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THE COSINE AND SINE ADDITION AND SUBTRACTION FORMULAS ON SEMIGROUPS

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Abstract. The cosine addition formula on a semigroup S is the functional equation $g(xy) = g(x)g(y) - f(x)f(y)$ for all $x, y \in S$. We find its general solution for $g, f\colon S \to \mathbb{C}$, using the recently found general solution of the sine addition formula $f(xy) = f(x)g(y) + g(x)f(y)$ on semigroups. A simpler proof of this latter result is also included, with some details added to the solution.

We also solve the cosine subtraction formula $g(x\sigma(y)) = g(x)g(y) + f(x)f(y)$ on monoids, where σ is an automorphic involution. The solutions of these functional equations are described mostly in terms of additive and multiplicative functions, but for some semigroups there exist points where f and/or g can take arbitrary values.

The continuous solutions on topological semigroups are also found.

1. Introduction

Let S be a (not necessarily commutative) semigroup, let K be a (commutative) field, and let $\sigma\colon S \to S$ be an automorphic involution. That σ is an *involution* means $\sigma \circ \sigma(x) = x$ for all $x \in S$. The *cosine addition formula*, *sine addition formula*, *cosine subtraction formula*, and *sine subtraction formula* for unknown functions $g, f\colon S \to K$ are, respectively, the functional equations

(1) $$g(xy) = g(x)g(y) - f(x)f(y),$$

(2) $$f(xy) = f(x)g(y) + g(x)f(y),$$

(3) $$g(x\sigma(y)) = g(x)g(y) + f(x)f(y),$$

(4) $$f(x\sigma(y)) = f(x)g(y) - g(x)f(y),$$

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for all $x, y \in S$. These equations obviously generalize familiar trigonometric identities if we think of $S = (\mathbb{R}, +)$ and $\sigma(y) = -y$.

These four functional equations have been investigated (separately or in systems) by many authors over the past century, beginning with the case $S = (\mathbb{R}, +)$, $\sigma(x) = -x$, and $K = \mathbb{R}$. Here we consider them separately, with primary focus on equations (1) and (3). We include equations (2) and (4) for two reasons. The first is that the solution of (2) is used to solve (1) and (3). The other reason is that we give a simpler proof (relative to the one in [5]) of the solution of (2) that also sharpens the descriptions of the solutions of (2) and (4).

The history of (1) and (3) parallels that of (2) and (4) as described in [5]. Equation (1) was solved on Abelian groups by Vincze [10], and on general groups by Chung, Kannappan, and Ng [4]. More recent results include [9, Theorem 6.1] by Stetkær, which describes the solution set of (1) on general semigroups in terms of the solutions of (2). Also Ajebbar and Elqorachi [3, Lemma 4.1 and Theorem 4.3] give the solutions of (1) and (3), respectively, on semigroups generated by their squares. The solution of (1) in this setting was also found independently by the author [6, Proposition 3.3]. For additional discussions of these equations and their history, the reader may consult [1, Section 3.2.3], [2, Ch. 13], [8, Ch. 4], and their references.

While (1) may be viewed as a special case of (3) by taking σ to be the identity function and replacing f by if, we treat the two equations separately. That is because we are able to solve (1) on all semigroups but we solve (3) only on monoids. A *monoid* is a semigroup containing an identity element that we denote e, so $ex = xe = x$ for all $x \in S$.

Section 2 contains the solution of the sine addition formula (2), along with other background, terminology, and notation. We present the general solution of (1) in section 3. The solution of (3) on monoids follows in section 4, as well as an updated solution of (4) with more detail about the solution forms. Section 5 contains a variety of examples illustrating some applications of the results.

Although $K = \mathbb{C}$ is chosen as the codomain, $\mathbb{C}$ can be replaced by any quadratically closed field of characteristic different from 2.

2. Background and setup

Two essential ingredients in solutions of the cosine and sine addition and subtraction formulas are the homomorphisms of S (or one of its subsemigroups) into the additive and multiplicative semigroups of $\mathbb{C}$. A function $A\colon S \to \mathbb{C}$ is said to be *additive* if

$$A(xy) = A(x) + A(y), \quad \text{for all } x, y \in S,$$

and a function $m\colon S \to \mathbb{C}$ is *multiplicative* if

$$m(xy) = m(x)m(y), \quad \text{for all } x, y \in S.$$

If $m \neq 0$ then we say that m is an *exponential*. For a multiplicative function $m\colon S \to \mathbb{C}$, we define the *nullspace* I_m by

$$I_m := \{x \in S \mid m(x) = 0\}.$$

If $I_m \neq \emptyset$ then it is a (two-sided) ideal of S called the *null ideal*.

Prime ideals of S are important to this story because of their connection with multiplicative functions. By an *ideal* of a semigroup S we mean a nonempty subset I such that $SI \subseteq I$ and $IS \subseteq I$. An ideal I is a *prime ideal* if $I \neq S$ and whenever $xy \in I$ it follows that either $x \in I$ or $y \in I$. That is, an ideal I is prime if and only if $S \setminus I$ is a proper nonempty subsemigroup of S. Thus for an exponential $m\colon S \to \mathbb{C}$ such that $I_m \neq \emptyset$, the null ideal I_m is a prime ideal. Conversely, if I is a prime ideal of S, then there exists an exponential $m\colon S \to \mathbb{C}$ such that $I = I_m$, namely $m(x) = 1$ for $x \in S \setminus I$ and $m(x) = 0$ for $x \in I$.

For any subset $T \subseteq S$ let $T^2 = \{t_1 t_2 \mid t_1, t_2 \in T\}$. We will *not* use the notation T^2 to denote the direct product $T \times T$.

In order to adequately describe some solutions of our functional equations, we partition the nullspace into the disjoint union $I_m = I_m^2 \cup P_m^{(1)} \cup P_m^{(1+)}$, where

$$P_m^{(1)} := \left\{p \in I_m \setminus I_m^2 \mid \text{ for all } w \in S \setminus I_m \text{ we have } pw \in I_m \setminus I_m^2\right\},$$
$$P_m^{(1+)} := \left\{p \in I_m \setminus I_m^2 \mid \text{ there exists } w_p \in S \setminus I_m \text{ such that } pw_p \in I_m^2\right\}.$$

(Note that $pw \in S \setminus I_m$ is impossible for $p \in I_m$ since I_m is an ideal if nonempty.)

A function $\varphi\colon S \to \mathbb{C}$ is said to be *Abelian* if

$$\varphi(x_{\pi(1)} \cdots x_{\pi(n)}) = \varphi(x_1 \cdots x_n)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in S$, and permutations π on $\{1, \dots, n\}$. We will see that the unknown functions f, g in our functional equations are Abelian. Note that additive and multiplicative functions from S to $\mathbb{C}$ (or into any commutative ring) are always Abelian.

Define the relation $\sim$ on a semigroup S by $x \sim y$ if and only if there exist $s_1, \dots, s_n \in S$ and a permutation π on $\{1, \dots, n\}$ such that $x = s_1 \cdots s_n$ and $y = s_{\pi(1)} \cdots s_{\pi(n)}$. It is clear that if $x \sim y$ then $\varphi(x) = \varphi(y)$ for any Abelian function $\varphi\colon S \to \mathbb{C}$. We read the statement $x \sim y$ as "*x rearranges to y*."

For a topological semigroup S, let $C(S)$ denote the algebra of continuous functions mapping S into $\mathbb{C}$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The following is almost the same as [5, Corollary 3.10], but here we give a much simpler proof. We also add more detail to the solution form of f in case (C), making it a bit stronger result than the one in [5].

THEOREM 2.1. *Let S be a semigroup, and suppose $f, g\colon S \to \mathbb{C}$ satisfy the sine addition law* (2) *with $f \neq 0$. Then f and g are Abelian and there exist multiplicative functions $m_1, m_2\colon S \to \mathbb{C}$ such that $g = (m_1 + m_2)/2$. In addition we have the following.*

(A) *If $m_1 \neq m_2$, then $f = c(m_1 - m_2)$ for some constant $c \in \mathbb{C}^*$.*

(B) *If $m_1 = m_2 = 0$, then $g = 0$, $S \neq S^2$, and f has the form*

$$f(x) = \begin{cases} f_0(x) & \text{for } x \in S \setminus S^2 \\ 0 & \text{for } x \in S^2 \end{cases} \tag{5}$$

where $f_0\colon S \setminus S^2 \to \mathbb{C}$ is an arbitrary nonzero function.

(C) *If $m_1 = m_2 =: m \neq 0$, then $g = m$ and f has the form*

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m \\ 0 & \text{for } x \in I_m^2 \cup P_m^{(1+)} \\ f_P(x) & \text{for } x \in P_m^{(1)} \end{cases} \tag{6}$$

where $A\colon S \setminus I_m \to \mathbb{C}$ is an additive function and f_P is the restriction of f to $P_m^{(1)}$. In addition f satisfies the following conditions.

(I) *If $x \sim pw$ with $p \in P_m^{(1+)}$ and $w \in S \setminus I_m$, then $f(x) = 0$.*

(II) *If $x = pw$ with $p \in P_m^{(1)}$ and $w \in S \setminus I_m$, then $x \in P_m^{(1)}$ and $f_P(x) = f_P(p)m(w)$.*

Note that some values of f_P may be chosen arbitrarily.

Conversely, if the formulas for (f, g) in (A), (B), *or* (C) *with* (I) *and* (II) *hold, and if f is Abelian in case* (C), *then (f, g) satisfies* (2).

Furthermore, if S is a topological semigroup and $f \in C(S)$, then

$$g, m_1, m_2, m \in C(S),\ A \in C(S \setminus I_m),\ f_0 \in C(S \setminus S^2) \text{ and } f_P \in C(P_m^{(1)}).$$

PROOF. The fact that f and g are Abelian, the representation $g = (m_1 + m_2)/2$ (and the continuity of m_1, m_2 in the topological case), and case (A) are all established in [8, Theorem 4.1].

In case (B) we immediately get $g = 0$, and (2) reduces to $f(xy) = 0$ for all $x, y \in S$. Since $f \neq 0$ we must have $S^2 \neq S$, and this yields (5) for an arbitrary (nonzero) function $f_0\colon S \setminus S^2 \to \mathbb{C}$.

In case (C) we see that m is an exponential and (2) becomes

$$f(xy) = f(x)m(y) + m(x)f(y), \quad x, y \in S. \tag{7}$$

If $x, y \in S \setminus I_m$ then $m(xy) = m(x)m(y) \neq 0$, so $xy \in S \setminus I_m$. In this case dividing (7) by $m(x)m(y)$ we find that

$$\frac{f(xy)}{m(xy)} = \frac{f(x)}{m(x)} + \frac{f(y)}{m(y)} \quad \text{for all } x, y \in S \setminus I_m.$$

Defining $A\colon S \setminus I_m \to \mathbb{C}$ by $A := f/m$, we have the top line of (6) where A is additive. If $I_m = \emptyset$ then we are finished, so suppose $I_m \neq \emptyset$ (thus I_m is a prime ideal).

Taking $x, y \in I_m$ in (7) we find that $f(xy) = f(x)m(y) + m(x)f(y) = 0$, since $m(x) = m(y) = 0$. This proves the case $x \in I_m^2$ of the middle line of (6). For the case $x \in P_m^{(1+)}$, by definition there exists $w_x \in S \setminus I_m$ such that $xw_x \in I_m^2$. Thus

$$0 = f(xw_x) = f(x)m(w_x) + f(w_x)m(x) = f(x)m(w_x),$$

since $m(x) = 0$. Now $m(w_x) \neq 0$ implies $f(x) = 0$, and we have completed the middle line of formula (6). For the bottom line of (6) there is nothing to prove.

Next suppose $x \sim pw$ with $p \in P_m^{(1+)}$ and $w \in S \setminus I_m$. Then by (7) (and using that f is Abelian) we get

$$f(x) = f(pw) = f(p)m(w) + f(w)m(p) = 0,$$

since $f(p) = m(p) = 0$. This proves condition (I).

Lastly let $x = pw$ with $p \in P_m^{(1)}$ and $w \in S \setminus I_m$. Now (7) yields

$$f(x) = f(pw) = f(p)m(w) + m(p)f(w) = f_P(p)m(w),$$

since $m(p) = 0$. We claim that $x \in P_m^{(1)}$. Certainly $x \in I_m \setminus I_m^2$ by the definition of $P_m^{(1)}$, since $p \in P_m^{(1)}$. If we suppose $x \in P_m^{(1+)}$, then by definition there exists a $w_x \in S \setminus I_m$ such that $xw_x \in I_m^2$. But then we would have $p(ww_x) = xw_x \in I_m^2$, contradicting $p \in P_m^{(1)}$. Therefore $x \in P_m^{(1)}$ and we have condition (II).

Noting the possibility that some values of f_P may be arbitrary, this completes the solution of (2) in case (C).

The converse in case (A) is easily verified by substitution, and in case (B) it is trivial. In case (C), suppose $g = m$ is an exponential, and f is an Abelian function of the form (6) with additive $A\colon S \setminus I_m \to K$ and restriction $f_P : P_m^{(1)} \to K$ such that conditions (I) and (II) hold. We begin the process of verifying that f satisfies (7) with cases in which $x \in S \setminus I_m$. For $y \in S \setminus I_m$ the verification is straightforward, so we omit it. Next, for $y \in I_m^2$ we have $xy \in I_m^2$, thus $f(xy) = f(y) = 0$ by the middle line of (6), so (7) holds since $m(y) = 0$.

Third, if $y \in P_m^{(1+)}$ then $f(y) = 0$ by (6) and $f(yx) = 0$ by condition (I). Hence

$$f(xy) = f(yx) = 0 = f(x)m(y) + f(y)m(x)$$

because $m(y) = 0$. Fourth, if $y \in P_m^{(1)}$ then by the definition of $P_m^{(1)}$ we have $yx \in P_m^{(1)}$, and by condition (II) we get

$$f(xy) = f_P(yx) = f_P(y)m(x) = f(y)m(x) + f(x)m(y)$$

since $m(y) = 0$, so again (7) holds. The mirror cases in which $y \in S \setminus I_m$ work the same way.

In all remaining cases $x, y \in I_m^2 \cup P_m^{(1+)} \cup P_m^{(1)}$, so we have $xy \in I_m^2$ and $f(xy) = m(x) = m(y) = 0$, therefore (7) holds.

In the topological case, the functions f_0 and f_P inherit continuity from f by restriction. For $m \neq 0$ the continuity of A follows from the definition $A := f/m$ on $S \setminus I_m$. $\square$

REMARK 2.2. Examples 5.1–5.3 below show why one cannot make a general declaration as to whether or not f_P takes arbitrary values in case (C). Those examples illustrate that all three possibilities arise: f_P can have arbitrary values at all points of $P_m^{(1)}$, at no points of $P_m^{(1)}$, or at some but not all points of $P_m^{(1)}$. One needs additional information about S and/or m in order to be more precise.

So we find that the solutions of (2) are described mostly in terms of additive and multiplicative functions. The new twist is the possibility of f having arbitrary values at some points.

3. General solution of the cosine addition formula

Recall the cosine addition formula (1):

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S.$$

Vincze [10] (for commutative semigroups) and Chung–Kannappan–Ng [4] (for semigroups in general) showed that the sine addition formula (2) "almost" implies (1) in the following sense. If $f, g \colon S \to K$ satisfy (2), then there exists a constant $\alpha \in K$ such that

$$g(xy) = g(x)g(y) - \alpha f(x)f(y), \quad x, y \in S.$$

This shows that there is a close connection between the two addition formulas. There is a weaker implication in the opposite direction. One can extract the following lemma from the proof of [4, Lemma 4] or [8, Theorem 4.15].

In both places it is stated for the case that S is a group, but a close reading of the proof(s) shows that it is actually true for semigroups. (Note that if $K = \mathbb{C}$, then (1) is equivalent to the equation

$$g(xy) = g(x)g(y) + f'(x)f'(y), \quad x, y \in S,$$

by taking $f' = if$.)

LEMMA 3.1. *If $f, g\colon S \to \mathbb{C}$ satisfy* (1), *then there exists a constant $\alpha \in \mathbb{C}$ such that*

$$f(xy) = f(x)g(y) + g(x)f(y) + \alpha f(x)f(y), \quad x, y \in S. \tag{8}$$

Now we present the general solution of (1) on semigroups. Note that if S is a monoid or a semigroup generated by its squares then $S^2 = S$, but this is not true in general for semigroups. A counterexample is $S = (\mathbb{N}, +)$, for which $S^2 = S \setminus \{1\}$. (The following can also be derived by combining [9, Theorem 6.1] with our Theorem 2.1, but we give a more direct proof.)

THEOREM 3.2. *The solutions $g, f\colon S \to \mathbb{C}$ of the cosine addition law* (1) *are the following families, where $m, m_1, m_2\colon S \to \mathbb{C}$ are multiplicative functions with $m \neq 0$ and $m_1 \neq m_2$.*

(i) $g = f = 0$.

(ii) $g = \frac{c^{-1}m_1 + cm_2}{c^{-1}+c}$ *and* $f = \frac{m_1 - m_2}{i(c^{-1}+c)}$, *where* $c \in \mathbb{C}^* \setminus \{\pm i\}$.

(iii) *If $S^2 \neq S$, then $g = \pm f$ where f has the form given in Theorem* 2.1(B).

(iv) $g = m \pm f$ *where f has the form given in Theorem* 2.1(C).

Note that g and f are Abelian in each case.

Furthermore, if S is a topological semigroup and $f, g \in C(S)$, then $m, m_1, m_2 \in C(S)$, $A \in C(S \setminus I_m)$, $f_0 \in C(S \setminus S^2)$, and $f_P \in C(P_m^{(1)})$.

PROOF. It is easy to check that each of the families (i)–(iv) is a solution of (1).

For the converse, if $g = 0$ then $f = 0$ and we have solution family (i). Henceforth we assume $g \neq 0$.

If $\{g, f\}$ is linearly dependent then $f = bg$ for some constant $b \in \mathbb{C}$. Putting this into (1) we get $g(xy) = (1 - b^2)g(x)g(y)$. In the case $b^2 = 1$ we have $g(xy) = 0$ for all $x, y \in S$. Since $g \neq 0$ this cannot happen unless $S^2 \neq S$, and then we are in solution family (iii). In the case $b^2 \neq 1$ the formula $m := (1 - b^2)g$ defines an exponential $m\colon S \to \mathbb{C}$. If $b = 0$ here, then we have $g = m$ and $f = 0$, which is the special case of solution family (iv) with $A = 0 = f_P$. If $b \neq 0$ then we get the special case of solution family (ii) with $c = ib$, $m_1 = m$, and $m_2 = 0$.

From here on we assume that $\{g, f\}$ is linearly independent. Considering the system (1), (8), we find after a simple calculation that for any $\lambda \in \mathbb{C}$,

$$(g - \lambda f)(xy) = (g - \lambda f)(x)(g - \lambda f)(y) - (\lambda^2 + \alpha\lambda + 1)f(x)f(y), \quad x, y \in S. \tag{9}$$

If λ_1 and λ_2 are the two roots of the equation $z^2+\alpha z+1=0$, then $\lambda_1\lambda_2=1$ so $\lambda_1\neq 0$ and $\lambda_2\neq 0$. Now (9) shows that $m_1:=g-\lambda_1 f$ and $m_2:=g-\lambda_2 f$ are multiplicative functions. In fact they are exponentials since $\{g,f\}$ is linearly independent.

If $\lambda_1\neq\lambda_2$, then $m_1\neq m_2$ and we get

$$g=\frac{\lambda_2 m_1-\lambda_1 m_2}{\lambda_2-\lambda_1}\quad\text{and}\quad f=\frac{m_1-m_2}{\lambda_2-\lambda_1}.$$

Since $\lambda_1\lambda_2=1$, putting $c:=i\lambda_1$ gives the forms in family (ii).

If $\lambda_1=\lambda_2=:\lambda$, then by $\lambda_1\lambda_2=1$ we have either $\lambda=1$ or $\lambda=-1$. We also have $g-\lambda f=m$, where again m is an exponential.

In the case $\lambda=1$, since $\lambda^2+\alpha\lambda+1=0$ we see that $\alpha=-2$. Also $g=m+f$, thus we find from (8) that

$$\begin{aligned}f(xy)&=f(x)(m+f)(y)+(m+f)(x)f(y)-2f(x)f(y)\\&=f(x)m(y)+m(x)f(y).\end{aligned}$$

Therefore the pair f,m satisfies (7). Since $m\neq 0$ we get the form of f from Theorem 2.1(C), and we have solution family (iv).

The case $\lambda=-1$ (where $\alpha=2$ and $g=m-f$) runs parallel to $\lambda=1$ and leads to solution family (iv) for $g=m-f$.

Finally, suppose S is a topological semigroup and $f,g\in C(S)$. It is clear that $m\in C(S)$ in family (iv); we also get $A\in C(S\setminus I_m)$ in case (iv) since $A=f/m$ on $S\setminus I_m$. In case (ii) we get the continuity of m_1 and m_2 from [8, Theorem 3.18(d)]. The functions f_0 and f_P inherit continuity from f by restriction. $\square$

Remark 3.3. In the topological case of Theorem 3.2, we note that if $f\neq 0$ then it is only necessary to assume $f\in C(S)$ to get the continuity of the other functions. For family (ii) it follows as at the end of the proof above. For family (iv) we see from the proof above that f,m satisfy (7). Putting $y=y_0$ there such that $f(y_0)\neq 0$ we get

$$m(x)=\frac{1}{f(y_0)}\left[f(xy_0)-f(x)m(y_0)\right],\quad x\in S,$$

so m is continuous. Therefore $g=m\pm f$ is continuous.

4. General solution of the cosine subtraction formula

Recall that the cosine subtraction formula (3) is

$$g(x\sigma(y))=g(x)g(y)+f(x)f(y),\quad x,y\in S.$$

The general solution of (3) on groups can be found (with a change of notation and terminology) in [8, Theorem 4.16]. Here we find the solution on monoids.

THEOREM 4.1. *Let M be a monoid, and let $\sigma\colon M \to M$ be an automorphic involution. The solutions $g, f\colon M \to \mathbb{C}$ of the cosine subtraction law* (3) *are the following pairs of functions, where $m, m_1, m_2\colon M \to \mathbb{C}$ are multiplicative functions such that $m \neq 0$, $m_1 \neq m_2$ and $m_j \circ \sigma = m_j$ for $j \in \{1, 2\}$.*

(a) $g = f = 0$.

(b) $g = \frac{c^{-1}m_1 + cm_2}{c^{-1}+c}$ *and* $f = \frac{m_2 - m_1}{c^{-1}+c}$, *where* $c \in \mathbb{C}^* \setminus \{\pm i\}$.

(c) $g = \frac{m + m\circ\sigma}{2}$ *and* $f = \frac{m - m\circ\sigma}{2i}$, *where* $m \circ \sigma \neq m$.

(d) $g = m \pm if$, *where f is as described in Theorem* 2.1(C), *with* $m \circ \sigma = m$, $f \circ \sigma = f$, *and* $A \circ \sigma = A$.

Note that f and g are Abelian in each case.

Moreover, if M is a topological monoid and $f, g \in C(M)$, then $m, m \circ \sigma, m_1, m_2 \in C(M)$, $A \in C(M \setminus I_m)$, and $f_P \in C(P_m^{(1)})$.

PROOF. Although [8, Theorem 4.16] is stated for groups, much of the proof works for monoids as well. It shows that $g = g \circ \sigma$ and $f \circ \sigma \in \{f, -f\}$. In the case $f \circ \sigma = f$ we get from (3) that the pair (g, if) satisfies the cosine addition formula (1). Alternatively, if $f \circ \sigma = -f$ then a similar calculation shows that the pair (g, f) satisfies (1).

When we apply Theorem 3.2, we may omit family (iii) since $M^2 = M$ for a monoid M.

Case 1: Suppose $f \circ \sigma = f$, so the pair (g, if) satisfies (1). Taking the solutions of (1) for the pair (g, if) from Theorem 3.2, we have to check the solution families (i), (ii), (iv) in equation (3). Family (i) is our (a). For the other families we may assume that $g \neq 0$.

For family (ii) recall that $m_1 \neq m_2$. Now $f \circ \sigma = f$ implies that

$$m_1 \circ \sigma - m_2 \circ \sigma = m_1 - m_2.$$

Writing this as $m_1 \circ \sigma + m_2 = m_1 + m_2 \circ \sigma$, we get $m_1 \circ \sigma = m_1$ and $m_2 \circ \sigma = m_2$ by an application of [8, Corollary 3.19], since $m_1 \neq m_2$. Therefore

$$g = \frac{c^{-1}m_1 + cm_2}{c^{-1}+c}, \quad if = \frac{m_1 - m_2}{(c^{-1}+c)i},$$

where $c \in \mathbb{C}^* \setminus \{\pm i\}$. This gives the forms in (b), and it is easy to check that they satisfy (3).

For family (iv) we have $g = m \pm if$ for some exponential $m\colon M \to \mathbb{C}$, where if has the form described in Theorem 2.1(C). Thus f has the same form after renaming A. Since g and f are both even we have $m \circ \sigma = m$, and from (6) we get that $A \circ \sigma = A$. Thus we have the forms given in (d). These forms satisfy (3) if and only if

$$\begin{aligned}0 &= g(x\sigma(y)) - g(x)g(y) - f(x)f(y)\\ &= m(x\sigma(y)) \pm if(x\sigma(y)) - [m(x) \pm if(x)][m(y) \pm if(y)] - f(x)f(y)\\ &= m(x)m(y) \pm if(x\sigma(y)) - m(x)m(y) \mp i[f(x)m(y) + m(x)f(y)],\end{aligned}$$

so we have a solution if and only if

$$f(x\sigma(y)) = f(x)m(y) + m(x)f(y), \quad x, y \in M.$$

Since f has the form (6), it is a solution of (7) (the sine addition formula with $g = m$) and therefore

$$f(x\sigma(y)) = f(x)m(\sigma(y)) + m(x)f(\sigma(y)) = f(x)m(y) + m(x)f(y)$$

since both f and m are even. Thus the forms in (d) indeed satisfy (3).

Case 2: Suppose $f \circ \sigma = -f$, so the pair (g, f) satisfies (1). Again we take the solution categories of (1) from Theorem 3.2, excluding case (iii), and check them in (3). Category (i) of Theorem 3.2 is again solution (a), and again we suppose $g \neq 0$ for the rest of the categories.

For category (ii), $f \circ \sigma = -f$ yields $m_1 \circ \sigma - m_2 \circ \sigma = -m_1 + m_2$. We claim that m_1, m_2 are exponentials. Indeed, if $m_2 = 0$ then $m_1 \circ \sigma + m_1 = 0 + 0$, and applying [8, Corollary 3.19] again we get $m_1 = 0$, contradicting $m_1 \neq m_2$. Reasoning as before, we find that $m_2 \circ \sigma = m_1$, $m_1 \circ \sigma = m_2$. Defining $m := m_1$ we have $m_2 = m \circ \sigma$, and

$$f = \frac{m - m \circ \sigma}{i(c^{-1} + c)}$$

for some $c \neq 0, \pm i$. From $g \circ \sigma = g$ we have also

$$c^{-1} m \circ \sigma + cm = c^{-1} m + cm \circ \sigma,$$

hence $(c^{-1} - c)(m \circ \sigma - m) = 0$. Since $m \circ \sigma \neq m$ (which follows from $m_1 \neq m_2$) this means $c = \pm 1$. For $c = 1$ we get $g = (m + m \circ \sigma)/2$ and $f = (m - m \circ \sigma)/2i$, and for $c = -1$ we get the same form with m replaced by $m' = m \circ \sigma$. Thus we have the forms shown in solution (c). It is easily checked that these forms satisfy (3).

In category (iv) we have $g = m \pm f$ with $g \circ \sigma = g$ and $f \circ \sigma = -f$. Thus

$$m \pm f = g = g \circ \sigma = m \circ \sigma \pm f \circ \sigma = m \circ \sigma \mp f,$$

and it follows that $2f = \pm(m \circ \sigma - m)$. Hence, relabeling $m \circ \sigma$ as m if necessary, we have

$$f = \frac{1}{2}(m \circ \sigma - m), \quad g = \frac{1}{2}(m \circ \sigma + m).$$

Substituting these forms into (3), we find after simplification that

$$0 = [m(x) - m \circ \sigma(x)][m(y) - m \circ \sigma(y)],$$

so $m - m \circ \sigma = 0 = f$. This is a special case of solution (d) with $f = 0$.

Finally, suppose M is a topological monoid and $g, f \in C(M)$. The continuity of m in (d) is immediately evident, and the continuity of A and f_P in that case follows as before. To get the continuity of m, $m \circ \sigma$, m_1, m_2 in (b) and (c) we apply [8, Theorem 3.18(d)] as before. □

Remark 4.2. It is not difficult to show that the variant

$$g(\sigma(y)x) = g(x)g(y) + f(x)f(y), \quad x, y \in M,$$

is equivalent to (3) on a monoid M. By symmetry of the right hand side we have $g(\sigma(y)x) = g(\sigma(x)y)$, and with $x = e$ we get that g is even, that is $g \circ \sigma = g$. Now

$$\begin{aligned} g(x\sigma(y)) = g \circ \sigma(x\sigma(y)) = g(\sigma(x)y) &= g(y)g(x) + f(y)f(x) \\ &= g(x)g(y) + f(x)f(y). \end{aligned}$$

The following strengthens [5, Theorem 4.2] by combining it with the extra details provided in Theorem 2.1(C).

Corollary 4.3. *Let M be a monoid, and let $\sigma\colon M \to M$ be an automorphic involution. The solutions $f, g\colon M \to \mathbb{C}$ of the sine subtraction formula* (4) *with $f \neq 0$ are the following pairs of functions, where $m\colon M \to \mathbb{C}$ is multiplicative, $b \in \mathbb{C}$, and $c \in \mathbb{C}^*$.*

(i) *For $m \neq m \circ \sigma$ (so $m \neq 0$) we have*

$$f = c(m - m \circ \sigma), \quad g = \frac{m + m \circ \sigma}{2} + b(m - m \circ \sigma).$$

(ii) *For $m = m \circ \sigma$ we have $g = m + bf$, where f is an odd function of the form given in Theorem* 2.1(C) *with* (*odd*) *additive $A\colon M \setminus I_m \to \mathbb{C}$. Here A, m, f_P must be chosen so that $f \neq 0$.*

Note that f and g are Abelian in each case.

Moreover, if M is a topological monoid and $f \in \mathcal{C}(M)$, then $g, m, m \circ \sigma \in \mathcal{C}(M)$ and $A \in \mathcal{C}(M \setminus I_m)$.

5. Examples

We begin with three examples illustrating Remark 2.2. Taking $g = m \neq 0$ in (2) yields (7) and puts us in case (C) of Theorem 2.1.

To set up these examples we introduce the notion of monogenic semigroup. If S is a semigroup and $a \in S$, then

$$\langle a \rangle := \{a^n \mid n \in \mathbb{N}\}$$

is called the *monogenic semigroup* generated by a. If a monogenic semigroup is infinite then it is isomorphic to $(\mathbb{N}, +)$. In the first two examples we use the same semigroup $(S, *)$, which is constructed using three disjoint monogenic semigroups $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$. Within each monogenic semigroup we omit the multiplication sign. Here $\langle a \rangle$ is isomorphic to $(\mathbb{N}, +)$, $\langle b \rangle = \{b, b^2\}$, and $\langle c \rangle = \{c, c^2\}$, where $b^k = b^2$ and $c^k = c^2$ for all $k \geq 2$. Define a binary operation $*\colon S \times S \to S$ by

$$x * y := \begin{cases} xy & \text{for } (x,y) \in \big(\langle a \rangle \times \langle a \rangle\big) \cup \big(\langle b \rangle \times \langle b \rangle\big) \cup \big(\langle c \rangle \times \langle c \rangle\big) \\ b^j & \text{for } \{x,y\} = \{a^k, b^j\} \\ c^2 & \text{for } \{x,y\} = \{a^k, c^j\} \text{ or } \{x,y\} = \{b^k, c^j\} \end{cases}$$

for all $k, j \in \mathbb{N}$. Then $(S, *)$ is a commutative semigroup. The first two examples use different exponential functions $m\colon S \to \mathbb{C}$, but in both cases we have $S \setminus I_m = \langle a \rangle$. The additive functions $A\colon S \setminus I_m \to \mathbb{C}$ have the form $A(a^k) = \lambda k$ for some constant $\lambda \in \mathbb{C}$.

EXAMPLE 5.1. Define $m\colon S \to \mathbb{C}$ by $m(x) = 1$ for all $x \in \langle a \rangle$ and $m(x) = 0$ for all $x \in \langle b \rangle \cup \langle c \rangle$. Then $I_m = \langle b \rangle \cup \langle c \rangle$, $I_m^2 = \{b^2, c^2\}$, and $I_m \setminus I_m^2 = \{b, c\}$. Since $c * a = c^2 \in I_m^2$ we have $c \in P_m^{(1+)}$, and $b \in P_m^{(1)}$ since $b * a^k = b$ for all $k \in \mathbb{N}$. Thus by Theorem 2.1(C) the solutions of (7) have the form

$$f(x) = \begin{cases} \lambda k & \text{for } x = a^k \in \langle a \rangle \\ 0 & \text{for } x \in \{b^2, c^2, c\} \\ \beta & \text{for } x = b \end{cases}$$

where $\beta = f_P(b)$ and λ are arbitrary complex constants.

In the second example we keep the same semigroup but change the exponential m.

EXAMPLE 5.2. This time define m on $\langle a \rangle$ by $m(a^k) = 2^k$, and $m(x) = 0$ for $x \in \langle b \rangle \cup \langle c \rangle$. Again we have $I_m^2 = \{b^2, c^2\}$, $P_m^{(1+)} = \{c\}$, and $P_m^{(1)} = \{b\}$. Taking the solutions of (7) from Theorem 2.1(C), condition (II) yields $f(b) = f(b * a^k) = f_P(b)2^k$ for all $k \in \mathbb{N}$. This is possible only if $f_P(b) = f(b) = 0$. So this time the solution reduces to

$$f(x) = \begin{cases} \lambda k & \text{for } x = a^k \in \langle a \rangle \\ 0 & \text{for } x \in \langle b \rangle \cup \langle c \rangle \end{cases}$$

where $\lambda \in \mathbb{C}$ is arbitrary.

In the third example we use a familiar semigroup.

EXAMPLE 5.3. Let $S = (\mathbb{N}, \cdot)$, and let $m\colon \mathbb{N} \to \mathbb{C}$ be the exponential function defined by $m(x) = 1$ if x is odd and $m(x) = 0$ if x is even. Then $I_m = 2\mathbb{N}$, $I_m^2 = 4\mathbb{N}$, $P_m^{(1)} = 2\mathbb{N} \setminus 4\mathbb{N}$, and $P_m^{(1+)} = \emptyset$. By Theorem 2.1(C) the form of $f\colon \mathbb{N} \to \mathbb{C}$ is

$$f(x) = \begin{cases} A(x) & \text{for } x \in \mathbb{N} \setminus 2\mathbb{N} \\ 0 & \text{for } x \in 4\mathbb{N} \\ f_P(x) & \text{for } x \in 2\mathbb{N} \setminus 4\mathbb{N} \end{cases}$$

for additive $A\colon \mathbb{N} \setminus 2\mathbb{N} \to \mathbb{C}$ and some $f_P\colon 2\mathbb{N} \setminus 4\mathbb{N} \to \mathbb{C}$. Here condition (II) provides the additional information that

$$f(2(2j+1)) = f_P(2) \quad \text{for all } j \in \mathbb{N} \cup \{0\}.$$

Defining $\tau := f_P(2)$ we get $f_P(x) = \tau$ for all $x \in 2\mathbb{N} \setminus 4\mathbb{N}$, therefore

$$f(x) = \begin{cases} A(x) & \text{for } x \in \mathbb{N} \setminus 2\mathbb{N} \\ 0 & \text{for } x \in 4\mathbb{N} \\ \tau & \text{for } x \in 2\mathbb{N} \setminus 4\mathbb{N} \end{cases}$$

where $\tau \in \mathbb{C}$ is arbitrary.

An example of an additive function on $\mathbb{N} \setminus 2\mathbb{N}$ (in fact on the whole semigroup) is $C_3(x) :=$ the number of times 3 occurs in the prime factorization of x.

We see from these three examples that f_P in formula (6) may take arbitrary values at all points of $P_m^{(1)}$ (Example 5.1), at no points of $P_m^{(1)}$ (Example 5.2), or at some but not all points of $P_m^{(1)}$ (Example 5.3). In general we cannot know which is the case without having additional information about S and/or m.

Now we turn to examples applying the results of Theorems 3.2 and 4.1 to solve (1) and (3). Examples on groups can be found in [8, Chapter 4], so we give examples on semigroups which are not groups. In the next example the semigroups have no prime ideals.

EXAMPLE 5.4. For any $k \in \mathbb{N}$ let S be the k-fold direct product $\mathbb{N} \times \cdots \times \mathbb{N}$ under (componentwise) addition. It is easy to see that S has no prime ideal. Any additive $A\colon S \to \mathbb{C}$ has the form $A(x_1, \dots, x_k) = \sum_{j=1}^{k} a_j x_j$, and any exponential $m\colon S \to \mathbb{C}$ has the form $m(x_1, \dots, x_k) = \prod_{j=1}^{k} (b_j)^{x_j}$, where $a_j \in \mathbb{C}$ and $b_j \in \mathbb{C}^*$. The solutions of (1) for $g, f\colon S \to \mathbb{C}$ are obtained by using these formulas in the solution forms of Theorem 3.2, where $f = Am$ in family (iv).

In preparation for our next example we quote the following from [7, Lemma 5.4]. We write $\mathfrak{R}(\alpha)$ for the real part of a complex number α.

LEMMA 5.5. *Let $M = M(2,\mathbb{C})$ be the monoid of 2×2 complex matrices under multiplication, and let $GL(2,\mathbb{C})$ be the subgroup of non-singular matrices in M.*

(i) *If $m\colon M \to \mathbb{C}$ is a continuous exponential, then either $m = 1$ or*

$$m(X) = \begin{cases} |\det(X)|^{\lambda-n}\,(\det(X))^n & \text{for } \det(X) \neq 0 \\ 0 & \text{for } \det(X) = 0, \end{cases} \tag{10}$$

for some $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ with $\mathfrak{R}(\lambda) > 0$.

(ii) *The continuous additive functions $A\colon GL(2,\mathbb{C}) \to \mathbb{C}$ have the form*

$$A(X) = \delta \log|\det(X)|, \quad X \in GL(2,\mathbb{C}),$$

for some $\delta \in \mathbb{C}$.

Now we are ready for an example where our semigroup has a prime ideal.

EXAMPLE 5.6. Consider the monoid $M = M(2,\mathbb{C})$. For any exponential $m \in C(M)$, it is clear by Lemma 5.5 that either $I_m = \emptyset$ or $I_m = M \setminus GL(2,\mathbb{C})$. In either case we have $I_m \setminus I_m^2 = \emptyset$. This is obvious if $I_m = \emptyset$; in the other case it follows from the fact that M is generated by its squares (see [7, p. 192]).

(a) We get the continuous solutions of (1) on M by substituting the appropriate forms from Lemma 5.5 into the formulas for f and g given in Theorem 3.2, where $I_m \setminus I_m^2 = \emptyset$ in family (iv) so (6) reduces to

$$f(X) = \begin{cases} A(X)m(X) & \text{for } X \in M \setminus I_m \\ 0 & \text{for } X \in I_m. \end{cases} \tag{11}$$

Family (iii) does not arise since M is a monoid.

(b) Let $\sigma\colon M \to M$ be the complex conjugation operator

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \quad \text{for all } a,b,c,d \in \mathbb{C}.$$

We get the continuous solutions of (3) on M by substituting the appropriate forms from Lemma 5.5 into the formulas for f and g given in Theorem 4.1, with $I_m \setminus I_m^2 = \emptyset$ in family (d) so (6) reduces to (11). Here m is even only if either $m = 1$ or $n = 0$ in (10). Every A is even.

(c) Let $\sigma\colon M \to M$ be the *adjugate-transpose operator* defined by

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad \text{for all } a, b, c, d \in \mathbb{C}.$$

We get the continuous solutions of (3) on M by substituting the appropriate forms from Lemma 5.5 into the formulas for f and g given in Theorem 4.1, using (11) in place of (6) again in family (d). Here every m and A are even.

In the preceding example our monoid had a zero, namely the zero matrix. We say that 0 is a *zero* of S if $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$. There can be at most one zero in S. Note that if S has a 0, then the only additive function on S is the zero function. (That follows from $A(0) = A(x \cdot 0) = A(x) + A(0)$ for all $x \in S$.)

Now consider the extended natural numbers $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ under addition, where $x + \infty = \infty + x = \infty + \infty = \infty$ for all $x \in \mathbb{N}$. Thus ∞ is a zero for $(\bar{\mathbb{N}}, +)$.

Example 5.7. Let $S = \bar{\mathbb{N}} \times \bar{\mathbb{N}}$ under addition. The prime ideals of S are $I_1 = \{\infty\} \times \bar{\mathbb{N}}$, $I_2 = \bar{\mathbb{N}} \times \{\infty\}$, and $I_1 \cup I_2$. Nullspaces with their corresponding exponentials $m\colon S \to \mathbb{C}$ and additive functions $A\colon S \setminus I_m \to \mathbb{C}$ are the following, for some $b_1, b_2 \in \mathbb{C}^*$ and $c_1, c_2 \in \mathbb{C}$.

(i) $I_m = \emptyset$, with $m = 1$ and $A = 0$.

(ii) $I_m = I_1$, with $m(x, y) = (b_1)^x$ and $A(x, y) = c_1 x$ for all $(x, y) \in S \setminus I_m$.

(iii) $I_m = I_2$, with $m(x, y) = (b_2)^y$ and $A(x, y) = c_2 y$ for all $(x, y) \in S \setminus I_m$.

(iv) $I_m = I_1 \cup I_2$, with $m(x, y) = (b_1)^x (b_2)^y$ and $A(x, y) = c_1 x + c_2 y$ for all $(x, y) \in S \setminus I_m$.

We get the solutions $f, g\colon S \to \mathbb{C}$ of (1) by plugging the appropriate forms above into the formulas of Theorem 3.2.

Note that since the semigroup operation is addition we have $I_m^2 = \{x + y \mid x, y \in I_m\}$, so $I_1 \setminus I_1^2 = \{(\infty, 1)\}$ in case (ii). In this case we see that $(\infty, 1) + (1, 1) = (\infty, 2) \in I_1^2$, where $(1, 1) \in S \setminus I_1$, so $(\infty, 1) \in P_m^{(1+)}$. Similar calculations show that $I_m \setminus I_m^2 = P_m^{(1+)}$ also in cases (iii) and (iv). Thus in Theorem 3.2(iv) we find that the form of f simplifies to

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m \\ 0 & \text{for } x \in I_m. \end{cases} \tag{12}$$

Note this corrects a mistake in [5, Example 5.7], where it was claimed that the form of f does not reduce to (12).

For the next example we note first that the continuous exponentials on the monoid $M = [-1,1]$ under multiplication have one of the three forms (13)

$$m_0 := 1,\ m_\alpha(x) := \begin{cases} |x|^\alpha & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases} \text{ or } m_\alpha^\pm(x) := \begin{cases} |x|^\alpha \operatorname{sgn}(x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

where $\Re(\alpha) > 0$. Since M has a zero, the only additive function on M is $A = 0$. On the sub-monoid $M \setminus \{0\}$ the continuous additive functions have the form $A(x) = c \log |x|$ for all $x \neq 0$, where c can be any complex constant.

Example 5.8. Let $S = [-1,1] \times [-1,1]$ under multiplication and the product topology. The continuous exponentials on S have the form $m(x,y) = m_1(x)m_2(y)$, denoted $m = m_1 \otimes m_2$, where each $m_j : [-1,1] \to \mathbb{C}$ has one of the three forms in (13) with $\Re(\alpha) > 0$. The null ideals corresponding to continuous exponentials on S are $I_1 = \{0\} \times [-1,1]$, $I_2 = [-1,1] \times \{0\}$, and $I_1 \cup I_2$. Exponentials $m \in C(S)$ and additive functions $A \in C(S \setminus I_m)$ corresponding to these ideals are as follows, where $c_1, c_2, \alpha, \beta \in \mathbb{C}$ with $\Re(\alpha), \Re(\beta) > 0$.

(a) For $I_m = \emptyset$, $m = m_0 \otimes m_0 = 1$ and $A = 0$.

(b) For $I_m = I_1$, $m \in \{m_\alpha \otimes m_0, m_\alpha^\pm \otimes m_0\}$ and $A(x,y) = c_1 \log |x|$.

(c) For $I_m = I_2$, $m \in \{m_0 \otimes m_\alpha, m_0 \otimes m_\alpha^\pm\}$ and $A(x,y) = c_2 \log |y|$.

(d) For $I_m = I_1 \cup I_2$, $m = m_1 \otimes m_2$ with $m_1, m_2 \in \{m_\alpha, m_\beta, m_\alpha^\pm, m_\beta^\pm\}$ and $A(x,y) = c_1 \log |x| + c_2 \log |y|$.

We get the continuous solutions of (1) on S by substituting the appropriate forms above into the formulas for f and g given in Theorem 3.2. Here $I_m \setminus I_m^2 = \emptyset$ in family (iv) so the form of f again reduces to (12). Family (iii) does not arise since S is a monoid.

Let $\sigma \colon S \to S$ be defined by $\sigma(x,y) = (y,x)$. We get the solutions $f, g \in C(S)$ of (3) by plugging the appropriate forms above into the formulas of Theorem 4.1, using (12) in place of (6) in family (d). An exponential m is even (that is $m = m \circ \sigma$) if and only if it has one of the forms $m = 1$, $m = m_\alpha \otimes m_\alpha$, or $m = m_\alpha^\pm \otimes m_\alpha^\pm$. For the cases $m = m_\alpha \otimes m_\alpha$ and $m = m_\alpha^\pm \otimes m_\alpha^\pm$ the corresponding additive function A is even if and only if $c_2 = c_1$.

We close with an example on a semigroup that is fundamental for computer scientists. Let T be a set containing n elements for some positive integer $n \geq 2$, and let T^* be the set of (possibly empty) finite sequences of elements from T. The elements of T^* are called *strings* (or words), and the elements of T are called *letters*. Under the operation of concatenation of strings, T^* becomes a monoid, called the *free monoid* on alphabet T, with the empty string serving as the identity element. We identify each $\ell \in T$ with the string in T^* consisting of exactly one copy of the letter ℓ, so $T \subset T^*$. For each letter $\ell \in T$, the set of all strings containing at least one copy of ℓ

is a prime ideal which we denote T^*_ℓ. Furthermore every proper nonempty subset $L \subset T$ generates a prime ideal denoted

$$T^*_L := \bigcup_{\ell \in L} T^*_\ell.$$

Thus T^* has $2^n - 2$ prime ideals.

EXAMPLE 5.9. Let T^* denote the free monoid on an alphabet T that contains $n \geq 2$ letters. The forms of multiplicative and additive functions on T^* are straightforward to compute in terms of their values on T. For each $\ell \in T$ and $x \in T^*$, let $C_\ell(x)$ denote the number of times letter ℓ appears in string x. This defines an additive function on T^*. For each $x \in T^*$ let $T_x = \{\ell \in T \mid C_\ell(x) \geq 1\}$. Then for any multiplicative $m : T^* \to \mathbb{C}$ we have

$$m(x) = \prod_{\ell \in T_x} m(\ell)^{C_\ell(x)}, \quad x \in T^*.$$

Such m is an exponential if and only if there exists an $\ell \in T$ such that $m(\ell) \neq 0$.

Given an exponential m, suppose $w \in I_m \setminus I_m^2$. Then there do not exist $u, v \in I_m$ such that $w = uv$. Therefore w must contain exactly one letter $\ell \in I_m$, and that letter must occur exactly once in w, so $C_\ell(w) = 1$. It is not hard to see that $I_m \setminus I_m^2 = P_m^{(1)}$ and $P_m^{(1+)} = \emptyset$. Condition (II) of Theorem 2.1(C) governs the form of f_P. For each letter $\ell \in T \cap P_m^{(1)}$ the value of $f_P(\ell)$ can be chosen arbitrarily, then condition (II) determines the values of $f_P(x)$ for all $x \sim \ell w$ with $w \in S \setminus I_m$.

If $m\colon T^* \to \mathbb{C}$ is an exponential with null ideal T^*_L for some $\emptyset \neq L \subset T$, then the additive functions $A\colon T^* \setminus T^*_L$ have the form

$$A(x) = \sum_{k \in T \setminus L} A(k) C_k(x), \quad x \in T^* \setminus T^*_L.$$

(a) We get the solutions of (1) on T^* by substituting the appropriate forms described above into the formulas for f and g given in Theorem 3.2. Family (iii) does not occur since T^* is a monoid.

(b) Let $\ell_1 \neq \ell_2 \in T$, and define $\sigma : T^* \to T^*$ by $\sigma(x) := \hat{x}$ where $\hat{x}$ is the word obtained from x by replacing each occurrence of ℓ_1 in x by ℓ_2 and vice versa. An exponential m is even if and only if $m(\ell_1) = m(\ell_2)$. Note that additive functions A enter the solutions in Theorem 4.1 only if m is even. Suppose m is even and has null ideal T^*_L for some nonempty $L \subset T$. Then from $m(\ell_1) = m(\ell_2)$ we get that either both or neither of ℓ_1, ℓ_2 belong to L. If both belong to L then an additive $A : T^* \setminus T^*_L$ is even by default. If neither of ℓ_1, ℓ_2 belongs to L then A is even if and only if $A(\ell_1) = A(\ell_2)$. Similar considerations apply to f_P (so that f is even) in Theorem 4.1(d). We get the solutions of (3) on T^* by plugging the appropriate forms into the formulas for f and g given in Theorem 4.1.

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