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COMMENTS ON THE PAPER "BEST PROXIMITY POINT RESULTS FOR P-PROXIMAL CONTRACTIONS"

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Abstract. Recently, Altun et al. [1] introduced the notion of p-proximal contractions and p-proximal contractive mappings and discussed about best proximity point results for this two classes of mappings. After that, Gabeleh and Markin [3] showed that the best proximity point theorem for p-proximal contractions proved in [1] follows from the same conclusion fixed point theory. In this short note, we show that the best proximity point theorem for p-proximal contractive mappings follows from the corresponding fixed point result for metric spaces.

1. Introduction

Let A and B be non-empty subsets of a metric space (M,d) and $Q\colon A\to B$ be a non-self mapping. A necessary condition, to guarantee the existence of solutions of the equation $Qx=x$, is $Q(A)\cap A\neq\phi$. If $Q(A)\cap A=\phi$ then the equation $Qx=x$ has no solutions. In this case, one seeks for an element in the domain space whose distance from its image is minimum i.e, one interesting problem is to minimize $d(x,Qx)$ such that $x\in A$. Since $d(x,Qx)\geq \operatorname{dist}(A,B)=\inf\{d(x,y): x\in A,\ y\in B\}$, so, one searches for an element $x\in A$ such that $d(x,Qx)=\operatorname{dist}(A,B)$. Best proximity point problems deal with this situation. Authors usually discover best proximity point theorems to generalize the corresponding fixed point results in metric spaces. Recently, Altun et al. [1] introduced the notion of p-proximal contractions and p-proximal contractive mappings and discussed about best proximity point results for these two classes of mappings. After that, Gabeleh and Markin [3] showed that the best proximity point theorem

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proved in [1] for p-proximal contractions follows from a result in fixed point theory. Then, we showed in [4] that if the p-proximal contraction constant $k < \frac{1}{3}$ then the best proximity point result for p-proximal contractions in [1] follows from the Banach contraction principle. In this short note, we consider the case of p-proximal contractive mappings and show that the best proximity point result for p-proximal contractive mappings follows from a fixed point result proved in [2] for metric spaces.

2. Main results

We first recall the following definition of p-proximal contractive mapping from [1] as follows.

Definition 2.1 [1, Definition 5]. Let (A, B) be a pair of non-empty subsets of a metric space (M, d). A mapping $T\colon A \to B$ is said to be a p-proximal contractive mapping if

$$\left.\begin{aligned} d(u_1, Tx_1) &= \operatorname{dist}(A, B) \\ d(u_2, Tx_2) &= \operatorname{dist}(A, B) \end{aligned}\right\} \implies d(u_1, u_2) < d(x_1, x_2) + |d(u_1, x_1) - d(u_2, x_2)|$$

for all $u_1, u_2, x_1, x_2 \in A$ with $x_1 \neq x_2$.

In this paper, the following notations will be needed. Let (M, d) be a metric space and A, B be non-empty subsets of M. Then

$$A_0 = \{x \in A : d(x, y) = \operatorname{dist}(A, B) \text{ for some } y \in B\};$$

$$B_0 = \{y \in B : d(x, y) = \operatorname{dist}(A, B) \text{ for some } x \in A\}.$$

In [1, Theorem 4], Altun et al. proved the following best proximity point result for p-proximal contractive mappings.

Theorem 2.2 [1, Theorem 4]. *Let (M, d) be a metric space, A, B be non-empty subsets of M and the mapping $T\colon A \to B$ be a p-proximal contractive mapping. Assume that (A, B) has the P-property and $T(A_0) \subseteq B_0$. If there exist $p, q \in A_0$ such that*

$$d(q, Tp) = \operatorname{dist}(A, B) \quad \text{and} \quad d(p, q) \leq d(Tp, Tq)$$

then $p = q$ and so T has a unique best proximity point.

In [2], Altun et al. proved the following fixed point theorem.

Theorem 2.3 [2, Theorem 2.9]. *Let (M, d) be a metric space. Let $T\colon M \to M$ be a p-contractive self mapping and f be a function defined by $f(x) = d(x, Tx)$. If there exists $x_0 \in M$ such that $f(x_0) \leq f(Tx_0)$, then T has a unique fixed point in M.*

Altun et al. proved Theorem 2.2 to generalize the above fixed point result in case of non-self mappings. In our next main result, we show that the best proximity point result (Theorem 2.2) follows from the corresponding fixed point result for metric spaces.

THEOREM 2.4. *Theorem* 2.2 *is a straightforward consequence of Theorem* 2.3.

PROOF. Let $x \in A_0$. As $T(A_0) \subseteq B_0$, so, $Tx \in B_0$. This implies that there exists $y \in A_0$ such that $d(y, Tx) = \operatorname{dist}(A, B)$. We show that $y \in A_0$ is unique. Suppose there exists $y_1, y_2 \in A_0$ such that $d(y_1, Tx) = \operatorname{dist}(A, B)$ and $d(y_2, Tx) = \operatorname{dist}(A, B)$. Since the pair (A, B) have the P-property so, we have

$$d(y_1, y_2) = d(Tx, Tx) \implies d(y_1, y_2) = 0 \implies y_1 = y_2.$$

Define a mapping $S\colon A_0 \to A_0$ by $Sx = y$ having the property that $d(Sx, Tx) = \operatorname{dist}(A, B)$. Now, we show that $S\colon A_0 \to A_0$ is a p-contractive mapping. Let $x_1, x_2 \in A_0$ with $x_1 \neq x_2$. Since $d(Sx_1, Tx_1) = \operatorname{dist}(A, B)$ and $d(Sx_2, Tx_2) = \operatorname{dist}(A, B)$ and $T\colon A \to B$ is a p-proximal contractive mapping, so we have,

$$d(Sx_1, Sx_2) < d(x_1, x_2) + |d(Sx_1, x_1) - d(Sx_2, x_2)|.$$

This shows that $S\colon A_0 \to A_0$ is a p-contractive mapping. It is given that there exist $p, q \in A_0$ such that

$$d(q, Tp) = \operatorname{dist}(A, B) \quad \text{and} \quad d(p, q) \leq d(Tp, Tq).$$

Now, since $d(Sp, Tp) = \operatorname{dist}(A, B)$ and the pair (A, B) has the P-property so,

$$d(Sp, q) = d(Tp, Tp) \implies q = Sp.$$

Also, since $d(Sp, Tp) = \operatorname{dist}(A, B)$ and $d(Sq, Tq) = \operatorname{dist}(A, B)$, so, we have

$$d(Sp, Sq) = d(Tp, Tq).$$

Let $f\colon A_0 \to \mathbb{R}$ be defined by $f(x) = d(x, Sx)$, $x \in A_0$. Then by the given condition we have

$$\begin{aligned} d(p, q) \leq d(Tp, Tq) &\implies d(p, Sp) \leq d(Sp, Sq) \\ \implies d(p, Sp) \leq d(Sp, S(Sp)) &\implies f(p) \leq f(Sp). \end{aligned}$$

This shows that there exists $p \in A_0$ such that $f(p) \leq f(Sp)$. So, by [2, Theorem 2.9], the mapping $S : A_0 \to A_0$ has a fixed point in A_0, i.e, there exists $z \in A_0$ such that $Sz = z$. So, $d(z, Tz) = d(Sz, Tz) = \operatorname{dist}(A, B)$. This shows that $z \in A_0$ is a best proximity point of the mapping $T\colon A \to B$. Uniqueness of best proximity points already proved in [1, Theorem 4], so omitted. $\square$

Now, we apply our result to [1, Example 2] to validate our claim. In, [1, Example 2], Altun et al. considered the metric space $M = [0,1] \times [0,1]$ with respect to usual standard metric, $A = [0,1] \times \{1\}$ and $B = [0,1] \times \{0\}$. It is clear that $\operatorname{dist}(A,B) = 1$ and $A_0 = A$. In [1, Example 2], authors considered the mapping $T\colon A \to B$ defined by

$$T(t,1) = \Big(\frac{t}{2}, 0\Big), \quad (t,1) \in A.$$

Now, we construct the function $S\colon A_0 \to A_0$. Let $(t_0,1) \in A_0$ and $(s,1) \in A_0$ be such that

$$d\big((s,1), T(t_0,1)\big) = \operatorname{dist}(A,B) \implies d\Big((s,1), \Big(\frac{t_0}{2}, 0\Big)\Big) = 1$$

$$\implies \sqrt{\Big(s - \frac{t_0}{2}\Big)^2 + 1} = 1 \implies s = \frac{t_0}{2}.$$

So, $S\colon A_0 \to A_0$ be defined by $S(t_0,1) = (\frac{t_0}{2}, 1)$, $(t_0,1) \in A_0$ and $(0,1)$ is the unique fixed point of the mapping S. So, $(0,1)$ is the unique best proximity point of the mapping $T\colon A \to B$.

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