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EXPONENTIAL SUMS WITH COEFFICIENTS OF THE LOGARITHMIC DERIVATIVE OF AUTOMORPHIC *L*-FUNCTIONS AND APPLICATIONS

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Abstract. Let $\Lambda_\pi(n)$ denote the nth coefficient in the Dirichlet series expansion of the logarithmic derivative of $L(s,\pi)$ associated with an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$. In this paper, for all α of irrational type 1 lying in the interval $[0,1]$, we investigate the best possible estimate for the sum $\sum_{n\le x}\Lambda_\pi(n)e(n\alpha)$ under a certain assumption. And we consider the metric result on the exponential sum involving automorphic L-functions without any assumptions. Let $\Lambda(n)$ be the von Mangoldt function. Then as an application, for $\varepsilon>0$ and all $0<\alpha<1$ in a set of full Lebesgue measure (depending on π), we obtain $\sum_{n\le x}\Lambda(n)\lambda_\pi(n)e(n\alpha)=O(x^{\frac{5}{6}+\varepsilon})$.

1. Introduction

Let $m\geqslant 2$ be an integer, and let $\mathcal{A}(m)$ be the set of all cuspidal automorphic representations of GL_m over $\mathbb{Q}$ with unitary central character. Fix $\pi\in\mathcal{A}(m)$. The standard L-function $L(s,\pi)$ associated to π is of the form

$$L(s,\pi)=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s},\quad \Re s>1. \tag{1.1}$$

The inverse function $L^{-1}(s,\pi)$ can be written as

$$L^{-1}(s,\pi)=\sum_{n=1}^{\infty}\frac{\mu_\pi(n)}{n^s}. \tag{1.2}$$

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Taking the logarithmic derivative for $L(s,\pi)$, we define, for $\Re s>1$,

$$-\frac{L'}{L}(s,\pi)=\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s}.$$

Suppose

$$\sum_{n\le x}\mu_\pi(n)=O(x^{\theta+\varepsilon}) \tag{1.3}$$

for $0<\theta<1$ and $\varepsilon>0$. Then by partial summation, we have

$$L^{-1}(s,\pi)=s\int_1^\infty\frac{\sum_{n\le x}\mu_\pi(n)}{x^{s+1}}\,dx.$$

By a well-known theorem of Landau (see [14, Theorem 10.1.4]), we obtain that $L^{-1}(s,\pi)$ converges for $\Re s>\theta$. Therefore, $L(s,\pi)\ne 0$ for $\Re s>\theta$, which we understand as an analytic meaning of the Grand Riemann Hypothesis (GRH). Actually, the GRH is equivalent to the assertion that (1.3) holds for $\theta=1/2$.

Let χ be a primitive Dirichlet character of modulus q. The arithmetic conductors of π and χ are coprime. According to [5, Proposition 5.14], we can consider

$$\sum_{n\le x}\Lambda_\pi(n)\chi(n)$$

to grasp the analytic meaning of the GRH. By Gauss sums, we write

$$\sum_{n\le x}\Lambda_\pi(n)\overline{\chi}(n)=\frac{1}{\tau(\chi)}\sum_{a \bmod q}\chi(a)\sum_{n\le x}\Lambda_\pi(n)e(na/q),$$

where $e(z)=e^{2\pi iz}$ for $z\in\mathbb{R}$. If we estimate $\sum_{n\le x}\Lambda_\pi(n)e(na/q)=O(x^\theta)$, we deduce that $\frac{L'}{L}(s,\pi\otimes\overline{\chi})$ has no zeros for $\Re s>\theta$. Naturally, we are motivated to consider the exponential sum

$$\sum_{n\le x}\Lambda_\pi(n)e(n\alpha) \tag{1.4}$$

for any real α.

In this paper, we follow the method in [15] to study the sum (1.4) for all α of irrational type 1. According to Kuipers and Niederreiter [11], we have the definition of an irrational number of type τ.

Definition 1.1. Let τ be a positive real number or infinity. The irrational number α is said to be of type τ if τ is the supremum of all t for which

$$\tau = \sup \Big\{ t \in \mathbb{R} : \liminf_{n\to\infty} n^t \|\alpha n\| = 0 \Big\},$$

where q runs through the positive integers. Here $\|\cdot\|$ denotes the distance to a nearest integer.

Our goal here is to establish the following result.

Theorem 1.2. *For every fixed* $\frac{1}{2} < \theta < 1$ *and for all* α *of irrational type* 1, *suppose that*

$$\sum_{n\le x} \lambda_\pi(n) e(n\alpha) \ll x^\theta. \tag{1.5}$$

Then

$$\sum_{n\le x} \Lambda_\pi(n) e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{4-3\theta}{5-4\theta}+\varepsilon}$$

for every $\varepsilon > 0$.

We expect to obtain a similar conclusion of the exponential twists of the coefficients in the Dirichlet series expansion for $L(\pi,s)^{-1}$.

Corollary 1.3. *Under the condition* (1.5) *we have*

$$\sum_{n\le x} \mu_\pi(n) e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{4-3\theta}{5-4\theta}+\varepsilon}$$

for every $\varepsilon > 0$.

Morever, applying to Theorem 1.2 , we can estimate sums of the form

$$S(x) = \sum_{n\le x} \Lambda(n)\lambda_\pi(n) e(n\alpha), \tag{1.6}$$

where $\Lambda(n)$ is the von Mangoldt function defined by

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

$\Lambda(n)$ is somehow the characteristic function of primes. Sums concerning prime numbers are important problems in analytic number theory. One can predict the asymptotic behavior of (1.6), and hence some properties of the distribution of primes. The study of the sum (1.6) is well understood when the $\lambda_\pi(n)$ are the normalized Fourier coefficients of a modular or Maass form on the upper half plane. Perelli [16] studied the exponential sums connected with Ramanujan's τ-function, and showed that the bound

$$\sum_{n\le N}\Lambda(n)\tau(n)e(n\alpha)\ll N^{11/2}\big(Nq^{-1/2}+N^{1/2}q^{1/2}+N^{5/6}\big)\log^c N,$$

holds for some suitable constant $c>0$. Here, $\alpha\in\mathbb{R}$ is such that

$$\Big|\alpha-\frac{l}{q}\Big|\le\frac{1}{q^2},\quad (l,q)=1,\ q\ge 1$$

for some integers l, q, and the implied constant depends only on the τ-function. It can be seen that his result also holds for the normalized Fourier coefficients of a primitive holomorphic cusp form. Fouvry and Ganguly [2] obtained a strong bound which states that there exists an effective constant $c>0$ such that, for any $\alpha\in\mathbb{R}$,

$$\sum_{n\le N}\Lambda(n)\lambda_\pi(n)e(n\alpha)\ll_\pi N\exp\big(-c\sqrt{\log N}\big),$$

where $\lambda_\pi(n)$ are the normalized Fourier coefficients of a primitive holomorphic or Maass cusp form.

Let $L(s,\pi)$ be the L-function associated to a Hecke–Maass form π for $SL(m,\mathbb{Z})$. Let $\lambda_\pi(n)$ denote the nth coefficient of the Dirichlet series for $L(s,\pi)$. Taking the logarithmic derivative for $L(s,\pi)$, we have

$$-\frac{L'}{L}(s,\pi)=\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s}=\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)}{n^s}.$$

Jiang and Lü [7] made the the following assumptions to investigate the best possible estimates for the sum $\sum_{n\le x}\Lambda(n)\lambda_\pi(n)e(n^k\theta)$:

(A) Weaker GRH: For any primitive Dirichlet character χ, there are no zeros of $L(\pi\otimes\chi,s)$ in the half plane $\sigma=\Re s>a$. Here $1/2\le a<1$.

(B) Hypothesis H: For any fixed $\nu\ge 2$,

$$\sum_p\frac{|a_\pi(p^\nu)|^2(\log p)^2}{p^\nu}<\infty.$$

(C) One has

$$\sum_{n\le x}\lambda_\pi(n)e(n^k\theta)\ll_\pi x^b$$

uniformly in θ, where the implied constant depends only on the form π and $1/2 < b < 1$.

Then under assumptions (A), (B) and (C), they have, for any $\theta \in \mathbb{R}$ and $\varepsilon > 0$,

$$\sum_{n \le x} \Lambda(n)\lambda_\pi(n)e(n^k\theta) \ll x^{\rho_k+\varepsilon},$$

where

$$\rho_1 = \begin{cases} a + 1/4, & \text{for } 1/2 \le a \le 7/12, \\ \max\left(\frac{a+1}{2}, \frac{5}{6}, \frac{1}{2}\left(1 + \frac{1}{3-2b}\right)\right), & \text{for } 7/12 \le a < 1, \end{cases}$$

$$\rho_k = \max\left(1 - \frac{2(1-a)}{4^{k-1}+2}, 1 - \frac{1}{3}\cdot\frac{1}{4^{k-1}}, 1 - \frac{1}{4^{k-1}(1-b)+1}\right), \quad \text{for } k \ge 2.$$

We establish the following stronger result for α of irrational type 1 by Theorem 1.2.

Corollary 1.4. *Under the condition* (1.5) *we have*

$$\sum_{n \le x} \Lambda(n)\lambda_\pi(n)e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{4-3\theta}{5-4\theta}+\varepsilon}$$

for every $\varepsilon > 0$.

We expect to remove the assumption in Theorem 1.2. In other words, we expect to obtain an upper bound for the sum

$$\sum_{n \le x} \lambda_\pi(n)e(n\alpha).$$

It is generally known that when $\lambda_\pi(n)$ are either the normalized Fourier coefficients of a modular form, or a Maass form on the upper half plane, i.e. an automorphic form on GL_2, one has the classical estimate on linear polynomials in [4, Theorem 8.1],

$$\sum_{n \le x} \lambda_\pi(n)e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{1}{2}+\varepsilon}$$

for any $\alpha \in \mathbb{R}$. For $m = 3$, $\lambda_\pi(n)$ being coefficients of the L-function of automorphic cusp form π on GL_m over $\mathbb{Q}$, Miller [13] showed that

$$\sum_{n \le x} \lambda_\pi(n)e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{3}{4}+\varepsilon}$$

for any $\alpha \in \mathbb{R}$. The key tools used in this proof are the Voronoï summation for GL_3 developed by Miller and Schmid [13] and Weil's estimate for Kloosterman sums. Concwerning the problem for $m \geq 3$, Jiang, Lü and Wang [9] showed that for any automorphic cuspidal representation π over GL_m,

$$S_{\lambda_\pi}(x) = \sum_{n \leq x} \lambda_\pi(n) e\,(n\alpha) \ll_\pi \frac{x}{\log x}$$

for any $\alpha \in \mathbb{R}$. The result provides a non-trivial, uniform estimate for $S_{\lambda_\pi}(x)$ without any assumptions.

In this paper, we obtain metric estimates for

$$S_\pi(x, \alpha) = \sum_{n \leqslant x} \lambda_\pi(n) e(n\alpha). \tag{1.7}$$

Our proof of Theorem 1.5 below partially follows the approach in [19]. We used Chebyshev's inequality, the well-known Carleson–Hunt inequality [3] and the Borel–Cantelli lemma to give that:

THEOREM 1.5. *For every fixed $\varepsilon > 0$ and for all $0 < \alpha < 1$ in a set of full Lebesgue measure (depending on π), we have,*

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll_{\pi,\alpha,\varepsilon} (\log x)^{\frac{1}{2}} (\log\log x)^{\frac{1}{2}+\varepsilon} x^{\frac{1}{2}}$$

uniformly for $x \geq 4$.

Applying Theorem 1.5, we have the following corollary.

COROLLARY 1.6. *For every fixed $\varepsilon > 0$ and for all $0 < \alpha < 1$ in a set of full Lebesgue measure (depending on π and ε), we have*

$$\sum_{n \leq x} \Lambda(n) \lambda_\pi(n) e(n\alpha) \ll_{\pi,\alpha,\varepsilon} x^{\frac{5}{6}+\varepsilon}$$

uniformly for $x \geq 4$.

2. Preliminaries

2.1. Automorphic L-functions. Given $\pi \in \mathcal{A}(m)$, the standard L-function is defined to be

$$L(s, \pi) = \prod_{p<\infty} L(s, \pi_p) = \prod_{p<\infty} \sum_{k=0}^{\infty} \frac{\lambda_\pi(p^k)}{p^{ks}} = \sum_{n=0}^{\infty} \lambda_\pi(n) e(n\alpha)$$

for $\Re s > 1$. For each (finite) prime p, the inverse of the local factor $L(s,\pi_p)$ is also a polynomial in p^s of degree $\leq m$,

$$L(s,\pi_p)^{-1} = \prod_{j=1}^{m} \left(1 - \alpha_{\pi,j}(p)p^{-s}\right),$$

where $\{\alpha_{\pi,j}(p)\}_{j=1}^{m}$ are the Satake parameters associated with π_p. We write $\lambda_\pi(n)$ in terms of the Satake parameters

$$\lambda_\pi(p^k) = \sum_{n_1+\cdots+n_m=k} \prod_{j=1}^{m} \alpha_{j,\pi}^{n_j}(p). \tag{2.1}$$

At the Archimedean place of $\mathbb{Q}$, there exist m complex Langlands parameters $\mu_\pi(j)$ from which we define

$$L(s,\pi_\infty) = \pi^{-\frac{ms}{2}} \prod_{j=1}^{m} \Gamma\Big(\frac{s+\mu_\pi(j)}{2}\Big).$$

Define the completed L-function

$$\Lambda(s,\pi) = N_\pi^{s/2} L(s,\pi) L(s,\pi_\infty).$$

Thus, $\Lambda(s,\pi)$ extends to an entire function and is bounded in the vertical strip. The generalized Ramanujan conjectures assert that

$$|\alpha_{\pi,j}(p)| = 1 \quad \text{and} \quad |\Re\mu_\pi(j)| = 0 \quad (1 \leq j \leq m).$$

Due to Kim and Sarnak [10] ($2 \leq m \leq 4$) and Luo, Rudnick and Sarnak [12] ($m \geq 5$), the best known record is

$$|\alpha_{\pi,j}(p)| \leq p^{\theta_m}, \quad \text{and} \quad -\Re\mu_\pi(j) \leq \theta_m \tag{2.2}$$

for all primes p and $1 \leq j \leq m$, where

$$\theta_2 = \frac{7}{64}, \quad \theta_3 = \frac{5}{14}, \quad \theta_4 = \frac{9}{22}, \quad \theta_m = \frac{1}{2} - \frac{1}{m^2+1} \ (m \geq 5). \tag{2.3}$$

With all the local factors defined as above, we can turn to the functional equation. Let $\widetilde{\pi}$ denote the contragredient of $\pi \in \mathcal{A}(m)$, which is also an irreducible cuspidal automorphic representation with unitary central character in $\mathcal{A}(m)$. We have the equalities

$$\left\{ \alpha_{j,\widetilde{\pi}}(p) : 1 \leqslant j \leqslant m \right\} = \left\{ \overline{\alpha_{j,\pi}(p)} : 1 \leqslant j \leqslant m \right\}$$

and

$$\{\mu_{\widetilde{\pi}}(j) : 1 \leqslant j \leqslant m\} = \{\overline{\mu_\pi(j)} : 1 \leqslant j \leqslant m\}.$$

Moreover, $W(\pi)$ is a complex number of modulus 1 and $\Lambda(s,\pi)$ satisfies a functional equation of the form

$$\Lambda(s,\pi) = W(\pi)\Lambda(1-s,\widetilde{\pi}).$$

Finally, the analytic conductor of π is defined by

$$C(\pi,t) = N_\pi \prod_{j=1}^{m} \left(1 + |it + \mu_\pi(j)|\right), \quad C(\pi) = C(\pi,0).$$

The inverse of $L(s,\pi)$ is (1.2) where

$$\mu_\pi(n) = \begin{cases} 0, & p^{m+1} \,|\, n \text{ for some prime } p, \\ \prod_{p^\ell \| n} (-1)^\ell & \\ \times \sum_{1 \le j_1 < \cdots < j_\ell \le m} \alpha_\pi(p,j_1) \cdots \alpha_\pi(p,j_\ell), & \text{for all } \ell \le m. \end{cases}$$

Clearly, $\mu_\pi(n)$ are multiplicative.

We can write the logarithmic derivative of $L(s,\pi)$, for $\Re s > 1$,

$$(2.4) \qquad -\frac{L'}{L}(s,\pi) = \sum_p \sum_{k=1}^{\infty} \frac{(\log p) a_\pi(p^k)}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n) a_\pi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s},$$

where

$$(2.5) \qquad a_\pi(n) = \begin{cases} \sum_{j=1}^m \alpha_{\pi,j}(p)^k, & \text{if } n = p^k, \\ 0, & \text{otherwise} \end{cases}$$

for all primes p and integers $k \geq 1$.

By (2.1) and (2.5), we note that $a_\pi(p) = \lambda_\pi(p)$ for suitable complex numbers $\alpha_{j,\pi}(p)$.

2.2. Rankin–Selberg L-functions and related conclusions. To prove Theorem 1.3, we need some results based on the Rankin–Selberg theory. The Rankin–Selberg L-function $L(s,\pi \times \tilde{\pi})$ associated to π and its contragredient $\tilde{\pi}$ is defined as a product of local factors

$$L(s,\pi \times \tilde{\pi}) = \prod_{p<\infty} L(s,\pi_p \times \tilde{\pi}_p).$$

The inverse of the local factor is given by

$$L(s,\pi_p\times\tilde{\pi}_p)^{-1}=\prod_{j=1}^{m}\prod_{k=1}^{m}\left(1-\alpha_{\pi,j}(p)\overline{\alpha_{\pi,k}(p)}p^{-s}\right).$$

By Jacquet and Shalika [6], the product $\prod_p L(s,\pi_p\times\tilde{\pi}_p)$ converges absolutely in $\Re s>1$. We write this product as a Dirichlet series

$$L(s,\pi\times\tilde{\pi})=\prod_p\sum_{k=0}^{\infty}\frac{\lambda_{\pi\times\tilde{\pi}}(p^k)}{p^{ks}}=\sum_{n=1}^{\infty}\frac{\lambda_{\pi\times\tilde{\pi}}(n)}{n^s}.$$

Let $\pi'=\bigotimes_p\pi'_p\in\mathcal{A}(m')$ and $\pi''=\bigotimes_p\pi''_p\in\mathcal{A}(m'')$. We define the Rankin–Selberg L-function $L\left(s,\pi'\times\pi''\right)$ associated to π' and π'' to be

$$L(s,\pi'\times\pi'')=\prod_p L(s,\pi'_p\times\pi''_p)=\sum_{n=1}^{\infty}\frac{\lambda_{\pi'\times\pi''}(n)}{n^s}$$

for $\Re(s)>1$. For each (finite) prime p, the inverse of the local factor $L\left(s,\pi'_p\times\pi''_p\right)$ is defined to be a polynomial in p^{-s} of degree $\leqslant m'm''$,

$$L(s,\pi'_p\times\pi''_p)^{-1}=\prod_{j'=1}^{m'}\prod_{j''=1}^{m''}\left(1-\frac{\alpha_{j',j'',\pi'\times\pi''}(p)}{p^s}\right)$$

for suitable complex numbers $\alpha_{j',j'',\pi'\times\pi''}(p)$. With θ_m as in (2.3), we have the pointwise bound

$$\left|\alpha_{j',j'',\pi'\times\pi''}(p)\right|\leqslant p^{\theta_{m'}+\theta_{m''}}\leqslant p^{1-\frac{1}{m'm''}}.$$

If $p\nmid N_{\pi'}N_{\pi''}$, we have the equality of sets

$$\left\{\alpha_{j',j'',\pi'\times\pi''}(p):j'\leqslant m',j''\leqslant m''\right\}=\left\{\alpha_{j',\pi'}(p)\overline{\alpha_{j'',\pi''}(p)}:j'\leqslant m',j''\leqslant m''\right\}.$$

It is known that $\lambda_{\pi\times\tilde{\pi}}(n)\geq 0$. Since $L(s,\pi\times\tilde{\pi})$ extends to the complex plane with a simple pole at $s=1$, we have,

$$\sum_{Nn<x}\lambda_{\pi\times\widetilde{\pi}}(n)\sim x\,\mathrm{Res}_{s=1}\,L(s,\pi\times\widetilde{\pi})\ll x, \tag{2.6}$$

which follows from a standard Tauberian argument.

In order to prove our results, we need to prove some inequalities which are based on Rankin–Selberg theory.

LEMMA 2.1. *With the above notation, we have,*

(i)
$$\sum_{n\le x} |\Lambda_\pi(n)| \ll x,$$

(ii)
$$\sum_{n\le x} \Big(\sum_{tr=n} |\lambda_\pi(t)\mu_\pi(r)| \Big)^2 \ll x(\log x)^{4m+3},$$

(iii)
$$\sum_{n\le x} |\mu_\pi(n)| \ll x,$$

where the implied constants depend on π only.

PROOF. (i) We write $a_{\pi\times\pi}(n)$ to be the coefficients of Dirichlet series $-\frac{L'}{L}(s,\pi\times\widetilde{\pi})$, namely

$$-\frac{L'}{L}(s,\pi\times\widetilde{\pi}) = \sum_{n=1}^{\infty} \frac{\Lambda(n)a_{\pi\times\tilde{\pi}}(n)}{n^s}.$$

By the appendix in [18], we have

$$|a_\pi(n)|^2 \leqslant a_{\pi\times\tilde{\pi}}(n).$$

Following from Shahidi's non-vanishing result of $L(s,\pi\times\tilde{\pi})$ at $\Re s = 1$ (see [17]), we obtain

$$\sum_{n\leqslant x} \Lambda(n)\, |a_\pi(n)|^2 \ll \sum_{n\leqslant x} \Lambda(n) a_{\pi\times\tilde{\pi}}(n) \sim x. \tag{2.7}$$

From the definition of $\Lambda_\pi(n)$, we know that

$$\sum_{n\le x} \Lambda_\pi(n) = \sum_{n\le x} \Lambda(n)a_\pi(n).$$

It is well known that

$$\sum_{n\le x} |\Lambda(n)| \ll x.$$

Then by the Cauchy–Schwarz inequality and the inequality (2.7), we have,

$$\sum_{n\le x} \Lambda(n)a_\pi(n) \ll \Big(\sum_{n\leqslant x} |\Lambda(n)| \Big)^{\frac{1}{2}} \Big(\sum_{n\leqslant x} \Lambda(n)\, |a_\pi(n)|^2 \Big)^{\frac{1}{2}} \ll_\pi x.$$

As for (ii), we obtain the desired result from [8, Lemma 5.4]. For (iii), it follows from the Cauchy–Schwarz inequality and [8, Lemma 5.4] that

$$\sum_{n\le x}|\mu_\pi(n)| \ll x^{\frac12}\Big(\sum_{n\le x}|\mu_\pi(n)|^2\Big)^{\frac12} \ll_\pi x. \quad \square$$

2.3. Vaughan's method. We will prove Theorem 1.2 by applying Vaughan's method for $\Lambda_\pi(n)$. Now let

$$L'(s,\pi)=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)\log n}{n^s}, \quad L(s,\pi)=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s},$$

$$-\frac{L'}{L}(s,\pi)=\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s}, \quad \frac{1}{L(s,\pi)}=\sum_{n=1}^{\infty}\frac{\mu_\pi(n)}{n^s},$$

$$F(s)=\sum_{n\le U}\frac{\Lambda_\pi(n)}{n^s}, \quad G(s)=\sum_{n\le V}\frac{\mu_\pi(n)}{n^s}$$

be defined for $\sigma=\Re s\ge 1$. Here U and V are arbitrary parameters to be chosen later satisfying $U,V\ge 1$.

Lemma 2.2. *Let $U,V\ge 1$ be arbitrary. Then we have*

$$\Lambda_\pi(n)=a_1(n)+a_2(n)+a_3(n)+a_4(n),$$

where

$$a_1(n)=\begin{cases}\Lambda_\pi(n) & \text{if } n\le U\\ 0 & \text{otherwise,}\end{cases}$$

$$a_2(n)=-\sum_{\substack{djr=n\\ j\le V, r\le U}}\lambda_\pi(d)\mu_\pi(j)\Lambda_\pi(r), \quad a_3(n)=\sum_{\substack{dj=n\\ j\le V}}\lambda_\pi(d)(\log d)\mu_\pi(j)$$

and

$$a_4(n)=\Big(\sum_{\substack{dj=n\\ d>U, j>V}}\Lambda_\pi(d)\Big(\sum_{\substack{tr=j\\ r\le V}}\lambda_\pi(t)\mu_\pi(r)\Big)\Big).$$

Proof. By Vaughan's method, we have

$$\frac{L'}{L}(s,\pi)=F(s)-L(s,\pi)G(s)F(s)+\frac{L'}{L}(s,\pi)G(s)L(s,\pi)$$
$$+\Big(\frac{L'}{L}(s,\pi)-F(s)\Big)(1-L'(s,\pi)G(s)), \quad \Re s>1.$$

Once we pick out the coefficients of n^{-s} on each side, we obtain the desired result. □

3. Proof of Theorem 1.2

From Lemma 2.2, we obtain that

$$\sum_{n\le x}\Lambda_\pi(n)e(n\alpha)=S_1(x)+S_2(x)+S_3(x)+S_4(x),$$

where $S_i(x)=\sum_{n\le x}a_i(n)e(n\alpha)$ for $i=1,\ 2,\ 3$ and 4. Now we begin our estimations of the sums $S_i(x)$.

Estimating $S_1(x)$. It directly follows from Lemma 2.2, the absolute inequality and Lemma 2.1 that

$$S_1(x)\le\sum_{n\le U}|\Lambda_\pi(n)|\ll_\pi U. \tag{3.1}$$

Estimating $S_2(x)$. Due to Lemma 2.2, we have

$$\begin{aligned}S_2(x)&=-\sum_{n\le x}\Bigg(\sum_{\substack{djr=n\\ j\le V,r\le U}}\lambda_\pi(d)\mu_\pi(j)\Lambda_\pi(r)\Bigg)e(n\alpha)\\&=-\sum_{r\le U}\Lambda_\pi(r)\Bigg(\sum_{j\le V}\mu_\pi(j)\Bigg)\sum_{d\le x/jr}\lambda_\pi(d)e(djr\alpha).\end{aligned}$$

By (1.5), partial summation and Lemma 2.1, we deduce that

$$S_2(x)\ll_\pi (UV)^{1-\theta}x^\theta. \tag{3.2}$$

Estimating $S_3(x)$. Using Lemma 2.2, we have

$$S_3(x)=\sum_{n\le x}\Bigg(\sum_{\substack{dj=n\\ j\le V}}\lambda_\pi(d)(\log d)\mu_\pi(j)\Bigg)e(n\alpha).$$

We can rewrite $S_3(x)$ as

$$\sum_{j\le V}\mu_\pi(j)\sum_{d\le\frac{x}{j}}\lambda_\pi(d)(\log d)e(dj\alpha).$$

Then it follows from (1.5), partial summation and Lemma 2.1 that

$$S_3(x)\ll_\pi V^{1-\theta}x^\theta\log x. \tag{3.3}$$

Estimating $S_4(x)$. We know from Lemma 2.2 that

$$S_4(x) = \sum_{n\le x}\Bigg(\sum_{\substack{dj=n\\ d>U,j>V}}\Lambda_\pi(d)\Bigg(\sum_{\substack{tr=j\\ r\le V}}\lambda_\pi(t)\mu_\pi(r)\Bigg)\Bigg)e(n\alpha).$$

We break the above sum into dyadic intervals:

$$J = \sum_{\substack{W<d<2W\\ d>U,j>V,dj\le x}}\Lambda_\pi(d)\sum_{\substack{tr=j\\ r\le V}}\lambda_\pi(t)\mu_\pi(r)e(dj\alpha).$$

Thus, applying the Cauchy–Schwarz inequality and Lemma 2.1 to J, we obtain

$$|J| \le W^{\frac{1}{2}}\Bigg(\sum_{\substack{W<d<2W\\ d>U,d\le x/V}}\Bigg|\sum_{\substack{j>V\\ j<x/d,j<x/W}} f(j,V)e(dj\alpha)\Bigg|^2\Bigg)^{\frac{1}{2}},$$

where

$$f(j,V) = \sum_{\substack{tr=j\\ r\le V}}\lambda_\pi(t)\mu_\pi(r). \tag{3.4}$$

Here the second sum over d is

$$\sum_{V<j\le x/W} f(j,V)\sum_{V<k\le x/W}\overline{f(k,V)}\sum_{\substack{W<d\le 2W\\ d>U,d\le x/j,d\le x/k}} e(dj\alpha)\overline{e(dk\alpha)}.$$

By the inequality of arithmetic and geometric means, we obtain that

$$|f(j,V)\overline{f(k,V)}| \le \frac{1}{2}|f(j,V)|^2 + \frac{1}{2}|f(k,V)|^2.$$

The penultimate formula above is

$$\ll \sum_{V<j\le x/W}|f(j,V)|^2\sum_{|h|\le x/W}\Bigg|\sum_{\substack{W<d\le 2W\\ d>U,d\le x/V}} e(dh\alpha)\Bigg|.$$

Define

$$\Delta = \sum_{|h|\le x/W}\Bigg|\sum_{\substack{W<d\le 2W\\ d>U,d\le x/V}} e(dh\alpha)\Bigg|.$$

Then we have

$$J \ll \Delta^{\frac{1}{2}} \Big(W \sum_{V<j\le x/W} |f(j,V)|^2 \Big)^{\frac{1}{2}}. \tag{3.5}$$

Using (3.4) and Lemma 2.1, we derive

$$\sum_{V<j\le x/W} |f(j,V)|^2 \ll_\pi \Big(\frac{x}{W}\Big)\Big(\log \frac{x}{W}\Big)^{4m+3}. \tag{3.6}$$

Here $\|\alpha h\|$ denotes the distance of αh to the nearest integer, [5, (8.6)] about exponential sums implies that

$$\Delta \ll \sum_{|h|\le x/W} \min\Big(2W, \frac{x}{V}, \frac{1}{||\alpha h||}\Big).$$

Then by [15, Lemma 3], for α of irrational type 1, we have

$$\Delta \ll \Big(\frac{x}{W}\Big)^{1+\varepsilon}.$$

Thus, following from the above inequality, (3.5) and (3.6), we obtain

$$S_4(x) \ll_{\pi,\varepsilon} x^{1+\varepsilon} U^{-\frac{1}{2}} (\log x)^{2m+3}. \tag{3.7}$$

Choosing parameters. By (3.1), (3.2), (3.3) and (3.7), we choose $U = V = x^{\frac{2-2\theta}{5-4\theta}}$ and obtain that, for all $\varepsilon > 0$ and α of irrational type 1 lying in the interval $[0,1]$,

$$\sum_{n\le x} \Lambda(n)\lambda_\pi(n)e(n\alpha) \ll_\pi x^{\frac{4-3\theta}{5-4\theta}+\varepsilon}.$$

4. Proof of Corollary 1.3

By Vaughan's method, we have

$$\mu_\pi(n) = b_1(n) + b_2(n) + b_3(n) + b_4(n),$$

where

$$b_1(n) = \begin{cases} \mu_\pi(n) & \text{if } n \le U \\ 0 & \text{otherwise,} \end{cases}$$

$$b_2(n) = -\sum_{\substack{djr=n \\ j\le V, r\le U}} \lambda_\pi(d)\mu_\pi(j)\mu_\pi(r), \quad b_3(n) = \sum_{\substack{dj=n \\ j\le V}} \mu_\pi(j),$$

and

$$b_4(n) = -\Bigg(\sum_{\substack{dj=n \\ d>U, j>V}} \mu_\pi(d)\Bigg(\sum_{\substack{tr=j \\ r\le V}} \lambda_\pi(t)\mu_\pi(r)\Bigg)\Bigg)$$

with $U, V \ge 1$. We define

$$\sum_{n\le x} \mu_\pi(n)e(n\alpha) = \sum_{n\le x} \big(b_1(n)+b_2(n)+b_3(n)+b_4(n)\big)e(n\alpha)$$

$$= S_1(x)+S_2(x)+S_3(x)+S_4(x).$$

By the same method as in the proof of Theorem 1.2, we have

$$S_1(x) \ll_\pi U,\ S_2(x) \ll_\pi (UV)^{1-\theta}x^\theta,\ S_4 \ll_{\pi,\varepsilon} x^{1+\varepsilon}U^{-\frac{1}{2}}(\log x)^{2m+3} \tag{4.1}$$

for every $\varepsilon > 0$.

As for $S_3(x)$, by the Cauchy–Schwarz inequality, we have

$$S_3(x) \le \Bigg(\sum_{j\le V} |\mu_\pi(j)|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{j\le V}\Bigg|\sum_{d\le \frac{x}{j}} e(dj\alpha)\Bigg|^2\Bigg)^{\frac{1}{2}}$$

$$= \Bigg(\sum_{j\le V} |\mu_\pi(j)|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\frac{x}{V}\le d_1, d_2\le x}\ \sum_{j\le \frac{x}{d_1}, \frac{x}{d_2}, V} e((d_1-d_2)j\alpha)\Bigg)^{\frac{1}{2}}.$$

Following from Lemma 2.1, [5, (8.6)] and [15, Lemma 3], we obtain

$$S_3(x) \ll_\pi x^{\frac{1}{2}}V^{\frac{1}{2}}, \tag{4.2}$$

where α is of irrational type 1.

Thus, by (4.1) and (4.2), we choose $U = V = x^{\frac{2-2\theta}{5-4\theta}}$ to get

$$\sum_{n\le x} \mu_\pi(n)e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{4-3\theta}{5-4\theta}+\varepsilon}.$$

5. Proof of Corollary 1.4

Following from (2.4), we obtain that

$$\sum_{n\le x} \Lambda(n)\lambda_\pi(n)e(n\alpha)$$

$$= \sum_{n\le x} \Lambda_\pi(n)e(n\alpha) + O\bigg(\sum_{k\ge 2, p^k\le x} (\log p)(\lambda_\pi(p^k) + a_\pi(p^k))e(p^k\alpha)\bigg).$$

By the Cauchy–Schwarz inequality, (2.6) and (2.7), we derive that

$$\sum_{k\ge 2, p^k\le x} (\log p)(\lambda_\pi(p^k) + a_\pi(p^k))e(p^k\alpha) \ll x^{\frac{3}{4}}(\log x)^2.$$

Thus, we obtain

$$\sum_{n\le x} \Lambda(n)\lambda_\pi(n)e(n\alpha) = \sum_{n\le x} \Lambda_\pi(n)e(n\alpha) + O(x^{\frac{3}{4}}(\log x)^2).$$

By the above equality and Theorem 1.2, we finish the proof of Corollary 1.4.

6. Proof of Theorem 1.5

Consider the sets

$$L_r = \bigg\{ 0 \le \alpha < 1 :$$
$$\max_{1\le x\le 2^r} \bigg| \sum_{n\le x} \lambda_\pi(n)e(\alpha n)\bigg| > r^{\frac{1}{2}}(\log r)^{\frac{1}{2}+\varepsilon} \bigg(\sum_{n\le 2^r} |\lambda_\pi(n)|^2 \bigg)^{\frac{1}{2}} \bigg\}$$

for $r \ge 2$ and $\varepsilon > 0$. Chebyshev's inequality shows that

$$\text{meas}(L_r) \le \frac{1}{r(\log r)^{1+\varepsilon} \sum_{n\le 2^r} |\lambda_\pi(n)|^2} \times \int_0^1 \bigg(\max_{1\le x\le 2^r} \bigg| \sum_{n\le x} \lambda_\pi(n)e(\alpha n)\bigg|\bigg)^2 \mathrm{d}\alpha, \tag{6.1}$$

where meas$(\cdot)$ denotes the Lebesgue measure. In order to deal with

$$\int_0^1 \bigg(\max_{1\le x\le 2^r} \bigg| \sum_{n\le x} \lambda_\pi(n)e(\alpha n)\bigg|\bigg)^2 \mathrm{d}\alpha,$$

we recall the Carleson–Hunt theorem:

THEOREM 6.1 [3]. *For any sequence $(c_k)_{k\in\mathbb{Z}}$ of complex numbers, and any positive integer Y,*

$$\int_0^1 \bigg(\max_{1\le y\le Y} \bigg| \sum_{1\le |k|\le y} c_k \exp(2\pi i k\lambda)\bigg|\bigg)^2 \mathrm{d}\lambda \ll \sum_{1\le |k|\le Y} |c_k|^2.$$

Then we immediately have

$$\int_0^1 \left(\max_{1\le x\le 2^r} \left| \sum_{n\le x} \lambda_\pi(n) e(\alpha n) \right| \right)^2 \mathrm{d}\alpha \ll \sum_{n\le 2^r} |\lambda_\pi(n)|^2. \tag{6.2}$$

Hence, it follows from (6.1) and (6.2) that $\mathrm{meas}(L_r) \ll 1/r(\log r)^{1+2\varepsilon}$, so that

$$\sum_{r=2}^{\infty} \mathrm{meas}(L_r) < \infty.$$

We recall the Borel–Cantelli lemma.

Lemma 6.2. *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$ and suppose $\{E_n\}_{n=1}^{\infty} \subset \Sigma$ is a collection of measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then*

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right) = 0.$$

This yields

$$\mathrm{meas}\left(\bigcap_{R=2}^{\infty} \bigcup_{r=R}^{\infty} L_r \right) = 0. \tag{6.3}$$

Hence, for every $\varepsilon > 0$, we have

$$\max_{1\le x\le 2^r} \left| \sum_{n\le x} \lambda_\pi(n) e(n\alpha) \right| \ll_{f,\alpha,\varepsilon} r^{\frac{1}{2}} (\log r)^{\frac{1}{2}+\varepsilon} \left(\sum_{n\le 2^r} |\lambda_\pi(n)|^2 \right)^{\frac{1}{2}}.$$

Since

$$\sum_{n\le x} |\lambda_\pi(n)|^2 \le \sum_{n\le x} \lambda_{\pi\times\hat{\pi}}(n) \sim c_\pi x,$$

we deduce that

$$\max_{1\le x\le 2^r} \left| \sum_{n\le x} \lambda_\pi(n) e(n\alpha) \right| \ll_{\pi,\alpha,\varepsilon} (\log x)^{\frac{1}{2}} (\log\log x)^{\frac{1}{2}+\varepsilon} x^{\frac{1}{2}},$$

where 2^r denotes the smallest power of 2 exceeding x.

7. Proof of Corollary 1.6

For fixed $\varepsilon > 0$. By Theorem 1.5, we have

$$\sum_{n\le x}\lambda_\pi(n)e(n\alpha)\ll_{\pi,\alpha,\varepsilon} x^{\frac{1}{2}+\varepsilon},$$

for all $0<\alpha<1$ in a set of full Lebesgue measure. Then following the same method as in the proof of Theorem 1.2, we have

$$S_1(x)=\sum_{n\le U}\Lambda_\pi(n)e(n\alpha)\ll_\pi U,$$

$$S_2(x)=-\sum_{n\le x}\Bigg(\sum_{\substack{djr=n\\ j\le V, r\le U}}\lambda_\pi(d)\mu_\pi(j)\Lambda_\pi(r)\Bigg)e(n\alpha)\ll_{\pi,\alpha,\varepsilon}(UV)^{\frac{1}{2}}x^{\frac{1}{2}+\varepsilon},$$

$$S_3(x)=\sum_{n\le x}\Bigg(\sum_{\substack{dj=n\\ j\le V}}\lambda_\pi(d)(\log d)\mu_\pi(j)\Bigg)e(n\alpha)\ll_{\pi,\alpha,\varepsilon}V^{\frac{1}{2}}x^{\frac{1}{2}+\varepsilon}\log x$$

for $0<\alpha<1$ in a set of full Lebesgue measure and

$$S_4(x)=\sum_{n\le x}\Bigg(\sum_{\substack{dj=n\\ d>U, j>V}}\Lambda_\pi(d)\Bigg(\sum_{\substack{tr=j\\ r\le V}}\lambda_\pi(t)\mu_\pi(r)\Bigg)\Bigg)e(n\alpha)$$
$$\ll_{\pi,\varepsilon} x^{1+\varepsilon}U^{-\frac{1}{2}}(\log x)^{2m+3}$$

for all α of irrational type 1 lying in the interval $[0,1]$. We denote by $\mathcal{B}$ the set of $0<\alpha<1$ satisfying Theorem 1.5 which is non-effective. Let $\mathcal{A}$ denote the set of those numbers α of irrational type 1 lying in the interval $[0,1]$. Following Davenport and Roth [1], $\mathcal{A}$ has Lebesgue measure 1. By the property of Lebesgue measure, the intersection of $\mathcal{A}$ and $\mathcal{B}$ also has Lebesgue measure 1. Hence, we choose $U=V=x^{\frac{1}{3}}$ to obtain that, for all $0<\alpha<1$ in a set of full Lebesgue measure (depending on π and ε),

$$\sum_{n\le x}\Lambda_\pi(n)e(n\alpha)\ll_{\pi,\alpha,\varepsilon} x^{\frac{5}{6}+\varepsilon}.$$

It follows by the same method as in the proof of Corollary 1.4 that

$$\sum_{n\le x}\Lambda(n)\lambda_\pi(n)e(n\alpha)\ll_{\pi,\alpha,\varepsilon} x^{\frac{5}{6}+\varepsilon}.$$

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