Jech [12, Equation (4.2), p. 47]). Let $x \in L$ and let $\phi \in \mathrm{fix}_G(E)$. By way of contradiction, assume $\phi(x) \neq x$. Then either $x < \phi(x)$ or $\phi(x) < x$. Since every element of G moves only finitely many atoms, there exists $k \in \omega \setminus \{0\}$ such that ϕ^k is the identity mapping on A. For such a k, and assuming $x < \phi(x)$, we obtain the following:

$$x < \phi(x) < \phi^2(x) < \cdots < \phi^{k-1}(x) < \phi^k(x) = x,$$

and thus $x < x$, which is a contradiction. If $\phi(x) < x$, then similarly we obtain the contradiction "$x < x$". Therefore, $\phi(x) = x$, and consequently $\mathrm{fix}_G(E) \subseteq \mathrm{fix}_G(L)$ as required. Hence, **LW** is true in $\mathcal{N}$. □

Claim. $\mathcal{N} \models (\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))$.

Proof. Fix $k \in \omega \setminus \{0,1\}$. Let $x \in \mathcal{N}$ be infinite and also let $c : [x]^{<\omega} \to k$ such that $c \in \mathcal{N}$. If x is well orderable in $\mathcal{N}$, then the conclusion follows immediately from Theorem 1(1). So assume x is not well orderable in $\mathcal{N}$.

Let $E = \bigcup\{A_i : i \in K\}$ for some finite $K \subset \omega$, be a support of x and c. Let z, E_z, f, $\mathcal{Y}$ $(= \{f^{-1}(\{A_i\}) : i \in \omega \setminus (K' \setminus \{i_0\})\})$ and y $(= \mathrm{dom}(f))$ be given as in the proof of Lemma 3. We consider the following two cases:

Case 1: for all but finitely many $u \in \mathcal{Y}$, $|u| = 2$ (recall that, by the proof of Lemma 3, y is amorphous in $\mathcal{N}$). Without loss of generality, we assume that, for every $u \in \mathcal{Y}$, $|u| = 2$; otherwise, we may follow the subsequent argument by merely replacing $\mathcal{Y}$ by $\mathcal{Y} \setminus \{u \in \mathcal{Y} : |u| = 1\}$.

Since $E_z \setminus A_{i_0}$ is a support of c (because E is a support of c and $E \subset E_z \setminus A_{i_0}$), it follows that the collection

$$\mathcal{Z} = \{u \cup v : u, v \in \mathcal{Y},\ u \neq v\},$$

which comprises 4-element subsets of x, is c-monochromatic. Indeed, fix any $i \in \omega \setminus K'$. First, note that

$$\mathcal{Z} = \mathrm{Orb}_{\mathrm{fix}_G(E_z \setminus A_{i_0})}\left(f^{-1}(\{A_{i_0}\}) \cup f^{-1}(\{A_i\})\right)$$

(and recall that, by the proof of Lemma 3, $E_z \setminus A_{i_0}$ is a support of f); hence $\mathcal{Z} \in \mathcal{N}$ since $E_z \setminus A_{i_0}$ is a support of $\mathcal{Z}$. Therefore, if $c(f^{-1}(\{A_{i_0}\})$

$\cup f^{-1}(\{A_i\})) = m$ for some $m \in k$, then since $E_z \setminus A_{i_0}$ is a support of c, we obtain that

$$\forall Z \in \mathcal{Z}(c(Z) = m). \tag{8}$$

We let

$$Y = \left\{ f^{-1}(\{A_{i_0}\}) \cup f^{-1}(\{A_j\}) : j \in \omega \setminus K' \right\}.$$

Then $Y \in \mathcal{N}$ since E_z is a support of Y, and $Y \subseteq \mathcal{Z} \subseteq [x]^{<\omega}$. Furthermore, by the definition of Y and the fact that $\mathcal{Y}$ is a partition of y, it follows that

$$\mathrm{FS}_{\leq 2}(Y) \subseteq \mathcal{Z}.$$

Thus, by equation (8), we conclude that

$$\forall t \in \mathrm{FS}_{\leq 2}(Y)(c(t) = m),$$

i.e. $\mathrm{FS}_{\leq 2}(Y)$ is c-monochromatic in $\mathcal{N}$ (and note that E_z is a support of $\mathrm{FS}_{\leq 2}(Y)$, so $\mathrm{FS}_{\leq 2}(Y) \in \mathcal{N}$).

Case 2: for all but finitely many $u \in \mathcal{Y}$, $|u| = 1$. Without loss of generality, assume that every member of $\mathcal{Y}$ is a singleton. We may work similarly to case 1 considering the corresponding families $\mathcal{Z}$ and Y, and thus concluding that $\mathrm{FS}_{\leq 2}(Y)$ is c-monochromatic in $\mathcal{N}$. We take the liberty to leave the details to the reader. □

Claim. $\mathcal{N} \models \neg\mathbf{HT}_3(2)$, *and thus* $\mathcal{N} \models \neg\mathbf{HT}_3(k)$ *for any integer* $k \geq 2$.

Proof. We will show that for the infinite set $\mathcal{A} = \{A_i : i \in \omega\}$, which is (an) amorphous (partition of the set A of atoms) in $\mathcal{N}$ (see Fact 2(1)), there is a colouring $c : [\mathcal{A}]^{<\omega} \to 2$ which is in $\mathcal{N}$ and is such that, for every infinite $Y \subseteq [\mathcal{A}]^{<\omega}$, $\mathrm{FS}_{\leq 3}(Y)$ is not c-monochromatic in $\mathcal{N}$. To this end, we define a colouring $c : [\mathcal{A}]^{<\omega} \to 2$ as follows: For every $X \in [\mathcal{A}]^{<\omega}$, let n, i with $0 \leq i < 4$ be such that $X \in [\mathcal{A}]^{4n+i}$ (this is simply Euclid's division algorithm), and then define

$$c(X) = \begin{cases} 0, & \text{if } i \in \{1,2\}; \\ 1, & \text{if } i \in \{0,3\}. \end{cases}$$

(In fact, every colouring c defined in terms of cardinality satisfying that, if $|Y| = |X| + 2$ then $c(X) \neq c(Y)$, works for this purpose.) Then, the colouring c is in $\mathcal{N}$ since $\mathrm{Sym}_G(c) = G \in \Gamma$ (i.e. c is fixed by every element of G). We now show that, for every infinite $Y \subseteq [\mathcal{A}]^{<\omega}$ which is in $\mathcal{N}$, $\mathrm{FS}_{\leq 3}(Y)$ is not c-monochromatic in $\mathcal{N}$.

Let Y be an infinite subset of $[\mathcal{A}]^{<\omega}$ which is in $\mathcal{N}$ (note that $\bigcup Y$ is cofinite in $\mathcal{A}$ since $\mathcal{A}$ is amorphous in $\mathcal{N}$). Let E be a finite subset of $\mathcal{A}$ such that $\bigcup E$ is a support of Y. As E is finite and Y is infinite, there exists $Z \in Y$ such that $Z \nsubseteq E$. Pick any $z \in Z \setminus E$ and pick $z', z'' \in \mathcal{A} \setminus (E \cup Z)$; it follows that z, z', z'' are pairwise disjoint and also disjoint from $\bigcup E$. Letting π' be a permutation of A exchanging z and z' and fixing all other atoms (since z and z' have two atoms each, there are several ways of doing this, but any will do) and π'' be another permutation of A exchanging z and z'', and fixing everything else, we define

$$Z' = \pi'(Z) = (Z \cup \{z'\}) \setminus \{z\} \text{ and } Z'' = \pi''(Z) = (Z \cup \{z''\}) \setminus \{z\}.$$

Since $Z \in Y$, $\pi', \pi'' \in \mathrm{fix}_G(\bigcup E)$ and $\bigcup E$ is a support of Y, we deduce that $Z' = \pi'(Z) \in \pi'(Y) = Y$, $Z'' = \pi''(Z) \in \pi''(Y) = Y$, and so $Z \triangle Z' \triangle Z'' \in \mathrm{FS}_{\leq 3}(Y)$. However, we have

$$Z \triangle Z' \triangle Z'' = Z \cup \{z', z''\},$$

and so

$$|Z \triangle Z' \triangle Z''| = |Z| + 2,$$

which yields $c(Z \triangle Z' \triangle Z'') \neq c(Z)$ by definition of c. Therefore, $\mathrm{FS}_{\leq 3}(Y)$ is not c-monochromatic in $\mathcal{N}$, and so $\mathbf{HT}_3(2)$ fails in $\mathcal{N}$, finishing the proof of the claim. □

The proof of the theorem is complete. □

By Fact 2 and Theorem 8, we immediately obtain the following result.

Theorem 9. *In* **ZFA**, $\mathbf{LW} \wedge \mathbf{CAC} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))]$ *does not imply* $\mathbf{HT}_3(2) \vee \mathbf{RT}$.

Theorem 10. *In* **ZF**, $\mathbf{CAC} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))]$ *does not imply* $\mathbf{HT}_3(2) \vee \mathbf{RT}$.

Proof. Consider the following statement:

$$\Phi = \mathbf{CAC} \wedge [(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))] \wedge \neg\mathbf{HT}_3(2) \wedge \neg\mathbf{RT}.$$

By Fact 2 and Theorem 8, we know that Φ is true in the permutation model $\mathcal{N}$ of Definition 9. In [20, proof of Theorem 2.3], it was shown that **CAC** is injectively boundable. Furthermore, $\neg\mathbf{HT}_3(2)$ and $\neg\mathbf{RT}$ are injectively boundable since they are boundable.

Claim. *The statement* $(\forall k \geq 2)(\mathbf{HT}_2(k))$ *is injectively boundable.*

Proof. First, we consider the following formula:

$$\Psi(x) = (\forall k \in \omega \setminus \{0,1\})(\forall c : [x]^{<\omega} \to k)(\exists \text{ infinite } y \subseteq [x]^{<\omega}) \\ (c \restriction \mathrm{FS}_{\leq 2}(y) \text{ is constant}).$$

It is not hard to verify that all bound variables of $\Psi(x)$ can be relativized to $\wp^{\omega+\omega}(x)$ (noting also that $[x]^{<\omega} \in \wp^2(x) \subseteq \wp^{\omega+\omega}(x)$ and that $\omega \in \wp^{\omega+\omega}(x)$). Thus, $\Psi(x)$ is equivalent to $\Psi^{\wp^{\omega+\omega}(x)}(x)$, i.e. $\Psi(x)$ is a boundable formula. Furthermore, note that "$(\forall k \in \omega \setminus \{0,1\})(\mathbf{HT}_2(k))$" is equivalent to

$$(\forall x)(|x|_- \leq \omega \rightarrow \Psi(x)),$$

and thus is injectively boundable. $\square$

In view of the above, we conclude that Φ is a conjunction of injectively boundable statements and since it has a permutation model (by Theorem 8), it follows from Theorem 2 (of Section 3) that Φ has a **ZF**-model. $\square$

5. Open questions and directions for further study

1. Does $\mathbf{HT}_3(2)$ imply **HT**?
2. Does $\mathbf{HT}_3(2)$ imply "There are no amorphous sets"?
3. Does $\mathbf{HT}_2(2)$ imply $\mathbf{HT}_2(3)$?
4. Does **CAC** imply $(\exists k \geq 2)(\mathbf{HT}_2(k))$ or $(\forall k \geq 2)(\mathbf{HT}_2(k))$? [Recall that, by Theorem 10, **CAC** does not imply $\mathbf{HT}_3(2)$ in **ZF**, and thus neither does it imply $\mathbf{HT}_3(k)$ in **ZF**, for any integer $k \geq 2$.]
5. Is **CAC** true in the Basic Cohen Model $\mathcal{M}1$ of [10]? [We recall that, in [8, Theorem 30], it was shown that $\mathbf{HT}_2(2)$ (and thus $\mathbf{HT}_2(k)$ for any integer $k \geq 2$) is false in $\mathcal{M}1$. We conjecture that **CAC** is true in $\mathcal{M}1$: It is known that, for every $X \in \mathcal{M}1$, there is, in $\mathcal{M}1$, an ordinal γ and a one-to-one function $f : X \rightarrow [A]^{<\omega} \times \gamma$, where A is the denumerable set of the added Cohen reals which is Dedekind-finite in $\mathcal{M}1$; see Jech [12, Lemmas 5.15, 5.25]. (Note that this readily yields (Form 82) $\wedge \neg\mathbf{HT}$ is true in $\mathcal{M}1$, and thus Form 82 does not imply **HT** in **ZF**.) So, in particular, every infinite poset $(P, \leq)$ in $\mathcal{M}1$ has a well-ordered partition $\{P_\alpha : \alpha < \gamma\}$ (γ some ordinal) into the Dedekind-finite sets P_α (each of which can be identified with a subset of $[A]^{<\omega}$). Our conjecture is that the P_α's are anti-chains in P. Hence, if all chains and anti-chains in P are finite in $\mathcal{M}1$, then since "The union of a well-orderable family of well-orderable sets is well orderable" is true in $\mathcal{M}1$ (see [12, Problem 22, p. 82]), it follows that P is well orderable in $\mathcal{M}1$. But then, it can be shown, without using any form of choice, that P has either an infinite chain or an infinite anti-chain (see, for example, [20, proof of Claim 5]); this is a contradiction. However, a complete argument justifying that (in $\mathcal{M}1$) the P_α's are anti-chains in P, still eludes us.]
6. Does $\mathbf{BPI} \wedge [(\forall k \geq 2)(\mathbf{HT}_2(k))]$ (where **BPI** denotes the Boolean prime ideal theorem, i.e. the statement "Every Boolean algebra has a prime ideal") imply **RT** in **ZF**? [**BPI** does not imply $\mathbf{HT}_2(2)$ in **ZF**; see [8, Theorem 30]. We also note that it is an *open problem* whether or not **BPI** implies **CAC** in **ZF**.]

Question (1) of the above list was originally posed in [3, Question 5.1(1)] and question (3) (as well as (1)) was posed in [8, Questions 27, 34].

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