Acta Math. Hungar. **164** (1) (2021), 312–325
https://doi.org/10.1007/s10474-021-01138-6
First published online May 5, 2021

ON THE ARITHMETIC MEAN OF THE SIZE OF CROSS-UNION FAMILIES

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(Received September 24, 2020; revised November 29, 2020; accepted December 1, 2020)

Abstract. Let $n > k > 1$ be integers, $[n] = \{1, \dots, n\}$ the standard n-element set and $\binom{[n]}{k}$ the collection of all its k-subsets. The families $\mathcal{F}_0, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are said to be *cross-union* if $F_0 \cup \dots \cup F_s \neq [n]$ for all choices of $F_i \in \mathcal{F}_i$. It is known [13] that for $n \le k(s+1)$ the geometric mean of $|\mathcal{F}_i|$ is at most $\binom{n-1}{k}$. We conjecture that the same is true for the arithmetic mean for the range $ks < n < k(s+1)$, $s > s_0(k)$ (Conjecture 8.1) and prove this in several cases. The proof for the case $n = ks + 2$ relies on a novel approach, a combination of shifting and Katona's cyclic permutation method.

1. Introduction

Let $n > k$ be positive integers and let $[n] = \{1, 2, \dots, n\}$ be the standard n-element set. Let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.

Erdős–Ko–Rado Theorem [4]. *Suppose that $n \ge 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting. Then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}. \tag{1.1}$$

This classical result has played a central role in the development of Extremal Set Theory. In particular, there are many different proofs (cf. e.g. [2,11,16–18,21]) and many generalizations (cf. e.g. [1,3,6,9,14,15,20]).

Two families $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are called *cross-intersecting* if $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$, $G \in \mathcal{G}$.

Research partially supported by the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant no. 075-15-2019-1926.

Key words and phrases: finite set, intersection, cyclic permutation.

Mathematics Subject Classification: 05D05.

PYBER THEOREM [22]. *Suppose that $n \geq 2k$ and $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are cross-intersecting. Then*

$$|\mathcal{F}|\,|\mathcal{G}| \leq \binom{n-1}{k-1}^2. \tag{1.2}$$

Setting $\mathcal{F} = \mathcal{G}$, (1.2) implies (1.1). For many applications a bound of the form

$$|\mathcal{F}| + |\mathcal{G}| \leq 2 \cdot \binom{n-1}{k-1} \tag{1.3}$$

would be much more convenient. Needless to say, (1.3) is stronger than (1.2). Unfortunately, for $n > 2k$, (1.3) is not true. The trivial example is $\emptyset$ and $\binom{[n]}{k}$.

EXAMPLE 1.1. Set $\mathcal{A} = \{[k]\}$, $\mathcal{B} = \left\{ B \in \binom{[n]}{k} : B \cap [k] \neq \emptyset \right\}$. Then $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting with

$$|\mathcal{A}| + |\mathcal{B}| = 1 + \binom{n}{k} - \binom{n-k}{k}. \tag{1.4}$$

Hilton and Milner [15] showed that this example is optimal for non-empty cross-intersecting families.

THEOREM 1.2 [15]. *Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are non-empty cross-intersecting families. Then*

$$|\mathcal{A}| + |\mathcal{B}| \leq 1 + \binom{n}{k} - \binom{n-k}{k}. \tag{1.5}$$

We invite the reader to check that for $n > 2k > 2$ the RHS of (1.5) is greater than the RHS of (1.3).

On the other hand in [7] the inequality

$$|\mathcal{A}| + |\mathcal{B}| \leq 2\binom{n-1}{k-1} \tag{1.6}$$

is established under the stronger assumption, $|\mathcal{B}| \geq |\mathcal{A}| \geq \binom{n-2}{k-2}$. In [12] it is used to give a very short proof of (1.2).

DEFINITION 1.3. The families $\mathcal{A}_1, \ldots, \mathcal{A}_r \subset \binom{[n]}{k}$ are called cross-intersecting if $A_1 \cap \cdots \cap A_r \neq \emptyset$ for all $A_1 \in \mathcal{A}_1, \ldots, A_r \in \mathcal{A}_r$. In the case $\mathcal{A} = \mathcal{A}_1 = \cdots = \mathcal{A}_r$, $\mathcal{A}$ is called *r-wise intersecting*.

THEOREM 1.4 [5]. *Suppose that* $\mathcal{A}\subset\binom{[n]}{k}$ *is* r*-wise intersecting,* $n\ge\frac{r}{r-1}k$. *Then*

$$|\mathcal{A}|\le\binom{n-1}{k-1}. \tag{1.7}$$

Note that for $n<\frac{r}{r-1}k$ the full $\binom{[n]}{k}$ is r-wise intersecting. In [8] it is shown that for $r\ge 3$ equality holds in (1.7) only if $\mathcal{A}=\left\{A\in\binom{[n]}{k}:x\in A\right\}$ for some fixed element $x\in[n]$.

The dual notion to cross-intersecting is *cross-union.* The families $\mathcal{H}_1,\dots,\mathcal{H}_r\subset\binom{[n]}{\ell}$ are called *cross-union* if $H_1\cup\dots\cup H_r\ne[n]$ holds for all choices of $H_i\in\mathcal{H}_i$, $i=1,\dots,r$. For a family $\mathcal{A}\subset\binom{[n]}{k}$ let $\mathcal{A}^c=\{[n]\setminus A:A\in\mathcal{A}\}$ be the family of complements, $\mathcal{A}^c\subset\binom{[n]}{n-k}$. It is easy to see that $\mathcal{A}_1,\dots,\mathcal{A}_r$ are cross-intersecting iff $\mathcal{A}_1^c,\dots,\mathcal{A}_r^c$ are cross-union.

THEOREM 1.5 [13]. *Suppose that* $k<n\le rk$. *Let* $\mathcal{H}_1,\dots,\mathcal{H}_r\subset\binom{[n]}{k}$ *be non-empty and cross-union. Then*

$$\prod_{1\le i\le r}|\mathcal{H}_i|\le\binom{n-1}{k}^r. \tag{1.8}$$

The aim of the present paper is to prove the *sum version* of (1.8) in certain situations.

DEFINITION 1.6. Let $1\le k<n\le k(s+1)$ where k,n,s are positive integers. We say that the triple (n,k,s) is *nice* if

$$\sum_{0\le i\le s}|\mathcal{F}_i|\le(s+1)\binom{n-1}{k} \tag{1.9}$$

holds for all choices of non-empty cross-union families $\mathcal{F}_0,\dots,\mathcal{F}_s\subset\binom{[n]}{k}$.

It is easy to show that $(k(s+1),k,s)$ is always nice (cf. Corollary 2.5).

THEOREM 1.7. $(ks+1,k,s)$ *is nice for* $s\ge 2$.

For the case $k=3$ we have:

THEOREM 1.8. $(3s+2,3,s)$ *is nice for* $s\ge 2$.

Note that these results settle the cases $k=3$, $3s<n\le 3(s+1)$. For $n=sk+2$ and $k\ge 4$ we prove

THEOREM 1.9. *Let* $n=sk+2$, $k\ge 5$. *If* $s\ge 4$ *or* $s=3$ *and* $k\le 47$ *then* (n,k,s) *is nice.*

By a "multiplying trick" we prove also:

THEOREM 1.10. *Let $d \geq 2$ be an integer. Suppose that (n, k, s) is nice. Then the triple (dn, dk, s) is nice as well.*

2. Preliminaries

The single most useful operation on families of sets, called *shifting*, was invented by Erdős, Ko and Rado [4]. In [8] it is shown that simultaneous shifting maintains the cross-union property. To explain its consequences let us introduce the notation $(a_1, a_2, \ldots, a_k)$ to denote the k-set $\{a_1, \ldots, a_k\}$ with the additional assumption $a_1 < \cdots < a_k$. One defines the *shifting partial order* $A \prec B$ on k-subsets where $(a_1, \ldots, a_k) \prec (b_1, \ldots, b_k)$ iff $a_i \leq b_i$ for all $1 \leq i \leq k$.

DEFINITION 2.1. The family $\mathcal{F} \subset \binom{[n]}{k}$ is called *shifted* if $F \in \mathcal{F}$ and $G \prec F$ always imply $G \in \mathcal{F}$.

Repeated applications of shifting produce shifted families (cf. [8] for details). Therefore it is sufficient to prove Theorems 1.7 and 1.8 for shifted families.

Since $(1, 2, \ldots, k)$ is the unique smallest element in the shifting partial order, $(1, 2, \ldots, k) \in \mathcal{F}$ for every non-empty shifted family $\mathcal{F} \subset \binom{[n]}{k}$.

DEFINITION 2.2. If $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s$ then the families are called *nested*.

CLAIM 2.3. *It is sufficient to prove Theorems* 1.7, 1.8 *and* 1.9 *for families that are shifted and nested.*

PROOF. The fact that we may suppose that $\mathcal{F}_i$ is shifted for $0 \leq i \leq s$ should be clear from the above discussion. Then $(1, 2, \ldots, k) \in \mathcal{F}$ for $0 \leq i \leq s$. For a fixed pair $0 \leq p < q \leq s$, replacing $\mathcal{F}_p$ and $\mathcal{F}_q$ by $\mathcal{F}_p \cap \mathcal{F}_q$ and $\mathcal{F}_p \cup \mathcal{F}_q$ will not alter $|\mathcal{F}_p| + |\mathcal{F}_q|$ and the cross-union property and shiftedness will be maintained. Iterating for all pairs $0 \leq p < q \leq k$ will eventually produce nested families. $\square$

The following statement follows easily by the methods of Kleitman [19].

LEMMA 2.4. *Let $k_0, \ldots, k_s$ be positive integers, $k_0 + \cdots + k_s \geq n$. Suppose that the families $\mathcal{G}_i \subset \binom{[n]}{k_i}$, $0 \leq i \leq s$, are cross-union. Then*

$$\sum_{0 \leq i \leq s} |\mathcal{G}_i| \Big/ \binom{n}{k_i} \leq s. \tag{2.1}$$

PROOF. Fix arbitrarily $A_0, \ldots, A_s$ satisfying $|A_0| = k_0, \ldots, |A_s| = k_s$ and $A_0 \cup \cdots \cup A_s = [n]$. Choose a permutation π of $[n]$ uniformly at random. Set $\pi(A) = \{\pi(a) : a \in A\}$.

The cross-union property implies that out of the $s+1$ sets $\pi(A_0), \ldots, \pi(A_s)$ at most s satisfy $\pi(A_i) \in \mathcal{G}_i$.

The probability of $\pi(A_i) \in \mathcal{G}_i$ is $|\mathcal{G}_i| / \binom{n}{k_i}$. Thus the LHS of (2.1) is the expected number of how many times $\pi(A_i) \in \mathcal{G}_i$ holds. Since this number is never more than s, (2.1) follows. $\square$

Corollary 2.5. *Suppose that $n = rk$, $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset \binom{[n]}{k}$ are cross-union. Then*

$$\sum_{1 \le i \le r} |\mathcal{F}_i| \le (r-1)\binom{n}{k} = r\binom{n-1}{k}. \tag{2.2}$$

Proof. The identity $(r-1)\binom{rk}{k} = r\binom{rk-1}{k}$ is easily checked. The inequality part of (2.2) follows from (2.1) by setting $s = r-1$ and $k_0 = \cdots = k_s = k$. $\square$

3. A general bound

Throughout this section let $\emptyset \ne \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s \subset \binom{[n]}{k}$ be non-empty, shifted cross-union families. Suppose that $n = ks + \ell$ with $1 \le \ell < k$. By Lemma 2.4 we have

$$|\mathcal{F}_0| + \cdots + |\mathcal{F}_s| \le s\binom{n}{k}. \tag{3.1}$$

Let us assume that

$$|\mathcal{F}_0| + \cdots + |\mathcal{F}_s| \ge (s+1)\binom{n-1}{k} \tag{3.2}$$

and derive a lower bound on $|\mathcal{F}_0|$. To this end define $\mathcal{G}_i = \mathcal{F}_i \cap \binom{[\ell+1, ks+\ell]}{k}$, $1 \le i \le s$.

Claim 3.1. *$\mathcal{G}_1, \ldots, \mathcal{G}_s$ are cross-union.*

Proof. Indeed, if $G_i \in \mathcal{G}_i$, $1 \le i \le s$, satisfy $G_1 \cup \cdots \cup G_s = [\ell+1, ks+\ell]$ then adding $[k] \in \mathcal{F}_0$ we get a contradiction with the cross-union property. $\square$

Corollary 3.2. *We have*

$$\sum_{1 \le i \le s} |\mathcal{F}_i| \le s\binom{n}{k} - \binom{ks}{k}. \tag{3.3}$$

PROOF. Define $\overline{\mathcal{F}}_i = \binom{[n]}{k} \setminus \mathcal{F}_i$, $1 \le i \le s$, the families of missing k-sets. Then $\overline{\mathcal{G}}_i := \overline{\mathcal{F}}_i \cap \binom{[\ell+1, sk+\ell]}{k}$ are the missing k-sets inside $[\ell+1, ks+\ell]$. Applying Lemma 2.4 with $k_0 = 0$, $\mathcal{G}_0 = \{\emptyset\}$ along with $\mathcal{G}_1, \ldots, \mathcal{G}_s$ yields

$$\sum_{1 \le i \le s} |\overline{\mathcal{G}}_i| \ge s\binom{ks}{k} - (s-1)\binom{ks}{k} = \binom{ks}{k}.$$

Since $\overline{\mathcal{G}}_i \subset \overline{\mathcal{F}}_i$ for $1 \le i \le s$, (3.3) follows. $\square$

Using (3.3) along with (3.1) and (3.2) we infer

$$|\mathcal{F}_0| \ge (s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{s}. \tag{3.4}$$

Let us evaluate (3.4) in the case $k = 3$, $n = 3s + 2$. We have

$$\begin{aligned} &s\binom{3s+2}{3} - (s+1)\binom{3s+1}{3} \\ &= \frac{(3s+1)s}{2}\big(s(3s+2) - (s+1)(3s-1)\big) = \frac{(3s+1)s}{2}. \end{aligned}$$

Thus we proved:

LEMMA 3.3. *In proving Theorem* 1.8 *one may assume*

$$|\mathcal{F}_0| \ge \binom{3s}{3} - \frac{(3s+1)s}{2} > \binom{3s-1}{3}. \tag{3.5}$$

PROOF. The first part of (3.5) follows from (3.4) and the above computation. The second part is equivalent to

$$\binom{3s-1}{2} > \frac{(3s+1)s}{2}.$$

Expanding and rearranging we obtain $3s^2 - 5s + 1 > 0$ which holds for $s \ge 2$. $\square$

Actually we only need the following consequence of (3.5).

CLAIM 3.4. *If* $\mathcal{F}_0 \subset \binom{[3s+2]}{3}$ *is shifted and* $|\mathcal{F}_0| > \binom{3s-1}{3}$ *then* $(1, 2, 3s) \in \mathcal{F}_0$.

PROOF. In view of $|\mathcal{F}_0| > \binom{3s-1}{3}$ there is some $(a, b, c) \in \mathcal{F}_0$ with $c \ge 3s$. As $a \ge 1$ and $b \ge 2$ are obvious, $(1, 2, 3s) \prec (a, b, c)$. Now $(1, 2, 3s) \in \mathcal{F}_0$ follows by shiftedness. $\square$

For general k let us show

PROPOSITION 3.5. *We have*

$$\sum_{0\le i\le s} |\mathcal{F}_i| \le s\binom{n}{k} + \binom{n-1}{k} - \binom{ks}{k}. \tag{3.6}$$

PROOF. In view of (3.3) we only need to show

$$|\mathcal{F}_0| \le \binom{n-1}{k}. \tag{3.7}$$

Since $\mathcal{F}_0 \subset \mathcal{F}_i$ for $1 \le i \le s$, $\mathcal{F}_0$ is $(s+1)$-wise union. Thus (3.7) follows by applying Theorem 1.4 to the family of complements, $\mathcal{F}_0^c = \big\{[n] \setminus F : F \in \mathcal{F}_0\big\}$. $\square$

4. The proof of Theorem 1.7

Let us set $X_0 = \bigcup_{F\in\mathcal{F}_0} F$ and let us apply Lemma 2.4 with $k_0 = 1$, $k_1 = \dots = k_s = k$; $\mathcal{G}_0 = \binom{X_0}{1}$, $\mathcal{G}_i = \mathcal{F}_i$ for $1 \le i \le s$. After rearranging we obtain

$$\sum_{1\le i\le s} |\mathcal{F}_i| \le \left(s - \frac{|X_0|}{n}\right)\binom{n}{k}. \tag{4.1}$$

We distinguish two cases:

(i) $X_0 = [n]$. Now the RHS of (4.1) is $(s-1)\binom{n}{k}$. Noting that $|\mathcal{F}_0| \le \dots \le |\mathcal{F}_s|$ implies $|\mathcal{F}_0| \le \frac{1}{s}\sum_{1\le i\le s} |\mathcal{F}_i|$, from (4.1) we infer

$$\sum_{0\le i\le s} |\mathcal{F}_i| \le \left(1 + \frac{1}{s}\right)(s-1)\binom{sk+1}{k} = (s+1)\frac{(s-1)}{s}\binom{sk+1}{k}.$$

To prove (1.9) we show $\frac{s-1}{s}\binom{sk+1}{k} < \binom{sk}{k}$. Indeed, dividing both sides by $\binom{sk}{k}$ yields

$$\frac{(s-1)(sk+1)}{s((s-1)k+1)} < 1,$$

which is true by

$$(s-1)(sk+1) = (s^2-s)k + s - 1 < (s^2-s)k + s = s\big((s-1)k+1\big).$$

(ii) $k \le |X_0| \le sk = n-1$. For notational convenience set $x = |X_0|$. Now (4.1) and $|\mathcal{F}_0| \le \binom{x}{k}$ yield

$$\sum_{0\le i\le s} |\mathcal{F}_i| \le s\binom{n}{k} + \binom{x}{k} - x\binom{n}{k} \Big/ n. \tag{4.2}$$

To conclude the proof, we show that the RHS of (4.2) is at most $(s+1)\binom{sk}{s}$. Noting that both $\binom{x}{k}$ and $-x$ are convex functions, it is sufficient to check it for $x = k$ and $x = sk$.

For $x = sk$ we have

$$s\binom{sk+1}{k} + \binom{sk}{k} - \frac{sk}{sk+1}\binom{sk+1}{k} = (s+1)\binom{sk}{k}.$$

Indeed, subtracting $\binom{sk}{k}$ from both sides and dividing by s yields

$$\binom{sk+1}{k}\left(1 - \frac{k}{sk+1}\right) = \binom{sk}{k}$$

which is true by

$$1 - \frac{k}{sk+1} = \frac{(s-1)k+1}{sk+1}.$$

In the case $x = k$ the necessary inequality reads

$$\left(s - \frac{k}{sk+1}\right)\binom{sk+1}{k} + 1 \le (s+1)\binom{sk}{k}. \tag{4.3}$$

Multiplying both sides by $(sk+1)$ and noting $(sk+1)\binom{sk}{k} = \big((s-1)k+1\big)\binom{sk+1}{k}$ (4.3) gets transformed to

$$(s^2k + s - k)\binom{sk+1}{k} + sk + 1 \le \big((s^2-1)k + s + 1\big)\binom{sk+1}{k}.$$

Since $sk+1 \le \binom{sk+1}{k}$ we can replace $sk+1$ by $\binom{sk+1}{k}$ on the LHS and note that the two sides coincide. This finishes the proof of (4.3) and thereby the proof of Theorem 1.7. $\square$

REMARK. It should be clear from the proof that equality can hold only if $x = sk$ and then $|\mathcal{F}_0| = \binom{sk}{k}$. This and nestedness imply $\mathcal{F}_0 = \mathcal{F}_1 = \cdots = \mathcal{F}_s$. Eventually, this entails that $\mathcal{F}_i = \binom{[sk]}{k}$, $i = 0, \dots, s$, is the unique optimal system.

5. The proof of Theorem 1.8

We are going to apply Katona's cyclic permutation method (cf. [18]). Let us fix a cyclic permutation $x_0x_1x_2\dots x_{n-1}x_0$ of $[n]$ and let us define $(s+1)n$ 3-element sets indexed by $0 \le t \le s$. The sets are $C_0^{(0)}$, $C_1^{(1)}$, …, $C_s^{(s)}$, $C_{s+1}^{(0)}$, … where $C_m^{(t)} = \bigl(x_{3m}, x_{3m+1}, x_{3m+2}\bigr)$ and the index t is the residue of m modulo $s+1$. Since $(3s+2, 3s+3) = 1$, each of the n circular arcs $x_a\, x_{a+1}\, x_{a+2}$ occurs exactly once with each index t, $0 \le t \le s$.

It should be clear from the definition that for each m, $0 \le m < (s+1)n$, $C_m^{(t)}$, $C_{m+1}^{(t+1)}$, …, $C_{m+s}^{(t+s)}$ cover $[n]$. Consequently at least one of $C_{m+i}^{(t+i)} \in \mathcal{F}^{(t+i)}$ fails ($t+i$ is computed modulo $s+1$).

Let q_j denote the number of m, $0 \le m < (s+1)n$ such that $m \equiv j \pmod{s+1}$, $C_m^{(j)} \in \mathcal{F}^{(j)}$.

In view of the above considerations $q_0 + q_1 + \dots + q_s \le sn$.

More importantly, equality holds only if the consecutive missing arcs are always at distance $s+1$ from each other. That is, one of the q_j is zero and all the others are equal to n. Taking into consideration that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_s$, $q_0 = 0$ follows.

Lemma 5.1. *We have*

$$q_0 + q_1 + \dots + q_s \le sn - 1. \tag{5.1}$$

Proof. In view of the above we must show only that $q_0 = 0$, $q_1 = \dots = q_s = n$ is impossible. Actually, this is the really novel part of the proof where we combine Katona's cycle method with shifting.

Suppose indirectly that $q_1 = \dots = q_s = n$. This means that all n cyclical 3-arcs are in each of $\mathcal{F}_1, \dots, \mathcal{F}_s$. Therefore for each consecutive pair x_m, x_{m+1} its complement is the union of s sets, one from each of $\mathcal{F}_i$, $1 \le i \le s$. Should $\{x_m, x_{m+1}\}$ be contained in some $F_0 \in \mathcal{F}_0$, we get the desired contradiction.

Choose m to satisfy $x_m = 1$. By Claim 3.4, $(1, 2, 3s) \in \mathcal{F}_0$. To avoid a contradiction, $\{x_{m-1}, x_{m+1}\} = (3s+1, 3s+2)$. However, not both can be neighbours of 2. Thus we can choose m' so that $\{x_{m'}, x_{m'+1}\} = \{2, p\}$ for some $3 \le p \le 3s$. By shiftedness $(1, 2, p) \in \mathcal{F}_0$ and we get the desired contradiction with the cross-union property. $\square$

The rest of the proof is simple averaging. Choose the cyclic permutation uniformly at random. Then the expected value of q_i is

$$E(q_i) = n \cdot |\mathcal{F}_i| \Big/ \binom{n}{3}, \quad 0 \le i \le s.$$

Summing this for $0 \le i \le s$ and using (5.1) along with the fact that expectation never exceeds the maximum we obtain

$$\frac{3s+2}{\binom{3s+2}{3}} \sum_{0 \le i \le s} |\mathcal{F}_i| \le s(3s+2) - 1 = (s+1)(3s-1).$$

Equivalently,

$$\sum_{0 \le i \le s} |\mathcal{F}_i| \le (s+1)\frac{3s-1}{3s+2}\binom{3s+2}{3} = (s+1)\binom{3s+1}{3}. \qquad \square$$

With a little more effort we can prove that equality is possible only if $\mathcal{F}_0 = \cdots = \mathcal{F}_s = \binom{[n-1]}{3}$.

6. The proof of Theorem 1.9

Since the underlying idea of combining shifting with Katona's cyclic permutation method is the same as in the proof of Theorem 1.8, we will be somewhat sketchy. Let $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ be nonempty, nested, shifted and cross-union. There are two new ingredients that we present first. Let $\pi = (x_1, x_2, \dots, x_n, x_1)$ be a fixed cyclic permutation. A *k-arc* is a k-set formed by consecutive elements: $\{x_i, x_{i+1}, \dots, x_{i+k-1}\}$. Let $\mathcal{A}(\pi)$ denote the collection of the k-arcs and note $|\mathcal{A}(\pi)| = n = sk+2$. Set $\mathcal{A}_i = \mathcal{F}_i \cap \mathcal{A}(\pi)$. Our aim is to prove

$$|\mathcal{A}_0| + \cdots + |\mathcal{A}_s| \le (s+1)(n-k) = (s+1)((s-1)k+2). \tag{6.1}$$

Note that $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_s$ implies $\mathcal{A}_0 \subset \cdots \subset \mathcal{A}_s$.

One of the ingredients is a recent statement from [10]. It states that (6.1) holds if $\mathcal{A}_0$ is non-empty. Thus in proving (6.1) we may assume that $\mathcal{A}_0 = \emptyset$.

The second new ingredient is:

LEMMA 6.1. *Suppose that $\{x_i, x_{i+1}\}$ is contained in a member of $\mathcal{F}_0$ for some i, $1 \le i \le n$. Then*

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_s| \le sn - s. \tag{6.2}$$

PROOF. Let us consider the k-arcs $B_1, \dots, B_s \in \mathcal{A}(\pi)$ that partition $[n] \setminus \{x_i, x_{i+1}\}$. Let us construct a bipartite graph, $\mathcal{G}$ with partite sets $\mathcal{B} := \{B_1, \dots, B_s\}$ and $[s]$, where we draw an edge between B_u and $v \in [s]$ iff $B_u \in \mathcal{A}_v$.

We claim that there is no perfect matching in this bipartite graph. Indeed the opposite means that there is a permutation $u(1)$, $u(2)$, …, $u(s)$

of $[s]$ such that $B_{u(j)} \in \mathcal{A}_j \subset \mathcal{F}_j$ for $1 \le j \le s$. Fixing an $F_0 \in \mathcal{F}_0$ containing $\{x_i, x_{i+1}\}$, we get the contradiction

$$F_0 \cup B_{u(1)} \cup \cdots \cup B_{u(s)} = [n].$$

Now it is a well-known and easy consequence of the König–Hall Theorem that $|\mathcal{G}| \le s(s-1) = s^2 - s$. That is, there are at least s edges missing. This implies (6.2). $\square$

Let us show that (6.2) implies (6.1). Since $\mathcal{A}_0 = \emptyset$, all we need to prove is

$$s(ks+2) - s \le (s+1)((s-1)k+2).$$

Expanding and rearranging yields $k \le s+2$, which is true by our assumptions.

To conclude the proof of the theorem we are going to show that independently of the permutation π, we can always find $\{x_i, x_{i+1}\}$ which is covered by some member of $\mathcal{F}_0$.

To achieve this we need:

Lemma 6.2. *We have*

$$|\mathcal{F}_0| \ge (s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{sk}{k} > \binom{sk-3}{k}. \tag{6.3}$$

Proof. The first inequality is (3.4). Thus we only have to show that

$$\binom{sk}{k} - \binom{sk-3}{k} > s\binom{n}{k} - (s+1)\binom{n-1}{k}. \tag{6.4}$$

Plugging in $n = sk+2$ and using $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ the RHS can be written as

$$\frac{s(sk+2) - (s+1)((s-1)k+2)}{(s-1)k+2}\binom{n-1}{k} = \frac{k-2}{(s-1)k+2}\binom{sk+1}{k}$$
$$= \frac{(k-2)(sk+1)}{((s-1)k+2)((s-1)k+1)}\binom{sk}{k} < \frac{s}{s-1}\,\frac{(k-2)}{(s-1)k+2}\binom{sk}{k}.$$

Note that

$$\binom{sk-3}{k} \Big/ \binom{sk}{k} = \frac{(s-1)k}{sk} \cdot \frac{(s-1)k-1}{sk-1} \cdot \frac{(s-1)k-2}{sk-2} < \Big(\frac{s-1}{s}\Big)^3.$$

Now (6.4) is reduced to

$$1 - \Big(\frac{s-1}{s}\Big)^3 > \frac{s}{s-1} \cdot \frac{k-2}{(s-1)k+2}$$

or equivalently

$$\frac{(s-1)(3s(s-1)+1)}{s^4} > \frac{k-2}{(s-1)k+2}. \tag{6.5}$$

Since the RHS is smaller than $1/(s-1)$, it is sufficient to prove

$$\frac{(s-1)^2(3s(s-1)+1)}{s^4} > 1.$$

For $s = 4$, $9 \times 37 > 256$ shows that it is true. As the LHS is easily seen to be an increasing function of s, it is true for all $s \geq 4$.

For $s = 3$, (6.5) is equivalent to

$$\frac{76}{81} > \frac{k-2}{k+1}.$$

That is, $\frac{5}{81} < \frac{3}{k+1}$ or $k < 47.6$, concluding the proof. □

In view of Lemma 6.2, $\mathcal{F}_0 \not\subset \binom{[sk-3]}{k}$. By shiftedness,

$$(1, 2, \ldots, k-1, sk-2) \in \mathcal{F}_0. \tag{6.6}$$

To conclude the proof is easy. We proceed similarly as in the $k = 3$ case.

Consider the location of 1, 2, 3, 4 in the cyclic permutation. If two of them are neighbours then choosing them as x_i, x_{i+1} we are done. Otherwise the four of them have altogether at least five neighbours. Not all of them are from $[sk-1, sk+2]$. Consequently, we find again $\{x_i, x_{i+1}\}$ with $x_i \leq 4$, $x_{i+1} \leq sk-2$ or $x_{i+1} \leq 4$, $x_i \leq sk-2$, guaranteeing that the pair is contained in some $F \in \mathcal{F}_0$. □

We should mention that for $k \geq 48$ it is sufficient to prove (6.3) with $\binom{sk-46}{k}$. This permits to prove the statement for $s = 3$, $k \geq 48$ as well.

7. A simple trick

Let us note first that for any positive integer d,

$$\binom{dn-1}{dk} \Big/ \binom{dn}{dk} = \frac{d(n-k)}{dn} = \frac{n-k}{n} = \binom{n-1}{k} \Big/ \binom{n}{k}. \tag{7.1}$$

It is easy to use (7.1) to prove Theorem 1.10. Let $\mathcal{F}_0, \ldots, \mathcal{F}_s \subset \binom{dn}{dk}$ be cross-union. Choose uniformly at random a partition $[dn] = X_1 \cup \cdots \cup X_n$ where $|X_i| = d$ for $1 \leq i \leq n$.

With a k-set $H \in \binom{[n]}{k}$ associate the dk-set $G(H) = \bigcup_{i \in H} X_i$. Define

$$\mathcal{G}_i = \left\{ H \in \binom{[n]}{k} : G(H) \in \mathcal{F}_i \right\}, \quad i = 0, 1, \dots, s.$$

It is immediate to check that $\mathcal{G}_0, \dots, \mathcal{G}_s$ are cross-union. By the assumptions,

$$\sum_{0 \le i \le s} |\mathcal{G}_i| \le (s+1)\binom{n-1}{k}. \tag{7.2}$$

Since for $H \in \binom{[n]}{k}$ the set $G(H)$ is a uniformly random dk-subset on $[dn]$, the expected size $E(|\mathcal{G}_i|)$ satisfies

$$E(|\mathcal{G}_i|) = \frac{|\mathcal{F}_i|}{\binom{dn}{dk}} \binom{n}{k}.$$

Using (7.2) we infer

$$\sum_{0 \le i \le s} \frac{|\mathcal{F}_i|}{\binom{dn}{dk}} \binom{n}{k} \le (s+1)\binom{n-1}{k},$$

or equivalently, invoking (7.1),

$$\sum_{0 \le i \le s} |\mathcal{F}_i| \le (s+1) \frac{\binom{n-1}{k}}{\binom{n}{k}} \binom{dn}{dk} = (s+1)\binom{dn-1}{dk}. \qquad \square$$

8. Concluding remarks

The results of the present paper motivate the following conjecture.

Conjecture 8.1. *Let $n = sk + \ell$ with $1 \le \ell \le k$, $s \ge 2$, $k \ge 2$. Suppose that $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are non-empty and cross-union. Then*

$$\frac{|\mathcal{F}_0| + \dots + |\mathcal{F}_s|}{s+1} \le \binom{n-1}{k} \quad \textit{holds for } s \ge s_0(\ell). \tag{8.1}$$

By Corollary 2.5 and Theorem 1.7, (8.1) holds with $s_0(\ell) = 2$ for both $\ell = k$ and $\ell = 1$. As to the case $\ell = 2$, for $k = 3$ and 4 it holds for $s \ge 3$ by Theorems 1.8 and 1.10. Theorem 1.9 establishes it for $s \ge 4$ as well as for $s = 3$, $5 \le k \le 47$. The statement of Conjecture 8.1 follows from Theorem 1.10 if $\ell = 3$ and k is a multiple of 3.

Consequently the first open cases are $\ell = 3$, $k = 5$, 7 or 8.

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