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THE SINE ADDITION AND SUBTRACTION FORMULAS ON SEMIGROUPS

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Abstract. The sine addition formula on a semigroup S is the functional equation $f(xy) = f(x)g(y) + g(x)f(y)$ for all $x, y \in S$. For some time the solutions have been known on groups, regular semigroups, and semigroups which are generated by their squares. The obstacle to finding the solution on all semigroups arose in the special case that g is a multiplicative function. We overcome this obstacle and find the general solution on all semigroups using a transfinite induction argument. A new type of solution appears which is not seen on regular semigroups or semigroups generated by their squares.

We also give the general solution of the sine subtraction formula $f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$ on monoids, where σ is an automorphic involution. The solutions of both equations can be described in terms of additive and multiplicative functions, with a slight new twist. The general continuous solutions on topological semigroups are also found. A variety of examples are given to illustrate the results.

1. Introduction

The *sine addition formula* on a semigroup S into a (commutative) field K is the functional equation

(1) $$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in S,$$

for two unknown functions $f, g \colon S \to K$. We use multiplicative notation for the semigroup operation, since S is not assumed to be commutative. This functional equation generalizes the trigonometric identity

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y), \quad x, y \in \mathbb{R}.$$

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The sine addition formula has attracted much interest, due both to its fundamental nature and to its connections with other trigonometric functional equations. It also has applications to other branches of mathematics such as the theory of function algebras (see [5, section 4.3]). It has been studied extensively (both individually and as part of a system of equations) for the past century, beginning with the case $S = (\mathbb{R}, +)$ and $K = \mathbb{R}$. For references to early works on this equation see [1, Section 3.2.3]. Over the years regularity assumptions on the functions have been weakened, and domains have been generalized to groups and eventually to semigroups. It was solved on Abelian groups by Vincze [6], and on general groups by Chung, Kannappan, and Ng [3]. The most current result on semigroups was given by Stetkær and the author [4] (see Lemma 2.1 below). Our main goal is to give the general solution of (1) on all semigroups.

A secondary goal is the solution of the *sine subtraction formula*

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad x, y \in S, \tag{2}$$

which generalizes the identity

$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y), \quad x, y \in \mathbb{R}.$$

We solve (2) for unknown functions $f, g \colon S \to \mathbb{C}$, where S is a monoid and $\sigma \colon S \to S$ is an automorphic involution. That σ is an *involution* means $\sigma \circ \sigma(x) = x$ for all $x \in S$.

For additional discussions of these functional equations and their history, see [2, Ch. 13], [5, Ch. 4], and their references.

The outline of the paper is as follows. In the next section we present some background information, including the current state of knowledge about (1). Lemma 2.1 shows that the missing piece of the solution of (1) occurs in the case that g is multiplicative. The complete solution of (1) for that case is given in Theorem 3.5, which shows that all solutions can be described in terms of additive and multiplicative functions. Corollary 3.10 sums up the general solution of (1) on semigroups. In section 4 we give the general solution of (2) on monoids. The paper concludes with a variety of applications and examples of these results on various kinds of semigroups.

2. Background and setup

The basic building blocks for solutions of the sine addition and subtraction formulas are the homomorphisms of S (or one of its sub-semigroups) into the additive and multiplicative semigroups of K. A function $A \colon S \to K$ is *additive* if

$$A(xy) = A(x) + A(y), \quad \text{for all } x, y \in S,$$

and a function $m\colon S \to K$ is *multiplicative* if

$$m(xy) = m(x)m(y), \quad \text{for all } x, y \in S.$$

If m is multiplicative and $m \neq 0$ then we call m an *exponential*. A significant role is played by the *nullspace*

$$I_m := \{x \in S \mid m(x) = 0\}$$

of a multiplicative function m. If the nullspace is nonempty then it is an ideal (two-sided) which we may call the *null ideal* of m.

For any field K let $K^* = K \setminus \{0\}$.

For a topological semigroup S and a topological field K, let $\mathcal{C}(S, K)$ denote the algebra of continuous functions mapping S into K. We abbreviate $\mathcal{C}(S, \mathbb{C})$ as $\mathcal{C}(S)$.

The current knowledge about (1) is summed up in the following, which is [4, Lemma 3.4 and Remark 3.5]. Recall that a *monoid* is a semigroup with an identity element. A semigroup S is *regular* if for every $a \in S$ there exists an $x \in S$ such that $axa = a$.

LEMMA 2.1. *Let S be a semigroup, and suppose $f, g\colon S \to \mathbb{C}$ satisfy the sine addition law* (1) *with $f \neq 0$. Then there exist multiplicative functions $m_1, m_2\colon S \to \mathbb{C}$ such that*

$$g = \frac{m_1 + m_2}{2}.$$

Additionally we have the following.

(i) *If $m_1 \neq m_2$, then $f = c(m_1 - m_2)$ for some constant $c \in \mathbb{C}^*$.*

(ii) *If $m_1 = m_2$, then letting $m := m_1$ we have $g = m$. If S is a semigroup such that $S = \{xy \mid x, y \in S\}$ (for instance a monoid), then $m \neq 0$ (i.e. m is an exponential).*

If S is a group, then there is a nonzero additive function $A\colon S \to \mathbb{C}$ such that $f = Am$.

If S is a semigroup which is regular or generated by its squares, then there exists an additive function $A\colon S \setminus I_m \to \mathbb{C}$ for which

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m \\ 0 & \text{for } x \in I_m. \end{cases}$$

Furthermore, if S is a topological group or semigroup (regular or generated by its squares), and $f, g \in \mathcal{C}(S)$, then $m_1, m_2, m \in \mathcal{C}(S)$. In the group case $A \in \mathcal{C}(S)$ and in the second case $A \in \mathcal{C}(S \setminus I_m)$.

Note that $\mathbb{C}$ can be replaced here by any quadratically closed commutative field of characteristic different from 2 (see e.g. [2, p. 212]).

So we see that (1) is completely solved (for complex valued functions) on semigroups in the case $m_1 \neq m_2$, that is, when g is the arithmetic mean of two distinct multiplicative functions. The remaining case, in which g is a multiplicative function m, reduces the sine addition formula to the special case

$$f(xy) = f(x)m(y) + m(x)f(y), \quad x, y \in S. \tag{3}$$

This equation also has been solved on groups and certain types of semigroups. Our primary goal is to find its solution on all semigroups.

Prime ideals of S play a key role in the study of (1) and (3), so we define them now. An ideal I of a semigroup S is called a *prime ideal* if $I \neq S$ and whenever $xy \in I$ it follows that either $x \in I$ or $y \in I$. This means that an ideal I is prime if and only if $S \setminus I$ is a (proper nonempty) sub-semigroup of S.

There is a very close relationship between prime ideals and exponentials on semigroups. If $m\colon S \to K$ is exponential, then the nullspace I_m is an ideal if it is nonempty. Furthermore if $I_m \neq \emptyset$ then it is easy to see that the null ideal I_m is a prime ideal. Conversely, if I is a prime ideal of S, then there exists an exponential $m\colon S \to K$ such that $I = I_m$ (just take $m(x) = 1$ for $x \in S \setminus I$ and $m(x) = 0$ for $x \in I$).

A function f on a semigroup S is said to be *Abelian* if $f(x_{\pi(1)} \cdots x_{\pi(n)}) = f(x_1 \cdots x_n)$ for all $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \in S^n$, and permutations π on $\{1, \ldots, n\}$. The next result shows that for any semigroup S and any solutions $f, m\colon S \to K$ of (3) with m multiplicative, the function f is Abelian. (Note that additive functions and multiplicative functions into a commutative ring are always Abelian.)

Lemma 2.2. *Let S be a semigroup, let R be a commutative ring, and let $f, m\colon S \to R$ be a solution of (3) with m multiplicative. Then for every $n \geq 2$,*

$$f(x_1 \cdots x_n) = \sum_{j=1}^{n} f(x_j) \prod_{k \in \{1,\ldots,n\} \setminus \{j\}} m(x_k), \quad \textit{for all } (x_1, \ldots, x_n) \in S^n. \tag{4}$$

Consequently f is Abelian.

Proof. We proceed by induction. For $n = 2$, equation (4) states that $f(x_1x_2) = f(x_1)m(x_2) + f(x_2)m(x_1)$, which is (3). Now suppose (4) is valid for some $n \geq 2$. Then by the inductive hypothesis and (3) we have

$$\begin{aligned} f(x_1 \cdots x_{n+1}) &= f((x_1 \cdots x_n)x_{n+1}) \\ &= f(x_1 \cdots x_n)m(x_{n+1}) + f(x_{n+1})m(x_1 \cdots x_n) \end{aligned}$$

$$= \sum_{j=1}^{n} f(x_j) \prod_{k\in\{1,\dots,n\}\setminus\{j\}} m(x_k)m(x_{n+1}) + f(x_{n+1}) \prod_{k=1}^{n} m(x_k)$$

$$= \sum_{j=1}^{n+1} f(x_j) \prod_{k\in\{1,\dots,n+1\}\setminus\{j\}} m(x_k)$$

for all $(x_1, \dots, x_{n+1}) \in S^{n+1}$. Therefore we have (4) for all $n \geq 2$. Clearly this implies that f is Abelian, since the right hand side of (4) is symmetric in all its variables. $\square$

Lemma 2.2 allows us to operate as if S were commutative, since our variables appear only as arguments of Abelian functions. To describe this license to permute variables in the argument of an Abelian function we introduce the following notion. Define the relation $\sim$ on semigroup S by $x \sim y$ if and only if there exist $s_1, \dots, s_n \in S$ and a permutation π on $\{1, \dots, n\}$ such that $x = s_1 \cdots s_n$ and $y = s_{\pi(1)} \cdots s_{\pi(n)}$. It is clear that if $x \sim y$ then $\varphi(x) = \varphi(y)$ for any Abelian function $\varphi\colon S \to K$. We read the statement $x \sim y$ as "*x rearranges to y.*"

The following simple observations will be useful in the next section.

Lemma 2.3. *Suppose $f, m\colon S \to K$ satisfy* (3) *with m multiplicative. If $x \sim pw$ with $p \in I_m$ and $w \in S \setminus I_m$, then $f(x) = f(p)m(w)$. Moreover, if $x \sim p_1 w_1$ and $x \sim p_2 w_2$ with $p_1, p_2 \in I_m$ and $w_1, w_2 \in S \setminus I_m$, then $f(p_1)m(w_1) = f(p_2)m(w_2)$.*

Proof. If $x \sim pw$ with $p \in I_m$ and $w \in S \setminus I_m$, then by (3) we get

$$f(x) = f(pw) = f(p)m(w) + f(w)m(p) = f(p)m(w),$$

since $m(p) = 0$. The second statement follows immediately since $f(p_i)m(w_i) = f(x)$ for $i = 1, 2$. $\square$

3. General solutions of (3) and (1)

For any $q \in S$, the *monogenic semigroup* generated by q is

$$\langle q \rangle := \left\{ q^n \mid n \in \mathbb{N} \right\}.$$

Note that for additive $A\colon S \to K$ and multiplicative $m\colon S \to K$ we have $A(q^n) = nA(q)$ and $m(q^n) = m(q)^n$, so the values of A and m on $\langle q \rangle$ are completely determined by their respective values at q.

If $\langle q \rangle$ is infinite then it is isomorphic to $(\mathbb{N}, +)$. This is sometimes called the free monogenic semigroup because it is a free semigroup with one generator.

If $\langle q \rangle$ is finite then we say that q is *periodic*. Then there exist positive integers $k \neq n$ such that $q^k = q^n$. If k is the smallest positive integer such that $q^k = q^n$ for some $n > k$, and if r is the smallest positive integer such that $q^k = q^{k+r}$, then we refer to k as the *index* and r as the *period* of the finite monogenic semigroup

$$\langle q \rangle = \{ q, q^2, \dots, q^{k+r-1} \}.$$

In this case $\{q^k, \dots, q^{k+r-1}\}$ is a cyclic subgroup of S.

LEMMA 3.1. *Let $q \in S$, and suppose $f, m \colon S \to K$ satisfy* (3) *with m multiplicative. Then we have the following.*

(i) *If $q \in S \setminus I_m$, then there exists an additive function $A_q \colon \langle q \rangle \to K$ such that $f = A_q m$ on $\langle q \rangle$.*

(ii) *If $q \in I_m$, then $f(q^n) = 0$ for all $n \geq 2$.*

Note that the value $f(q)$ may be unspecified for some $q \in I_m$.

PROOF. For part (i) suppose $q \in S \setminus I_m$. Then $m(q) \neq 0$ and thus $m(x) \neq 0$ for all $x \in \langle q \rangle$. Dividing (3) by $m(xy) = m(x)m(y) \neq 0$ for $x, y \in \langle q \rangle$, we see that

$$A_q(x) := \frac{f(x)}{m(x)} \quad \text{for all } x \in \langle q \rangle$$

defines an additive function $A_q \colon \langle q \rangle \to K$.

For part (ii) suppose $q \in I_m$, so $m(q) = 0$. For any $n \geq 2$, putting $x_1 = \dots = x_n = q$ in (4) we get

$$f(q^n) = nf(q)m(q)^{n-1} = 0.$$

□

Now define

$$I_m^2 := \{ yz \mid y, z \in I_m \} \quad \text{and} \quad P_m := I_m \setminus I_m^2.$$

We can think of P_m as the set of "prime-like" elements of I_m, in the sense that they cannot be realized as the product of two elements of I_m.

We introduce the notion of a sub-solution pair for (3).

DEFINITION 3.2. Let $f, m \colon S \to K$ be a solution of (3) with m multiplicative, and let $\mathcal{E}$ be the empty function from the empty set to K. Suppose

(i) T is a sub-semigroup of S; and

(ii) if $T \setminus I_m \neq \emptyset$ then $A_T : T \setminus I_m \to K$ is additive, otherwise $A_T = \mathcal{E}$.

We call (T, A_T) a *sub-solution pair* for (3) if

$$(5)\quad f(x) = \begin{cases} A_T(x)m(x) & \text{for } x \in T \setminus I_m \\ f(p)m(w) & \text{for } x \sim pw \in P_m \text{ with } p \in T \cap P_m,\ w \in T \setminus I_m \\ 0 & \text{for } x \in T \cap I_m^2. \end{cases}$$

The middle line of (5) is justified and consistent by an application of Lemma 2.3 on T.

By Lemma 3.1 we see that if $q \in S$ with $m(q) \neq 0$ then $(\langle q\rangle, f/m)$ is a sub-solution pair for (3). In this case the last two lines of (5) are vacuous. On the other hand if $m(q) = 0$ then $(\langle q\rangle, \mathcal{E})$ is a sub-solution pair for (3), where the first two lines of (5) are vacuous and the value of $f(q)$ is unspecified (unless $\langle q\rangle$ is a cyclic group, in which case $f(q) = 0$).

For a sub-semigroup $T \subseteq S$ and an element $y \in S \setminus T$, let $\langle T, y\rangle$ denote the semigroup generated by $T \cup \{y\}$. We show that a sub-solution pair for (3) on T can be extended to a sub-solution pair on $\langle T, y\rangle$.

Proposition 3.3. *Let $f, m\colon S \to K$ be a solution of* (3) *with m multiplicative, and let (T, A_T) be a sub-solution pair. Let $q \in S\setminus T$ and $T' = \langle T, q\rangle$. Then there exists an additive function $A_{T'} : T' \setminus I_m \to K$ such that $(T', A_{T'})$ is a sub-solution pair.*

Proof. We divide the proof into two cases, depending on the location of q.

Case 1: $q \in S \setminus I_m$. By Lemma 3.1 there is an additive function $A_q\colon \langle q\rangle \to K$ such that $f = A_q m$ on $\langle q\rangle$. Since T' contains both T and $\langle q\rangle$, our first step is to extend A_T to $\bar{A}_T\colon (T \setminus I_m) \cup \langle q\rangle \to K$ by the definition $\bar{A}_T := A_T \cup A_q$. This is not a problem if $(T \setminus I_m) \cap \langle q\rangle = \emptyset$. If there exist $t \in T \setminus I_m$ and $q^j \in \langle q\rangle$ such that $t = q^j$, then we have to check our definition of $\bar{A}_T$ for consistency. In this case we have

$$A_T(t)m(t) = f(t) = f(q^j) = A_q(q^j)m(q^j) = A_q(q^j)m(t).$$

Since $m(t) \neq 0$ this implies $A_T(t) = A_q(q^j)$, so the additive functions A_T and A_q agree at points where the underlying sub-semigroups intersect, thus $\bar{A}_T$ is well-defined.

Next we will extend $\bar{A}_T$ to an additive function $A_{T'}\colon T' \setminus I_m \to K$. For this we need to consider arbitrary products of elements from $T \setminus I_m$ and $\langle q\rangle$. Since these elements only appear as inputs to Abelian functions, any such product x can be rearranged to an element of the form tq^j for some $t \in T \setminus I_m$ and $j \in \mathbb{N}$, thus $x \sim tq^j$. For any $x \sim tq^j \in T' \setminus (T \cup \langle q\rangle \cup I_m)$, by (3) we have

$$\begin{aligned} f(tq^j) &= f(t)m(q^j) + f(q^j)m(t) \\ &= \bar{A}_T(t)m(t)m(q^j) + \bar{A}_T(q^j)m(q^j)m(t) = \left[\bar{A}_T(t) + \bar{A}_T(q^j)\right] m(tq^j). \end{aligned}$$

Defining $A_{T'}\colon T' \setminus I_m \to K$ by $A_{T'} := \bar{A}_T$ on $(T \setminus I_m) \cup \langle q\rangle$, and by

$$A_{T'}(tq^j) := \bar{A}_T(t) + \bar{A}_T(q^j)$$

on $T' \setminus (T \cup \langle q \rangle \cup I_m)$, we see that $A_{T'}$ is additive and $f = A_{T'} m$ on $T' \setminus I_m$. This proves the top line of (5) on T'.

Verifying the third line of (5) on T' is quite simple. If $x \in T' \cap I_m^2$, then $x = y'z'$ for some $y', z' \in I_m$. Then (3) yields

$$f(x) = f(y'z') = f(y')m(z') + f(z')m(y') = 0,$$

since $m(y') = m(z') = 0$.

Verification of the second line of (5) on T' is even easier. Suppose $x \sim p'w' \in T' \cap P_m$ with $p' \in T' \cap P_m$ and $w' \in T' \setminus I_m$. Then applying Lemma 2.3 on T' we get $f(x) = f(p')m(w')$.

This completes Case 1.

Case 2: $q \in I_m$. Since I_m is an ideal we have in this case $T' \setminus I_m = T \setminus I_m$, so defining $A_{T'} := A_T$ we immediately get $f = A_{T'} m$ on $T' \setminus I_m$. This gives the first line of (5) on T'.

The second and third lines of (5) hold on T' by the same calculations as in Case 1.

Therefore $(T', A_{T'})$ is a sub-solution pair. □

Define a partial order $\leq$ on the set of sub-solution pairs by $(T_1, A_1) \leq (T_2, A_2)$ if $T_1 \subseteq T_2$ and A_2 agrees with A_1 on T_1. The next step is to show that every chain of sub-solution pairs has an upper bound.

LEMMA 3.4. *Let Λ be a linearly ordered set and suppose $\{(T_\lambda, A_\lambda) \mid \lambda \in \Lambda\}$ is a chain of sub-solution pairs for* (3). *Put $T = \bigcup T_\lambda$ and $A = \bigcup A_\lambda$. Then (T, A) is a sub-solution pair for* (3), *and $(T_\lambda, A_\lambda) \leq (T, A)$ for all $\lambda \in \Lambda$.*

PROOF. The relation $(T_\lambda, A_\lambda) \leq (T, A)$ for all $\lambda \in \Lambda$ is obvious, so we just have to show that (T, A) is a sub-solution pair for (3).

To verify the first line of (5) on T, suppose $x \in T \setminus I_m$. Then there exists $\lambda \in \Lambda$ such that $x \in T_\lambda \setminus I_m$, so

$$f(x) = A_\lambda(x)m(x) = A(x)m(x).$$

For the second line of (5) on T, suppose $x \sim pw \in T \cap P_m$ with $p \in T \cap P_m$ and $w \in T \setminus I_m$. Then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $p \in T_{\lambda_1} \cap P_m$ and $w \in T_{\lambda_2} \setminus I_m$. Taking $\lambda = \max\{\lambda_1, \lambda_2\}$ we have $p \in T_\lambda \cap P_m$, $w \in T_\lambda \setminus I_m$, so $pw \in T_\lambda \cap P_m$ and therefore $f(x) = f(p)m(w)$.

For the third line of (5) on T, suppose $x \in T \cap I_m^2$. Then there exists $\lambda \in \Lambda$ such that $x \in T_\lambda \cap I_m^2$, thus $f(x) = 0$.

Therefore (T, A) is a sub-solution pair for (3). □

Now we come to our first main result.

THEOREM 3.5. *Let* $m\colon S\to K$ *be multiplicative. Then* $f\colon S\to K$ *satisfies* (3) *if and only if* f *is an Abelian function of the form*

$$f(x)=\begin{cases}A(x)m(x) & \text{for } x\in S\setminus I_m\\ f(p)m(w) & \text{for } x\sim pw\in P_m \text{ with } p\in P_m,\ w\in S\setminus I_m\\ 0 & \text{for } x\in I_m^2\end{cases}\tag{6}$$

for some additive function $A\colon S\setminus I_m\to K$.

Furthermore, if S *is a topological semigroup,* K *is a topological field, and* $f,m\in C(S,K)$, *then* $A\in C(S\setminus I_m,K)$.

PROOF. Suppose $f,m\colon S\to K$ is a solution of (3) with m multiplicative. Lemma 2.2 shows that f is Abelian. If $m=0$ then that same lemma shows that $f(x_1\cdots x_n)=0$ for all $n\geq 2$ and all $x_1,\dots,x_n\in S=I_m$. In this case the first two lines of (6) are vacuous and the third line holds.

Now we assume $m\neq 0$, so m is an exponential. By Lemma 3.1 we have for any q such that $m(q)\neq 0$ a sub-solution pair $(\langle q\rangle, f/m)$ for (3). Lemma 3.4 shows that every chain of sub-solution pairs for (3) has a least upper bound, so by Zorn's Lemma the collection of all sub-solution pairs has a maximal element, say (T,A). We claim that $T=S$. If this is not the case then there exists an element $y\in S\setminus T$, and by Proposition 3.3 there exists a sub-solution pair (T',A') such that $(T,A)\leq(T',A')$ with T a proper subset of T', contradicting the maximality of (T,A). Thus we have a sub-solution pair (S,A) of the form (6) for additive $A\colon S\setminus I_m\to K$.

Conversely, suppose m is multiplicative and f is Abelian of the form (6) for additive $A\colon S\setminus I_m\to K$. The verifications that the pair (f,m) is a solution of (3) are not very difficult. We omit the case $x,y\in S\setminus I_m$, which is straightforward. Suppose $x\in S\setminus I_m$ and $y\sim qw$ with $q\in P_m$ and $w\in S\setminus I_m$. Then $wx\in S\setminus I_m$ and we have

$$\begin{aligned}f(xy)&=f(xqw)=f(q(wx))=f(q)m(wx)\\&=f(q)m(w)m(x)=f(y)m(x)=f(x)m(y)+f(y)m(x)\end{aligned}$$

since $m(y)=0$.

Next, suppose $x\in S\setminus I_m$ and $y\in I_m^2$. Then $y=qr$ with $q,r\in I_m$, so

$$f(xy)=f(xqr)=f((xq)r)=0=f(x)m(y)+f(y)m(x),$$

since $(xq)r\in I_m^2$ and $m(y)=f(y)=0$.

Thirdly, suppose $x\sim pv$ and $y\sim qw$ for $p,q\in P_m$ and $v,w\in S\setminus I_m$. Then $xy\in I_m^2$ so we have

$$f(xy)=0=f(x)m(y)+f(y)m(x),$$

since $m(x)=m(y)=0$.

We leave the remaining verifications to the reader.

The topological statement is vacuous if $m = 0$. For $m \neq 0$ it follows immediately from the equation $f(x) = A(x)m(x)$ for $x \in S \setminus I_m$, since $m(x) \neq 0$ shows that $A = f/m$ is continuous on $S \setminus I_m$. □

REMARK 3.6. For the topological part of Theorem 3.5, if $f \neq 0$ then we need only assume that f is continuous. Indeed, if $f(y_0) \neq 0$ for some $y_0 \in S$ then

$$m(x) = \frac{1}{f(y_0)} \left[f(xy_0) - f(x)m(y_0) \right] \quad \text{for all } x \in S,$$

so m is continuous.

We have a small improvement to the part of Lemma 2.1(ii) dealing with semigroups generated by their squares.

COROLLARY 3.7. *Let $f, m\colon S \to K$ satisfy* (3) *with m multiplicative. If S is generated by its n-th powers for any $n \geq 2$, then there exists an additive function $A\colon S \setminus I_m \to K$ for which*

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m \\ 0 & \text{for } x \in I_m. \end{cases}$$

Furthermore, if S is a topological semigroup, K is a topological field, and $f, m \in C(S, K)$, then $A \in C(S \setminus I_m, K)$.

PROOF. By Theorem 3.5 it suffices to show that P_m is empty (or equivalently $I_m^2 = I_m$). For any $x \in I_m$ there exist $y_1, \dots, y_k \in S$ for some $k \in \mathbb{N}$ such that $x = \prod_{j=1}^{k} y_j^n$. Since $m(x) = 0$ we have $0 = m(y_1)^n \cdots m(y_k)^n$, so $m(y_j) = 0$ for some j. Writing $z_1 = y_1^n \cdots y_{j-1}^n y_j^{n-1}$ and $z_2 = y_j y_{j+1}^n \cdots y_k^n$, we have $x = z_1 z_2 \in I_m^2$. Therefore $P_m = \emptyset$ and the result follows. □

EXAMPLE 3.8. Let $S = (-1, 0) \cup (0, 1)$ under multiplication, and suppose $f, m\colon S \to K$ satisfy (3) with m multiplicative. It is easy to see that S has no prime ideals. By Corollary 3.7 we have $f = Am$ for some additive function $A\colon S \to K$, since S is generated by its cubes. (Note that S is not regular and is not generated by its squares.)

If in addition K is a topological field and $f, m \in \mathcal{C}(S, K)$, then $A \in \mathcal{C}(S, K)$.

The next result (which also has a simple direct proof) follows immediately from Theorem 3.5 and gives the general solution of (3) for the case $m = 0$.

Corollary 3.9. *If $m = 0$ then $f\colon S \to K$ satisfies* (3) *if and only if f has the form*

$$f(x) = \begin{cases} f(p) & \text{for } x = p \in P_m \\ 0 & \text{for } x \in S \setminus P_m. \end{cases}$$

In this case $P_m = S \setminus \{xy \mid x, y \in S\}$.

In view of this result we will generally assume that m is an exponential in applications of Theorem 3.5.

Now we arrive at our primary objective. Combining Lemma 2.1 with Theorem 3.5 we get the general solution of the sine addition formula (1). We choose $K = \mathbb{C}$ as co-domain here for convenience. Part (i) of the next result requires that the co-domain be a quadratically closed field of characteristic different from 2, while part (ii) is valid for any field as co-domain. If $f = 0$ in (1) then g is arbitrary, so as in Lemma 2.1 we exclude that trivial case.

Corollary 3.10. *Let S be a semigroup, and suppose $f, g\colon S \to \mathbb{C}$ satisfy the sine addition law* (1) *with $f \neq 0$. Then f is Abelian and there exist multiplicative functions $m_1, m_2\colon S \to \mathbb{C}$ such that $g = (m_1 + m_2)/2$. In addition we have the following.*

(i) *If $m_1 \neq m_2$, then $f = c(m_1 - m_2)$ for some constant $c \in \mathbb{C}^*$.*

(ii) *If $m_1 = m_2 =: m$, then $g = m$ and f is given by* (6) *for some additive function $A\colon S \setminus I_m \to \mathbb{C}$.*

The converse statements are also true if in part (*ii*) *we choose A and/or $f(p)$ for some $p \in P_m$ so that $f \neq 0$.*

Furthermore, if S is a topological semigroup and $f \in \mathcal{C}(S)$, then $g, m_1, m_2, m \in \mathcal{C}(S)$ and $A \in \mathcal{C}(S \setminus I_m)$.

Proof. Most of this follows immediately from Lemma 2.1 and Theorem 3.5. The continuity of g follows from that of f and the hypothesis $f \neq 0$ as in Remark 3.6. Then the continuity of m_1, m_2 in part (i) follows from the linear independence of distinct non-zero multiplicative functions (see [5, Theorem 3.18(d)]). The only other thing needing justification is the claim that f is Abelian in part (i), but the form of f makes this obvious. □

4. General solution of the sine subtraction formula

Now we come to our secondary objective.

Recall that the sine subtraction formula is the functional equation

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad x, y \in S, \tag{7}$$

for unknown functions $f, g\colon S \to K$, where $\sigma\colon S \to S$ is an automorphic involution. For any function $\varphi\colon S \to K$ let $\varphi_e = \frac{1}{2}(\varphi + \varphi \circ \sigma)$, $\varphi_o = \frac{1}{2}(\varphi - \varphi \circ \sigma)$

be the *even* and *odd* parts of φ, respectively (with respect to σ). Then $\varphi = \varphi_e + \varphi_o$.

We start with a small lemma bringing out a few facts about σ when m is even.

LEMMA 4.1. *Let S be a semigroup with automorphic involution σ, and let $m\colon S \to K$ be a multiplicative function such that $m \circ \sigma = m$. Then we have the following.*

(a) *If $x \in S \setminus I_m$, then $\sigma(x) \in S \setminus I_m$.*
(b) *If $x \in I_m$, then $\sigma(x) \in I_m$.*
(c) *If $x \in P_m$, then $\sigma(x) \in P_m$.*
(d) *If $x \in I_m^2$, then $\sigma(x) \in I_m^2$.*

PROOF. Parts (a) and (b) are obvious. For part (c), if $x \in P_m$ then $\sigma(x) \in I_m$ by (b). Suppose (for a contradiction) that there exists $x \in P_m$ with $\sigma(x) \in I_m \setminus P_m$, so $\sigma(x) = st \in I_m^2$. Then

$$x = \sigma \circ \sigma(x) = \sigma(st) = \sigma(s)\sigma(t) \in I_m^2,$$

contradicting $x \in P_m$. For (d), if $x = st \in I_m^2$ then $\sigma(x) = \sigma(s)\sigma(t) \in I_m^2$ by (b). □

The most current result for (7) on monoids is a combination of [5, Theorem 4.12] and [4, Proposition 3.6], which give the solutions in the case $m = m \circ \sigma$ only if S is a group or a monoid generated by its squares. Here we present the general solution for all monoids by incorporating the findings of Corollary 3.10.

THEOREM 4.2. *Let M be a monoid, and let $\sigma\colon M \to M$ be an automorphic involution. The solutions $f, g\colon M \to \mathbb{C}$ of the sine subtraction formula (7) with $f \neq 0$ are the following pairs of functions, where $m\colon M \to \mathbb{C}$ is multiplicative, $b \in \mathbb{C}$, and $c \in \mathbb{C}^*$.*

(i) *For $m \neq m \circ \sigma$ (so $m \neq 0$) we have*

$$f = c(m - m \circ \sigma), \quad g = \frac{m + m \circ \sigma}{2} + b(m - m \circ \sigma).$$

(ii) *For $m = m \circ \sigma$ we have $g = m + bf$ where f has the form (6) with additive $A\colon M \setminus I_m \to \mathbb{C}$. Furthermore $A \circ \sigma = -A$, $f(\sigma(p)) = -f(p)$ for $p \in P_m$, and either $A \neq 0$ or $f(p) \neq 0$ for some $p \in P_m$.*

Note that f and g are Abelian in each case.

Moreover, if M is a topological monoid and $f \in \mathcal{C}(M)$, then $g, m, m \circ \sigma \in \mathcal{C}(M)$ and $A \in \mathcal{C}(M \setminus I_m)$.

PROOF. We follow the general outline of the proof of [4, Proposition 3.6] (which in turn is based on [5, Theorem 4.12]). That proof starts by writing g as $g_e + g_o$ and showing that f is odd. Next it shows that $g_o = bf$ for some constant $b \in \mathbb{C}$, and that

$$f(xy) = f(x)g_e(y) + g_e(x)f(y), \quad x, y \in M.$$

We now take the solution forms of f and g_e from Corollary 3.10, so f is Abelian and there exist multiplicative functions $m_1, m_2 \colon M \to \mathbb{C}$ such that $g_e = (m_1 + m_2)/2$. If $m_2 \neq m_1$ then we have $f = c(m_1 - m_2)$ for some constant $c \in \mathbb{C}^*$, since $f \neq 0$. Since f is odd (and nonzero) while g_e is even, we infer that $m_2 = m_1 \circ \sigma$. Recalling $g_o = bf$, renaming bc as a new b, and defining $m := m_1$ we obtain the formulas in (i), which show that g is Abelian. Moreover since $m \neq m \circ \sigma$ we have $m \neq 0$, so m is exponential and we have part (i). Conversely, it is easy to check that these formulas define a solution of (7) with $f \neq 0$.

Now consider the case $m_1 = m_2 =: m$. Then $g_e = m$ where $m \colon M \to \mathbb{C}$ is multiplicative, and the form of f is given by (6) for some additive function $A \colon M \setminus I_m \to \mathbb{C}$. Moreover, since g_e is even we find that $m = m \circ \sigma$. Furthermore, since f is odd we get from (6) that $A \circ \sigma = -A$, and $f(\sigma(p)) = -f(p)$ for $p \in P_m$ by Lemma 4.1. Since $g_o = bf$ we have $g = m + bf$, so g is Abelian. This gives part (ii).

We have to check that the formulas for f and g in (ii) define solutions of (7). Here we have $g = m + bf$, so equation (7) is equivalent to

$$f(x\sigma(y)) = f(x)m(y) - m(x)f(y), \quad x, y \in M, \tag{8}$$

and this is the equation we check. By Lemma 4.1 and the fact that σ is an automorphism, σ preserves the location of variables in each case to be checked.

For the case $x, y \in M \setminus I_m$ it is easy to verify that the given form of f satisfies (8), so we omit that. Now consider the case $x \in M \setminus I_m$ and $y \sim qw$ for $q \in P_m$ and $w \in M \setminus I_m$. Then using Lemma 4.1 (and that f is Abelian and $m \circ \sigma = m$) we see that

$$\begin{gathered} f(x\sigma(y)) = f(x\sigma(q)\sigma(w)) = f(\sigma(q)x\sigma(w)) \\ = f(\sigma(q))m(x\sigma(w)) = -f(q)m(x)m(\sigma(w)) = -f(q)m(w)m(x) \\ = -f(qw)m(x) = f(x)m(y) - f(y)m(x), \end{gathered}$$

since $x\sigma(w) \in M \setminus I_m$ and $m(y) = 0$. The verification for the case $x \sim pv \in P_m$ and $y \in M \setminus I_m$ is similar, so we omit it.

Next consider the case $x \sim pv$, $y \sim qw$ for $p, q \in P_m$ and $v, w \in M \setminus I_m$. Then $x\sigma(y) \in I_m^2$, so we have $f(x\sigma(y)) = 0 = f(x)m(y) - m(x)f(y)$ in accord with (8), since $m(x) = m(y) = 0$.

If $x \in M \setminus I_m$ and $y \in I_m^2$, let $y = st$ with $s, t \in I_m$. In this case we have $x\sigma(y) = x\sigma(s)\sigma(t) = (x\sigma(s))\sigma(t) \in I_m^2$, so $f(x\sigma(y)) = 0 = f(x)m(y) - m(x)f(y)$ because $f(y) = m(y) = 0$.

The verifications of the remaining cases are similar. □

REMARK 4.3. It is easy to check that the variant $f(\sigma(y)x) = f(x)g(y) - g(x)f(y)$ has the same solutions as (7).

In the remaining sections we present some examples illustrating the application of our results to a variety of semigroups.

5. Semigroups with relatively few prime ideals

For the rest of the paper our main focus is on applications of Theorem 3.5 to various kinds of semigroups, with secondary attention given to illustrations of Theorem 4.2. We will not mention applications of Corollary 3.10 at all, since they follow readily. When discussing applications of Theorem 3.5 we generally assume that the multiplicative function m is an exponential, since the (almost trivial) case $m = 0$ is handled by Corollary 3.9.

Several examples illustrating solutions of the sine addition and subtraction formulas on groups and semigroups which are regular or generated by their squares were presented in [4], so we do not do that here. Instead we present examples on semigroups which are not regular (hence not groups) and not generated by their squares, so they are not covered by Lemma 2.1 or [4, Proposition 3.6].

When our examples involve a direct product of semigroups we use componentwise operations, and for topological semigroups we use the product topology.

The fewest number of prime ideals a semigroup can have is none, so we begin there. For semigroups with no prime ideals, the solutions of (3) have the simplest possible form: If m is an exponential then $f = Am$ for some additive function A. All groups and infinite monogenic semigroups are of this type, but there are others such as the following.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\text{char}(K)$ denote the characteristic of K.

EXAMPLE 5.1. (a) For any $k \in \mathbb{N}$ let $S = \mathbb{N}^k$ under (componentwise) addition, and suppose $f, m \colon S \to K$ satisfy (3) with m exponential. It is easy to see that S has no prime ideal. Therefore by Theorem 3.5 there exists an additive function $A \colon S \to K$ such that $f = Am$. The additive functions on S are of the form $A(x_1, \dots, x_k) = \sum_{j=1}^k A_j(x_j)$ for additive functions $A_j \colon \mathbb{N} \to K$, and the exponential functions have the form $m(x_1, \dots, x_k) = \prod_{j=1}^k m_j(x_j)$ for exponentials $m_j \colon \mathbb{N} \to K$. If $\text{char}(K) = 0$ then $A_j(x) = a_j x$ and $m_j(x) = (b_j)^x$ for constants $a_j \in K$, $b_j \in K^*$.

(b) Let $M = \mathbb{N}_0 \times \mathbb{N}_0$ under addition, and let $\sigma \colon M \to M$ be the switching involution defined by $\sigma(x, y) = (y, x)$ for all $(x, y) \in M$. The solutions $f, g \colon M \to \mathbb{C}$ of the sine subtraction formula (7) with m exponential and $f \neq 0$ are given by Theorem 4.2, where in part (ii) we have $f = Am$ since

M has no prime ideal. The additive functions such that $A \circ \sigma = -A \neq 0$ are of the form $A(x, y) = c(x - y)$ for some $c \in \mathbb{C}^*$, and the exponentials m such that $m = m \circ \sigma$ have the form $m(x, y) = b^{x+y}$ for some $b \in \mathbb{C}^*$.

In fact, if $S_1, \dots, S_k$ are semigroups with no prime ideals then their direct product $S_1 \times \cdots \times S_k$ also has no prime ideal. So the previous example generalizes to other direct products.

Other semigroups that may not have many prime ideals are semigroups with a zero. We say a semigroup S contains a *zero element*, usually denoted 0, if $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$. If S has a 0 then the only additive function on S is the zero function. (That follows from $A(0) = A(x \cdot 0) = A(x) + A(0)$ for all $x \in S$.) Moreover for any exponential m on S it follows from $m(0) = m(x \cdot 0) = m(x)m(0)$ for all $x \in S$ that either $m(0) = 0$ or $m = 1$. Note also that if m is an exponential with null ideal $I_m = \{0\}$, then P_m is empty since $0 = 0^2 \in I_m^2$.

In the next result and the following two examples, the singleton set containing the zero element is the only prime ideal of S.

COROLLARY 5.2. *Let S be a semigroup with zero element 0 and unique prime ideal $\{0\}$, and let $f, m \colon S \to K$ satisfy (3) where m is exponential. Then there exists an additive function $A \colon S \setminus I_m \to K$ for which*

$$f(x) = \begin{cases} A(x)m(x) & \textit{for} x \in S \setminus I_m \\ 0 & \textit{for} x \in I_m. \end{cases}$$

If S is a topological semigroup, K is a topological field, and $f, m \in \mathcal{C}(S, K)$, then $A \in \mathcal{C}(S \setminus I_m, K)$.

PROOF. The result follows from Theorem 3.5 after showing that P_m is empty for every m. If $I_m = \emptyset$ this is obvious. If $I_m \neq \emptyset$ then the only remaining choice is $I_m = \{0\}$, so $P_m = \emptyset$ and the result follows. (Note that if $I_m = \emptyset$ then $m = 1$ and $f = 0$.) $\square$

Our next example is the additive semigroup of extended natural numbers $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, where $n + \infty = \infty + n = \infty + \infty = \infty$ for all $n \in \mathbb{N}$. Thus ∞ is the zero element for $\bar{\mathbb{N}}$.

EXAMPLE 5.3. Let $S = (\bar{\mathbb{N}}, +)$, let $\operatorname{char}(K) = 0$, and let $f, m \colon S \to K$ be a solution of (3) with m exponential. Either $m = 1$, or $m(\infty) = 0$ and there exists $b \in K^*$ such that $m(s) = b^s$ for all $s \in \mathbb{N}$. The unique prime ideal of S is $\{\infty\}$. The additive functions on $S \setminus \{\infty\} = \mathbb{N}$ are of the form $A(x) = cx$ for some $c \in K$. Applying Corollary 5.2, if $I_m = \{\infty\}$ we have

$$f(x) = \begin{cases} cxb^x & \text{for } x \in \mathbb{N} \\ 0 & \text{for } x = \infty, \end{cases}$$

for some constants $b, c \in K$ with $b \neq 0$. In the case $I_m = \emptyset$ we have $m = 1$ and $f = 0$.

Now we consider an innocuous-looking semigroup, but it carries some rather wild additive and multiplicative functions.

EXAMPLE 5.4. Let $0 < r < 1$, let $S = ([0, r], \cdot)$ with the usual topology, and suppose $f, m \in \mathcal{C}(S, \mathbb{R})$ is a solution of (3) with m exponential. Here $\{0\}$ is the unique prime ideal of S. Applying Corollary 5.2 we get $f = 0$ and $m = 1$ if $I_m = \emptyset$. In the case $I_m = \{0\}$ we have

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in (0, r] \\ 0 & \text{for } x = 0, \end{cases}$$

where $A \in \mathcal{C}((0, r], \mathbb{R})$. In this case there are exorbitant numbers of unusual continuous additive functions on $(0, r]$ and exponentials on $[0, r]$. They can be chosen rather arbitrarily on the interval $(r^2, r]$, subject only to being continuous on $(r^2, r]$ and satisfying constraints at the endpoints (r and r^2) guaranteeing that they have continuous extensions to the respective full intervals $(0, r]$ and $[0, r]$. (For example the constraints on m are $0 < m(r) < 1$ and $\lim_{x \to (r^2)^+} m(x) = m(r)^2$.)

The wildness of the additive and exponential functions in the previous example is not merely due to the fact that S does not contain 1. For example, the continuous exponentials from $([0, 1), \cdot)$ into $\mathbb{R}$ are rather tame: $m = 1$ or $m(x) = x^c$ for some real constant $c > 0$.

Corollary 5.2 extends to direct products if we add one more assumption.

COROLLARY 5.5. *For some integer $n \geq 2$, let $S_1, \ldots, S_n$ be semigroups such that each S_j has unique prime ideal $\{0_j\}$ consisting of a zero element. Let $S = S_1 \times \cdots \times S_n$, and suppose $f, m \colon S \to K$ satisfy (3) where m is an exponential. If $S_j = \{xy \mid x, y \in S_j\}$ (for instance if S_j is a monoid) for each j, then there exists an additive function $A \colon S \setminus I_m \to K$ such that*

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m \\ 0 & \text{for } x \in I_m. \end{cases} \tag{9}$$

Moreover if each S_j is a topological semigroup, K is a topological group, and $f, m \in \mathcal{C}(S, K)$, then $A \in \mathcal{C}(S \setminus I_m, K)$.

PROOF. Again the result follows from Theorem 3.5 if we show that P_m is empty for every m. If $I_m = \emptyset$ this is obvious, so assume now that $I_m \neq \emptyset$. The list of prime ideals in S consists of

$$I_j := S_1 \times \cdots S_{j-1} \times \{0_j\} \times S_{j+1} \cdots \times S_n$$

for each $j \in \{1, \ldots, n\}$ and all nonempty unions of the I_j. By our hypothesis on the S_j we have $I_j^2 = I_j$ for $1 \le j \le n$. Suppose $I_m = \bigcup_{j \in J} I_j$ for some subset $J \subset \{1, \ldots, n\}$. Then

$$I_m^2 = \Big[\bigcup_{j \in J} I_j^2\Big] \cup \Big[\bigcup_{j \neq k \in J} I_j I_k\Big] = \bigcup_{j \in J} I_j^2 = \bigcup_{j \in J} I_j = I_m,$$

since $I_j I_k \subset I_j = I_j^2$. Therefore $P_m = \emptyset$ and we have (9) for additive $A \colon S \setminus I_m \to K$ defined by $A := f/m$. $\square$

To set up the next example, we first identify the exponentials and additive functions we will need. On the monoid $M = ([-1,1], \cdot)$ the continuous exponentials $m \colon M \to \mathbb{C}$ have one of the three forms

(10)
$$m = 1, \quad m(x) = \begin{cases} |x|^\alpha & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases} \text{ or } m(x) = \begin{cases} |x|^\alpha \operatorname{sgn}(x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

where $\alpha \in \mathbb{C}$ has positive real part (written $\Re(\alpha) > 0$). Since M has a 0, the only additive function on M is $A = 0$. On the sub-monoid $M \setminus \{0\}$ the continuous additive functions are of the form $A(x) = c \log |x|$ for some $c \in \mathbb{C}$.

EXAMPLE 5.6. Let $S = [-1,1] \times [-1,1]$ under (componentwise) multiplication. The continuous exponentials have the form $m(x,y) = m_1(x) m_2(y)$ where each $m_j : [-1,1] \to \mathbb{C}$ has one of the three forms in (10) with $\Re(\alpha) > 0$. The prime ideals of S are $I_1 = \{0\} \times [-1,1]$, $I_2 = [-1,1] \times \{0\}$, and $I_1 \cup I_2$. For a given exponential $m \in \mathcal{C}(S)$, the corresponding continuous additive functions $A \colon S \setminus I_m \to \mathbb{C}$ are as follows, where $c_1, c_2 \in \mathbb{C}$.

(a) If $I_m = \emptyset$ then $A = 0$.
(b) If $I_m = I_1$ then $A(x,y) = c_1 \log |x|$.
(c) If $I_m = I_2$ then $A(x,y) = c_2 \log |y|$.
(d) If $I_m = I_1 \cup I_2$ then $A(x,y) = c_1 \log |x| + c_2 \log |y|$.

(i) By Corollary 5.5 we get the continuous solutions of (3) on S for m exponential and $f, m \in \mathcal{C}(S)$ by plugging the above forms into (9).

(ii) Let $\sigma \colon S \to S$ be defined by $\sigma(x,y) = (y,x)$. By Theorem 4.2 and Corollary 5.5 we get the continuous solutions of (7) for $f \in \mathcal{C}(S)$ and $f \neq 0$ by plugging the above forms into the formulas of Theorem 4.2 using (9) in place of (6). In Theorem 4.2 (ii) the exponential m satisfies $m = m \circ \sigma$, so it must have the form $m = 1$ or

$$m(x,y) = \begin{cases} |xy|^\alpha & \text{for } xy \neq 0 \\ 0 & \text{for } xy = 0, \end{cases} \text{ or } m(x,y) = \begin{cases} |xy|^\alpha \operatorname{sgn}(xy) & \text{for } xy \neq 0 \\ 0 & \text{for } xy = 0, \end{cases}$$

where $\Re(\alpha) > 0$. Since $m \neq m \circ \sigma$ in Theorem 4.2(i), in that part m must have a form different from these. Also in part (ii), the condition $A \circ \sigma = -A$ is satisfied only if $A(x, y) = \beta(\log|x| - \log|y|)$ for some $\beta \in \mathbb{C}$.

Some semigroups with a zero do not satisfy the condition $S = \{xy \mid x, y \in S\}$ imposed in Corollary 5.5. The following is an example to which Corollary 5.5 does not apply.

EXAMPLE 5.7. Let $T = S \times S$ where $S = (\bar{\mathbb{N}}, +)$, and suppose $\mathrm{char}(K) = 0$. S has unique prime ideal $\{\infty\}$, and the exponentials on S and additive functions on $S \setminus \{\infty\} = \mathbb{N}$ were given in Example 5.3.

The prime ideals of T are $I_1 = \{\infty\} \times \bar{\mathbb{N}}$, $I_2 = \bar{\mathbb{N}} \times \{\infty\}$, and $I_1 \cup I_2$.

Nullspaces and exponentials on T, with corresponding additive functions on $T \setminus I_m$ are the following, where $b_1, b_2 \in K^*$ and $c_1, c_2 \in K$.

(a) $I_m = \emptyset$, $m = 1$, $A = 0$.

(b) $I_m = I_1$, with $m(x, y) = (b_1)^x$ and $A(x, y) = c_1 x$ for all $(x, y) \in T \setminus I_m$.

(c) $I_m = I_2$, with $m(x, y) = (b_2)^y$ and $A(x, y) = c_2 y$ for all $(x, y) \in T \setminus I_m$.

(d) $I_m = I_1 \cup I_2$, with $m(x, y) = (b_1)^x (b_2)^y$ and $A(x, y) = c_1 x + c_2 y$ for all $(x, y) \in T \setminus I_m$.

By Theorem 3.5 we get the solutions $f, m : T \to K$ of (3) for m exponential by plugging the above forms into (6).

Note that in cases (b),(c),(d) we have nonempty P_m given respectively by $P_m = \{(\infty, 1)\}$, $P_m = \{(1, \infty)\}$, and $P_m = \{(\infty, 1), (1, \infty)\}$, so (6) does not reduce to (9). Thus for example in (b) the value of $f(\infty, 1) \in K$ is unconstrained.

Next we consider the upper-triangular matrix semigroup

$$T^+(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \text{ with } a, d \geq 0 \right\}.$$

A few calculations reveal that $S = T^+(2, \mathbb{R})$ has exactly three prime ideals $I_1, I_2, I_1 \cup I_2$, where

$$I_1 = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \;\middle|\; b \in \mathbb{R}, d \geq 0 \right\} \text{ and } I_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; a \geq 0,\ b \in \mathbb{R} \right\}.$$

The prime ideal $I_1 \cup I_2$ consists of the singular matrices in S.

To prepare for the next example we identify the requisite exponentials and additive functions.

LEMMA 5.8. *Let $S = T^+(2, \mathbb{R})$ with the usual topology. The exponentials $m \in \mathcal{C}(S, \mathbb{R})$, corresponding nullspaces I_m, and additive functions $A \in \mathcal{C}(S \setminus I_m, \mathbb{R})$ are the following, where $p, q, \delta, \gamma \in \mathbb{R}$ with $p, q > 0$.*

(a) *If $I_m = \emptyset$, then $m = 1$ and $A = 0$.*

(b) *If* $I_m = I_1$, *then for all* $a, d \geq 0$, $b \in \mathbb{R}$, *and* $c > 0$,

$$m\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a^p \quad \textit{and} \quad A\begin{pmatrix} c & b \\ 0 & d \end{pmatrix} = \gamma \log c.$$

(c) *If* $I_m = I_2$, *then for all* $a, d \geq 0$, $b \in \mathbb{R}$, *and* $c > 0$,

$$m\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = d^q \quad \textit{and} \quad A\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \delta \log c.$$

(d) *If* $I_m = I_1 \cup I_2$, *then for all* $a, d \geq 0$, $b \in \mathbb{R}$, *and* $c_1, c_2 > 0$,

$$m\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a^p d^q, \quad \textit{and} \quad A\begin{pmatrix} c_1 & b \\ 0 & c_2 \end{pmatrix} = \gamma \log c_1 + \delta \log c_2.$$

Proof. Let $m \in \mathcal{C}(S, \mathbb{R})$ be an exponential, and let $\mathbf{O}$, $\mathbf{I}$ be the zero and identity matrices in S, respectively. Then $m(\mathbf{I}) = 1$, and either $m(\mathbf{O}) = 0$ or $m = 1$.

If $I_m = \emptyset$, then $m(\mathbf{O}) \neq 0$ so $m = 1$. The only additive function on S is the zero function, since S contains a zero. Thus we have part (a). Henceforth we assume that $I_m \neq \emptyset$, so $m(\mathbf{O}) = 0$.

For the remaining calculations, to save space we use the abbreviation

$$(a, b, d) := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

for elements of S.

If $I_m = I_1$ then for all $a \geq 0$ and $b \in \mathbb{R}$ we have

$$m(1, 0, 0)m(a, b, 0) = m((1, 0, 0)(a, b, 0)) = m(a, b, 0),$$

so taking $a \neq 0$ we get $m(1, 0, 0) = 1$. Now for all $b \in \mathbb{R}$ and $a, d \geq 0$ we see that

$$m(a, b, d) = m(a, b, d)m(1, 0, 0) = m((a, b, d)(1, 0, 0)) = m(a, 0, 0) =: \chi(a)$$

defines a continuous function $\chi\colon [0, \infty) \to \mathbb{R}$ with $\chi(1) = 1$. Since χ is clearly exponential and $\chi(0) = m(\mathbf{O}) = 0$ we have the form of m given in (b) with $p > 0$.

Now suppose $A \in \mathcal{C}(S \setminus I_1, \mathbb{R})$ is additive. For $d \geq 0$, $b \in \mathbb{R}$, and $a > 0$ we have

$$A(a, b, d) + A(1, 0, 0) = A((a, b, d)(1, 0, 0)) = A(a, 0, 0) =: \alpha(a),$$

for some continuous function $\alpha : (0, \infty) \to \mathbb{R}$. With $(a, b, d) = (1, 0, 0)$ the preceding equation gives $2\alpha(1) = \alpha(1)$, so $\alpha(1) = 0$. Since α is clearly additive we have the form of A given in part (b).

Part (c) is parallel to part (b).

Finally, suppose $I_m = I_1 \cup I_2$. Then for any $a_j, d_j \geq 0$ and $b_j \in \mathbb{R}$ $(j = 1, 2)$ we have

$$m(a_1, b_1, d_1)m(a_2, b_2, d_2) = m(a_1a_2, a_1b_2 + b_1d_2, d_1d_2).$$

Interchanging (a_1, b_1, d_1) and (a_2, b_2, d_2) yields

$$m(a_1a_2, a_1b_2 + b_1d_2, d_1d_2) = m(a_2a_1, a_2b_1 + b_2d_1, d_2d_1).$$

For $a_1, d_1, a, d > 0$, let $a_2 = a/a_1$, $d_2 = d/d_1$, and suppose $a \neq d$. Given any $y \in \mathbb{R}$, choosing $b_2 = dy/[d_1(d-a)]$ and $b_1 = -a_1y/(d-a)$ in the last equation we get

$$\psi(a, d) := m(a, 0, d) = m(a, y, d), \quad a \neq d > 0, \ y \in \mathbb{R}, \tag{11}$$

for some function $\psi\colon (0, \infty) \times (0, \infty) \setminus \{(x, x) \mid x > 0\} \to \mathbb{R}^*$. Since m is multiplicative we see that ψ is componentwise multiplicative, so as long as $a \neq d$ we have $\psi(a, d) = \chi_1(a)\chi_2(d)$ for multiplicative functions $\chi_1, \chi_2 : (0, \infty) \to \mathbb{R}^*$. Furthermore, for any $a, d > 0$ with $a \neq d$ and any $y \in \mathbb{R}$ we have

$$\begin{aligned} \chi_1(a)\chi_2(d) = m(a, yd/a, d) &= m(a, y, a)m(1, 0, d/a) \\ &= m(a, y, a)\chi_1(1)\chi_2(d/a). \end{aligned}$$

Thus $m(a, y, a) = \chi_1(a)\chi_2(a)$, showing that (11) holds also for $a = d$. Definition (11) also yields that χ_1, χ_2 are continuous (and nonzero) on $(0, \infty)$, so they have the form $\chi_1(x) = x^p$, $\chi_2(x) = x^q$ for some constants $p, q \in \mathbb{R}$. Thus at this point we have $m(a, y, d) = a^p d^q$ for all $a, d > 0$ and $y \in \mathbb{R}$. Since m is continuous and $m(0, y, c) = m(c, y, 0) = 0$ for all $y \in \mathbb{R}$, $c \geq 0$, this representation extends to $a = 0$ and $d = 0$ with $p, q > 0$. Thus we have the form of m given in part (d).

If $A \in \mathcal{C}(S \setminus (I_1 \cup I_2), \mathbb{R})$ is additive, then by the same reasoning we get the form for A shown in (d). □

Thus we have the following.

Example 5.9. Let $S = T^+(2, \mathbb{R})$, and let $f, m \in \mathcal{C}(S, \mathbb{R})$ be a solution of (3) with m exponential. We show that P_m is empty. For any element $\mathbf{X} \in I_1$ we have

$$\mathbf{X} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \mathbf{X} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so $\mathbf{X} \in I_1^2$. Similarly, if $\mathbf{Y} \in I_2$ then

$$\mathbf{Y} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Y},$$

therefore P_m is empty in all cases.

By Theorem 3.5 we have

$$f(\mathbf{X}) = \begin{cases} A(\mathbf{X})m(\mathbf{X}) & \text{for } \mathbf{X} \in S \setminus I_m \\ 0 & \text{for } \mathbf{X} \in I_m\,, \end{cases}$$

where $A \in C(S \setminus I_m, \mathbb{R})$ is additive. The forms of A and m are given in Lemma 5.8.

6. Semigroups with many prime ideals

Let B^+ be the set of nonempty finite sequences of elements from a nonempty set B. The elements of B^+ are called *strings* or *words*, and the elements of B are called *letters*. We identify the letter $\ell \in B$ with the string of length one in B^+ consisting of that letter alone, so $B \subset B^+$. Under the operation of concatenation of strings, B^+ becomes a semigroup called the *free semigroup* on B. For each letter $\ell \in B$, the set of all strings containing ℓ is a prime ideal of B^+ if B contains more than one letter. Furthermore, every proper nonempty subset of B generates a prime ideal of B^+, so the number of prime ideals is $2^{|B|} - 2$, where $|B|$ denotes the cardinality of B.

EXAMPLE 6.1. Let B^+ denote the free semigroup on alphabet B. If $f, m \colon B^+ \to K$ satisfy (3) with m exponential, then Theorem 3.5 provides the solution form (6) for f in terms of an additive function $A \colon B^+ \setminus I_m \to K$. The forms of multiplicative and additive functions on B^+ are straightforward to compute in terms of their values on the generating set B. For any string $w \in B^+$ and any letter $\ell \in B$, define $N_\ell(w)$ to be the number of times ℓ appears in w. Then $m(w) = \prod_{\ell \in w} m(\ell)^{N_\ell(w)}$ for every $w \in B^+$, so m is generated by its values $m(\ell)$ for $\ell \in B$. Similarly, if $A \colon B^+ \to K$ is additive then $A(w) = \sum_{\ell \in w} A(\ell) N_\ell(w)$ for all $w \in B^+$.

For a given exponential m, suppose $w \in P_m$. Then w cannot be decomposed into a product uv such that $m(u) = m(v) = 0$. Thus w must contain exactly one letter belonging to I_m, and that letter must occur exactly once in w.

If $\operatorname{char}(K) = 0$ then a simple example of an additive function on B^+ (and by restriction to its sub-semigroups) is $A(w) = L(w)1_K$, where 1_K is the multiplicative identity in K and $L \colon B^+ \to \mathbb{N}$ is the length function, defined by $L(w)$ equals the number of letters in the string $w \in B^+$, counting multiplicities. This is the additive function generated by taking $A(\ell) = 1_K$ for each letter ℓ.

For our next application we return to the notion of periodicity, which was introduced in Section 3. A semigroup S is said to be a *periodic semigroup* if every element of S is periodic. Although each element has finite order, the semigroup S itself may be infinite, in which case it has infinitely many prime ideals.

COROLLARY 6.2. *Let S be a periodic semigroup, and let* $\mathrm{char}(K) = 0$. *Then $f, m \colon S \to K$ satisfy* (3) *with m exponential if and only if f is Abelian and*

$$f(x) = \begin{cases} f(p)m(w) & \text{for } x \sim pw \in P_m \text{ with } p \in P_m,\ w \in S \setminus I_m \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

PROOF. By Theorem 3.5, f has the form (6). Let $x \in S \setminus I_m$. Since x is periodic, the monogenic semigroup $\langle x \rangle$ generated by x is finite. If $A \colon S \setminus I_m \to K$ is additive, then $A(x^n) = nA(x)$ for every $n \in \mathbb{N}$, but the set $A(\langle x \rangle)$ is finite, so $A(x) = 0$. Thus we have (12). □

For our final example we consider a familiar semigroup with infinitely many prime ideals, namely the multiplicative semigroup $S = (\mathbb{N}, \cdot)$ of positive integers. Since S is commutative the statement $x \sim y$ becomes $x = y$.

Let P denote the set of prime numbers. Then $p\mathbb{N}$ is a prime ideal of S for each $p \in P$. If $m \colon S \to K$ is multiplicative then the nullspace I_m is the union of all $p\mathbb{N}$ for primes p such that $m(p) = 0$.

COROLLARY 6.3. *Let $S = (\mathbb{N}, \cdot)$. The functions $f, m \colon S \to K$ satisfy* (3) *with m exponential if and only if and there exists an additive function $A \colon S \setminus I_m \to K$ such that*

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m \\ f(p)m(w) & \text{for } x = pw \text{ with } p \in P \cap I_m,\ w \in S \setminus I_m \\ 0 & \text{for } x \in I_m^2. \end{cases}$$

PROOF. By Theorem 3.5, if $I_m = \emptyset$ then we have $f = Am$ and we are finished. Assuming I_m is nonempty, we need to establish the middle line of the form of f. If $x = p'w'$ with $p' \in P_m$ and $w' \in S \setminus I_m$, then consider the prime factorization of p'. Since $m(p') = 0$ there exists a prime factor p of p' such that $m(p) = 0$. Thus we have $p' = py$ for some $y \in S$. Since $p' \in P_m$ it follows that $y \in S \setminus I_m$. Now we have $x = pw$ with $w := yw' \in S \setminus I_m$ as claimed. □

Note that P_m is generally not equal to $P \cap I_m$. For example if $m(3) = 0$ and $m(2) \neq 0$ then $6 \in P_m$, but obviously $6 \notin P \cap I_m$. In this case we have $f(6) = f(3)m(2)$, where $f(3)$ can be any element of K.

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