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ON PRODUCTS OF CONSECUTIVE ARITHMETIC PROGRESSIONS. III

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Abstract. Let $f(x,k,d)=x(x+d)\cdots(x+(k-1)d)$ be a polynomial with $k\geq 2$, $d\geq 1$. We consider the Diophantine equation $\prod_{i=1}^{r} f(x_i,\,k_i,d)=y^2$, $r\geq 1$. Using the theory of Pell equations, we affirm a conjecture of Bennett and van Luijk [3]; extend some results of this Diophantine equation for $d=1$, and give a positive answer to Question 3.2 of Zhang [19].

1. Introduction

Let us define the polynomial

$$f(x,k,d)=x(x+d)\cdots(x+(k-1)d)$$

with $k\geq 2$, $d\geq 1$. Many authors have studied the Diophantine equation

$$\prod_{i=1}^{r} f(x_i,k_i,d)=y^2, \tag{1.1}$$

where $r\geq 1$, $f(x_i,k_i,d)=x_i(x_i+d)\cdots(x_i+(k_i-1)d)$ are disjoint for $i=1,\dots,r$, and $2\leq k_1\leq k_2\leq\cdots\leq k_r$.

(1) *The case* $r=1$, $d\geq 1$. There are many results about (1.1) and the more general Diophantine equation

$$f(x,k,d)=by^l,$$

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where $b > 0$, $l \geq 3$ and the greatest prime factor of b does not exceed k; we can refer to [2,5–9,12,13].

(2) *The case* $r \geq 2$, $d = 1$. When $r = 2$, $d = 1$, $k_i = 3$, Sastry [6] showed that (1.1) has infinitely many positive integer solutions (x_1, x_2, y), where x_1, x_2 satisfying $x_2 = 2x_1 - 1$ and $(x_1 + 1)(2x_1 - 1)$ is a square.

Erdős and Graham [4, p. 67] asked whether (1.1) has, for fixed $r \geq 1$, $d = 1$ and k_1, k_2, …, k_r with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many positive integer solutions $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i - 1 < x_{i+1}$ for $1 \leq i \leq r - 1$. Skałba [14] obtained a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [17] answered the above question of Erdős and Graham in the negative when either $r = 4$, $d = 1$, $k_i = 4$, $i = 1, 2, 3, 4$, or $r \geq 6$, $d = 1$, $k_i = 4$, $1 \leq i \leq r$. Bauer and Bennett [1] extended Ulas's result to the cases $r = 3$ and $r = 5$.

For the case $r = 2$, $d = 1$, $k_1 = k_2 = 4$, (1.1) has a positive integer solution $(x_1, x_2, y) = (33, 1680, 3361826160)$. Luca and Walsh [11] studied this case by using the identity $(x - 1)x(x + 1)(x + 2) = (x^2 + x - 1)^2 - 1$ to reduce the original problem to a Pell equation $(x^2 + x - 1)^2 - dy^2 = 1$, where $d > 1$ is a square-free integer. Tengely [15] provided an upper bound for the size of the solutions and determined all solutions up to some bounds for this case.

Bennett and van Luijk [3] constructed an infinity of positive integer solutions of (1.1) for $r \geq 5$, $d = 1$, $k_i = 5$. Tengely and Ulas [16] studied (1.1) for $r = 2$, $d = 1$, $k_1 = 2$, $k_2 = 3, 4, 5$. Zhang [19] gave infinitely many positive integer solutions of (1.1) with $d = 1$ for the cases $r = 2$, $k_1 = 3$, $k_2 = 4$; $r = 3$, $k_1 = 3$, $k_2 = k_3 = 4$; $r = 3$, $k_1 = 3$, $k_2 = 4$, $k_3 = 5$; $r \geq 2$, $k_1 = 3$, $k_i = 4$, $i = 2, \ldots, r$; and $r \geq 3$, $k_1 = 3$, $k_2 = 5$, $k_i \geq 5$, $i = 3, \ldots, r$. Yıldız and Gürel [18] presented new algorithms generating new polynomial parameterizations that extend the ones given by Bennett and van Luijk [3], and produced the first examples of (1.1) for $r = 7, 8$, $d = 1$, $k_i = 6, 7$.

(3) *The case* $r \geq 2$, $d \geq 2$. We are looking for the positive integer solutions of (1.1) which satisfy $d \nmid x_i$ for some i. If the solutions $(x_1, \ldots, x_r, y)$ satisfy $d \mid x_i$, $i = 1, \ldots, r$, we call them trivial. For $r = 2$, $k_i = 3$ and even number d, Zhang and Cai [20] have proved that (1.1) has infinitely many nontrivial positive integer solutions. For $r = 2$, $k_i = 3$, $d \geq 2$, Katayama [10] showed that (1.1) also has infinitely many nontrivial positive integer solutions when the integers d is divisible by a prime p ($\equiv \pm 1 \pmod 8$). Zhang [19] gave infinitely many nontrivial positive integer solutions of (1.1) with $d \geq 2$ for some cases, such as $r \geq 2$, $k_1 = 2$, $k_i \geq 2$, $i = 2, \ldots, r$, and $r \geq 2$, $k_1 = k_2 = 3$, $k_i \geq 3$, $i = 3, \ldots, r$ with $d \geq 2$; $r = 3$ or $r \geq 5$, $k_i = 4$, $i = 1, \ldots, r$ with even number d; and $r = 3$ or $r \geq 5$, $k_i = 4$, $i = 1, \ldots, r$ with $d = 3$.

For more information on (1.1), we can refer to the references above and the related ones they cited.

In this paper, firstly, we study the case $r = 4$, $d = 1$, $k_i = 5$ and affirm a conjecture of Bennett and van Luijk [3].

THEOREM 1.1. *For* $r = 4$, $d = 1$, $k_i = 5$, $i = 1, 2, 3, 4$, (1.1) *has infinitely many positive integer solutions.*

Secondly, we consider the cases for $r \geq 2$, $d = 1$ with k_i are different. Let us recall that Bennett and van Luijk [3] showed that (1.1) has infinitely many positive integer solutions for the cases $r \geq 2$, $d = 1$, $k_1 = 2$, $k_i \geq 2$, $i = 2, \dots, r$, and $r \geq 2$, $d = 1$, $k_1 = k_2 = 3$, $k_i \geq 3$, $i = 3, \dots, r$. Tengely and Ulas [16] proved that (1.1) has infinitely many solutions in the ring $\mathbb{Z}[t]$ for the cases $r = 2$, $d = 1$, $k_1 = 2$, $k_2 = 3, 4$, and at least two solutions in the ring $\mathbb{Z}[t]$ for the case $r = 2$, $d = 1$, $k_1 = 2$, $k_2 = 5$.

Now we extend Theorem 1.1, Corollary 1.2 and Theorem 1.3 of [19] and have the following theorems.

THEOREM 1.2. *For* $r = 3$, $d = 1$, $k_1 = k_2 = 4$, $k_3 = 5, 6, 7, 8, 9$, (1.1) *has infinitely many positive integer solutions.*

THEOREM 1.3. *For* $r \geq 3$, $d = 1$, $k_1 = 3$, $k_2 = 4$, $k_i \geq 4$, $i = 3, \dots, r$, *if the Pell equation* $U^2 - AV^2 = 5$ *has a positive integer solution* (U_0, V_0) *satisfying the condition*

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2,$$

then (1.1) *has infinitely many positive integer solutions.*

THEOREM 1.4. *For* $r \geq 4$, $d = 1$, $k_1 = 4$, $k_2 = 6$, $k_i \geq 4$, $i = 3, \dots, r$, *if the Pell equation* $U^2 - AV^2 = 17$ *has a positive integer solution* (U_0, V_0) *satisfying the condition*

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2,$$

then (1.1) *has infinitely many positive integer solutions.*

For $r = 3$, $d = 1$, $k_1 = 4$, $k_2 = 6$, $k_3 \geq 5$, we cannot give infinitely many positive integer solutions of (1.1).

Thirdly, we investigate the case $r = 4$, $d \geq 2$, $k_i = 4$, and give a positive answer to Question 3.2 of [19] in the following theorem.

THEOREM 1.5. *For* $r = 4$, $k_i = 4$, $i = 1, 2, 3, 4$, *and even number* $d \geq 2$, (1.1) *has infinitely many nontrivial positive integer solutions*, *i.e.*, *the Diophantine equation*

$$\begin{aligned} &x(x+d)(x+2d)(x+3d)y(y+d)(y+2d)(y+3d) \\ &\times z(z+d)(z+2d)(z+3d)w(w+d)(w+2d)(w+3d) = t^2 \end{aligned} \tag{1.2}$$

has infinitely many nontrivial positive integer solutions for even number d.

Moreover, we generalize the results in Theorem 1.8 and Corollary 1.9 of [19] to $r = 4$ and any positive integer d with $3 \mid d$.

THEOREM 1.6. *For $r=4$, $k_i=4$, $i=1,2,3,4$, and any positive integer $d \geq 2$ with $3 \mid d$,* (1.1) *has infinitely many nontrivial positive integer solutions, i.e.,* (1.2) *has infinitely many nontrivial positive integer solutions for d with $3 \mid d$.*

Combining the above results, Theorem 1.6 and Corollary 1.7 of [19], we get

COROLLARY 1.7. *For any positive integer $d \geq 2$ with $2 \mid d$ or $3 \mid d$, if $r \geq 3$, $k_i=4$, $i=1,\dots,r$, then* (1.1) *has infinitely many nontrivial positive integer solutions.*

Lastly, we study the case $r=4$, $d \geq 2$, $k_i=5$, and obtain

THEOREM 1.8. *For $r=4$, $k_i=5$, $i=1,2,3,4$, and even number $d \geq 2$,* (1.1) *has infinitely many nontrivial positive integer solutions, i.e., the Diophantine equation*

$$\begin{aligned}(1.3)\quad & x(x+d)(x+2d)(x+3d)(x+4d)y(y+d)(y+2d)(y+3d)(y+4d)\\ & \times z(z+d)(z+2d)(z+3d)(z+4d)w(w+d)(w+2d)(w+3d)(w+4d)=t^2\end{aligned}$$

has infinitely many nontrivial positive integer solutions for even number d.

For even number $d \geq 2$, the Diophantine equation

$$x(x+d)(x+2d)(x+3d)(x+4d)y(y+d)(y+2d)(y+3d)(y+4d)=z^2$$

has integer solutions

$$(x,y,z)=\Big(\frac{d}{2},6d,945d^5\Big),\Big(\frac{5d}{2},11d,\frac{45045d^5}{2}\Big).$$

Since each even number $r \geq 4$ is of the form $4s+2$, $4s+4$, then we have

COROLLARY 1.9. *For $r \geq 4$, $d \geq 2$, $k_i=5$, $i=1,\dots,r$, where r and d are even numbers,* (1.1) *has infinitely many nontrivial positive integer solutions.*

2. Proofs

PROOF OF THEOREM 1.1. For $r=4$, $d=1$, $k_i=5$, $i=1,2,3,4$, take

$$x_1=u,\quad x_2=2u+4,\quad x_3=v,\quad x_4=\frac{v-2}{2},$$

then (1.1) leads to

$$\begin{aligned}(2.1)\qquad & u(u+1)(2u+5)(2u+7)(v-2)(v+1)(v+3)(v+6)\\ & \times\frac{(v(v+2)(v+4)(u+2)(u+3)(u+4))^2}{4}=y^2.\end{aligned}$$

Using the same method as Bauer and Bennett [1], if we let

$$u(2u+7)=\frac{1}{3}(v-2)(v+6), \tag{2.2}$$

then

$$(u+1)(2u+5)=\frac{1}{3}(v+1)(v+3),$$

and (2.1) has positive integer solutions.

(2.2) is equivalent to the Pell equation

$$U^2-6V^2=57,$$

where $U=12u+21,\ V=2v+4$. An infinity of positive integer solutions of $U^2-6V^2=57$ are given by

$$U_n+V_n\sqrt{6}=\left(9+2\sqrt{6}\right)\left(5+2\sqrt{6}\right)^n,\quad n\ge 0.$$

Thus,

$$\begin{cases} U_n=10U_{n-1}-U_{n-2}, & U_0=9,\ U_1=69,\ U_2=681;\\ V_n=10V_{n-1}-V_{n-2}, & V_0=2,\ V_1=28,\ V_2=278.\end{cases}$$

From

$$u=\frac{U-21}{12},\quad v=\frac{V-4}{2},$$

we have

$$\begin{cases} u_n=10u_{n-1}-u_{n-2}+14, & u_0=-1,\ u_1=4,\ u_2=55;\\ v_n=10v_{n-1}-v_{n-2}+16, & v_0=-1,\ v_1=12,\ v_2=137.\end{cases}$$

It is easy to check that

$$u_n, v_n\in\mathbb{Z}^+,\quad n\ge 1,$$

and

$$v_{2n+1}\equiv 0 \pmod 2.$$

Then

$$x_4=\frac{v_{2n+1}-2}{2}\in\mathbb{Z}^+,\quad n\ge 0.$$

Thus, we have

$$y_{2n+1}=\frac{1}{2}v_{2n+1}(v_{2n+1}+1)(v_{2n+1}+2)(v_{2n+1}+3)(v_{2n+1}+4)$$
$$\times\, u_{2n+1}(u_{2n+1}+2)(u_{2n+1}+3)(u_{2n+1}+4)(2u_{2n+1}+7)\in\mathbb{Z}^+,\quad n\ge 0.$$

From the recurrence relations of u_n and v_n, we can check the following inequalities

$$u_{2n+1}+4<\frac{v_{2n+1}-2}{2},\ \frac{v_{2n+1}-2}{2}+4<2u_{2n+1}+4,\ 2u_{2n+1}+8<v_{2n+1},$$

for $n\geq 1$.

Therefore, for $r=4$, $d=1$, $k_i=5$, $i=1,2,3,4$, (1.1) has infinitely many positive integer solutions

$$\Big(u_{2n+1},2u_{2n+1}+4,v_{2n+1},\frac{v_{2n+1}-2}{2},y_{2n+1}\Big),$$

where $n\geq 1$, such that $x_i(x_i+1)(x_i+2)(x_i+3)(x_i+4)$, $i=1,\dots,4$ are disjoint. □

Example 2.1. From Theorem 1.1, take $n=1,2,3$, the Diophantine equation

$$\prod_{i=1}^{4} x_i(x_i+1)(x_i+2)(x_i+3)(x_i+4)=y^2$$

has three positive integer solutions

$$(x_1,x_2,x_3,x_4,y)=(560,1124,1374,686,27777732057250795327099392000 0),$$

$$(55044,110092,134832,67415,$$

$$2252400654710414627638221525370979214670751324160 0),$$

$$(5393920,10787844,13212354,6606176,$$

$$18383358056019034711442787441621459833184100287199098681120282894336 00).$$

For the cases $r=3$, $d=1$, $k_1=k_2=4$, $k_3=5,6,7,8,9$, we have not a unified method to deal with (1.1), so we have to give the proofs one by one.

Proof of Theorem 1.2. 1) For $r=3$, $d=1$, $k_1=k_2=4$, $k_3=5$, (1.1) reduces to

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3)$$
$$\times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)=y^2.$$

Let $x_1=u$, $x_2=v$, $x_3=2v$, then we get

$$8u(u+1)(u+2)(u+3)(v+3)(2v+1)(2v+3)v^2(v+1)^2(v+2)^2=y^2.$$

One only needs to consider

$$2u(u+1)(u+2)(u+3)(v+3)(2v+1)(2v+3)=z^2.$$

Take

$$v=\frac{3(u^2+3u-2)}{4},$$

then

$$\frac{3u^2+9u-4}{2}\,\frac{[3u(u+1)(u+2)(u+3)]^2}{4}=z^2.$$

Considering

$$\frac{3u^2+9u-4}{2}=w^2,$$

it is equivalent to the Pell equation

$$U^2-6W^2=129,$$

where $U=6u+9, W=2w$.

An infinity of positive integer solutions of $U^2-6W^2=129$ is given by

$$U_n+W_n\sqrt{6}=\left(15+4\sqrt{6}\right)\left(5+2\sqrt{6}\right)^n,\quad n\geq 0.$$

Thus,

$$\begin{cases} U_n=10U_{n-1}-U_{n-2}, & U_0=15,\ U_1=123,\ U_2=1215;\\ W_n=10W_{n-1}-W_{n-2}, & W_0=4,\ W_1=50,\ W_2=496.\end{cases}$$

From

$$u=\frac{U-9}{6},\quad w=\frac{W}{2},$$

we have

$$\begin{cases} u_n=10u_{n-1}-u_{n-2}+12, & u_0=1,\ u_1=19,\ u_2=201;\\ w_n=10w_{n-1}-w_{n-2}, & w_0=2,\ w_1=24,\ w_2=248.\end{cases}$$

Note that

$$v=\frac{3(u^2+3u-2)}{4}$$

is a positive integer, so we need $u\equiv 2,3 \pmod 4$. By the recurrence relation of u_n, it is easy to show that $u_{2n+1}\equiv 3 \pmod 4$. Thus,

$$v_{2n+1}=\frac{3(u_{2n+1}^2+3u_{2n+1}-2)}{4}\in\mathbb{Z}^+,\quad n\geq 0.$$

Then

$$y_{2n+1} = 3w_{2n+1}u_{2n+1}(u_{2n+1}+1)(u_{2n+1}+2)(u_{2n+1}+3) \times v_{2n+1}(v_{2n+1}+1)(v_{2n+1}+2) \in \mathbb{Z}^+, \quad n \geq 0.$$

Therefore, for $r=3,\ d=1,\ k_1=k_2=4,\ k_3=5$, (1.1) has infinitely many positive integer solutions

$$(u_{2n+1}, v_{2n+1}, 2v_{2n+1}, y_{2n+1}), \quad n \geq 1,$$

such that $x_1(x_1+1)(x_1+2)(x_1+3)$, $x_2(x_2+1)(x_2+2)(x_2+3)$ and $x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)$ are disjoint.

2) For $r=3$, $d=1$, $k_1=k_2=4$, $k_3=6$, (1.1) leads to

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3) \times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5) = y^2.$$

Let $x_1 = 16$ and

$$x_2 = \frac{u(u+5)}{2}, \quad x_3 = u,$$

then we have

$$646(u^2+5u+2)(3u(u+1)(u+2)(u+3)(u+4)(u+5))^2 = y^2.$$

Considering

$$u^2+5u+2 = 646v^2,$$

it is equivalent to the Pell equation

$$U^2 - 646V^2 = 17,$$

where $U = 2u+5$, $V = 2v$.

An infinity of positive integer solutions of $U^2 - 646V^2 = 17$ is given by

$$U_n + V_n\sqrt{646} = \left(51 + 2\sqrt{646}\right)\left(305 + 12\sqrt{646}\right)^n, \quad n \geq 0.$$

Thus,

$$\begin{cases} U_n = 610U_{n-1} - U_{n-2}, & U_0 = 51,\ U_1 = 31059,\ U_2 = 18945939; \\ V_n = 610V_{n-1} - V_{n-2}, & V_0 = 2,\ V_1 = 1222,\ V_2 = 745418. \end{cases}$$

From

$$u = \frac{U-5}{2}, \quad v = \frac{V}{2},$$

we have

$$\begin{cases} u_n = 610u_{n-1} - u_{n-2} + 1520, & u_0 = 23,\ u_1 = 15527,\ u_2 = 9472967; \\ v_n = 610v_{n-1} - v_{n-2}, & v_0 = 1,\ v_1 = 611,\ v_2 = 372709. \end{cases}$$

By the recurrence relation of u_n, we have $u_n \equiv 1 \pmod 2$, then

$$x_2 = \frac{u_n(u_n+5)}{2} \in \mathbb{Z}^+, \quad n \geq 0.$$

Thus,

$$y_n = 1938u_n(u_n+1)(u_n+2)(u_n+3)(u_n+4)(u_n+5)v_n \in \mathbb{Z}^+, \quad n \geq 0.$$

So the Diophantine equation

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3) \\ \times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5) = y^2$$

has infinitely many positive integer solutions

$$\Big(16, \frac{u_n(u_n+5)}{2}, u_n, y_n\Big), \quad n \geq 0,$$

such that

$$x_1(x_1+1)(x_1+2)(x_1+3), \quad x_2(x_2+1)(x_2+2)(x_2+3), \\ x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)$$

are disjoint.

3) For $r=3$, $d=1$, $k_1=k_2=4$, $k_3=7$. Let

$$x_1 = \frac{p^2(p^2+7)}{4}, \quad x_2 = 2x_1+2,\ x_3 = p^2+1,$$

where p is a positive integer. Then

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3) \\ \times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6) = y^2,$$

where

$$y = \frac{1}{64}p(p^2+1)(p^2+2)(p^2+3)(p^2+4)(p^2+5)(p^2+6)(p^2+7) \\ \times (p^4+7p^2+4)(p^4+7p^2+8),$$

and $p \geq 2$.

It is easy to check that $x_1, y \in \mathbb{Z}^+$, and

$$x_1(x_1+1)(x_1+2)(x_1+3), \quad x_2(x_2+1)(x_2+2)(x_2+3),$$
$$x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6)$$

are disjoint for $p \geq 2$.

4) For $r=3$, $d=1$, $k_1=k_2=4$, $k_3=8$. Take

$$x_1 = p(4p+7), \quad x_2 = 2x_1+2, \quad x_3 = 4p.$$

Then

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3)$$
$$\times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6)(x_3+7) = y^2,$$

where

$$y = 16p(p+1)(2p+1)(2p+3)(4p+1)(4p+3)(4p+5)(4p+7)$$
$$\times (4p^2+7p+1)(4p^2+7p+2),$$

and $p \geq 1$.

We can also put

$$x_1 = (p+2)(4p+1), \quad x_2 = 2x_1+2, \quad x_3 = 4p+1.$$

Then

$$y = 16(p+1)(p+2)(2p+1)(2p+3)(4p+1)(4p+3)(4p+5)(4p+7)$$
$$\times (4p^2+9p+3)(4p^2+9p+4),$$

and $p \geq 2$.

It is easy to show that

$$x_1(x_1+1)(x_1+2)(x_1+3), \quad x_2(x_2+1)(x_2+2)(x_2+3),$$
$$x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6)(x_3+7)$$

are disjoint for $p \geq 2$.

5) For $r=3$, $d=1$, $k_1=k_2=4$, $k_3=9$. Let

$$x_1 = (p^2+2)(4p^2+1), \quad x_2 = 2x_1+2, \quad x_3 = 4p^2.$$

Then

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3)$$
$$\times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6)(x_3+7)(x_3+8) = y^2,$$

where

$$y = 32p(p^2+1)(p^2+2)(p^2+3)(2p^2+1)(2p^2+3)(4p^2+1)(4p^2+3)$$
$$\times (4p^2+5)(4p^2+7)(4p^4+9p^2+3)(4p^4+9p^2+4),$$

and $p \geq 1$.

It is easy to check that

$$x_1(x_1+1)(x_1+2)(x_1+3), \quad x_2(x_2+1)(x_2+2)(x_2+3),$$
$$x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6)(x_3+7)(x_3+8)$$

are disjoint for $p \geq 2$. $\square$

REMARK 2.2. In fact, for $r=3,\ d=1,\ k_1=k_2=4,\ k_3=9$, we have another simple method to give infinitely many positive integer solutions of (1.1). By the solutions of the case $r=3,\ d=1,\ k_1=k_2=4,\ k_3=8$, we can take $4p+8=(2q)^2$ or $4p+9=(2q+1)^2$, then

$$p = q^2-2 \quad \text{or} \quad q^2+q-2.$$

1) When $p = q^2-2$, let

$$x_1 = p(4p+7), \quad x_2 = 2x_1+2, \quad x_3 = 4p.$$

Then

$$x_1(x_1+1)(x_1+2)(x_1+3)x_2(x_2+1)(x_2+2)(x_2+3)$$
$$\times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)(x_3+5)(x_3+6)(x_3+7)(x_3+8) = y^2,$$

where

$$y = 32qp(p+1)(2p+1)(2p+3)(4p+1)(4p+3)(4p+5)(4p+7)$$
$$\times (4p^2+7p+1)(4p^2+7p+2),$$

and $q \geq 2$.

2) When $p = q^2+q-2$, take

$$x_1 = (p+2)(4p+1), \quad x_2 = 2x_1+2, \quad x_3 = 4p+1.$$

Then

$$y = 16(2q+1)(p+1)(p+2)(2p+1)(2p+3)(4p+1)(4p+3)(4p+5)(4p+7)$$
$$\times (4p^2+9p+3)(4p^2+9p+4),$$

and $q \geq 2$.

Proof of Theorem 1.3. For $r \geq 3$, $d=1$, $k_1=3$, $k_2=4$, $k_i \geq 4$, $i=3,\dots,r$, let

$$\prod_{i=3}^{r} x_i(x_i+1)\cdots(x_i+k_i-1)=Aw^2.$$

Choose $x_i \in \mathbb{Z}^+$, $k_i \geq 4$, $i=3,\dots,r$ such that $x_i(x_i+1)\cdots(x_i+k_i-1)$ are disjoint, A is not a perfect square, and the Pell equation $U^2-AV^2=5$ has a positive integer solution (U_0,V_0) satisfying the condition

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2.$$

By the transformation

$$x_2=u, \quad x_1=u(u+3),$$

(1.1) leads to

$$A(u^2+3u+1)u^2(u+1)^2(u+2)^2(u+3)^2w^2=y^2.$$

Let

$$u^2+3u+1=Av^2,$$

then $U^2-AV^2=5$, where $U=2u+3$, $V=2v$.

If (U',V') is the fundamental solution of the Pell equation $U^2-AV^2=1$, then an infinity of positive integer solutions of $U^2-AV^2=5$ is given by

$$U_n+V_n\sqrt{A}=\left(U'+V'\sqrt{A}\right)^n\left(U_0+V_0\sqrt{A}\right), \quad n \geq 0.$$

Thus,

$$\begin{cases} U_n=2U'U_{n-1}-U_{n-2}, & U_0=U_0,\ U_1=U'U_0+AV'V_0,\\ & U_2=(2U'^2-1)U_0+2AU'V'V_0;\\ V_n=2U'V_{n-1}-V_{n-2}, & V_0=V_0,\ V_1=U'V_0+V'U_0,\\ & V_2=(2U'^2-1)V_0+2U'V'U_0. \end{cases}$$

From

$$u=\frac{U-3}{2}, \quad v=\frac{V}{2},$$

we have

$$\begin{cases} u_n=2U'u_{n-1}-u_{n-2}+3(U'-1), & u_0=\frac{U_0-3}{2},\ u_1=\frac{U'U_0+AV'V_0-3}{2},\\ & u_2=U'(U'U_0+AV'V_0)-\frac{U_0+3}{2};\\ v_n=2U'v_{n-1}-v_{n-2}, & v_0=\frac{V_0}{2},\ v_1=\frac{U'V_0+V'U_0}{2},\\ & v_2=U'(U'V_0+V'U_0)-\frac{V_0}{2}. \end{cases}$$

In view of the condition of U_0 and V_0, we obtain $u_0 \in \mathbb{Z}$, $v_0 \in \mathbb{Z}^+$. By the recurrence relation of u_n and v_n, we have $u_{2n} \in \mathbb{Z}^+$, $v_{2n} \in \mathbb{Z}^+$, and

$$y_{2n} = Awu_{2n}(u_{2n}+1)(u_{2n}+2)(u_{2n}+3)v_{2n} \in \mathbb{Z}^+, \quad n \geq 1.$$

Therefore, for $r \geq 3,\ d=1,\ k_1=3,\ k_2=4,\ k_i \geq 4,\ i=3,\dots,r,$ (1.1) has infinitely many positive integer solutions

$$(u_{2n}(u_{2n}+3), u_{2n}, x_3, \dots, x_r, y_{2n}),$$

where $n \geq 1$, such that $x_i(x_i+1)\cdots(x_i+k_i-1)$, $i=1,\dots,r$ are disjoint. □

EXAMPLE 2.3. 1) For $r=3$, $d=1$, $k_1=3$, $k_2=4$, $k_3=4$, we can take $x_3=8$, then

$$8 \cdot (8+1) \cdot (8+2) \cdot (8+3) = 55 \cdot 12^2.$$

It is easy to see that $(U_0, V_0) = (15, 2)$ is a positive integer solution of the Pell equation $U^2 - 55V^2 = 5$ satisfying the condition

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2.$$

Hence, (1.1) has infinitely many positive integer solutions

$$(u_n(u_n+3), u_n, 8, y_n),$$

such that

$$x_1(x_1+1)(x_1+1), \quad x_2(x_2+1)(x_2+2)(x_2+3), \quad x_3(x_3+1)(x_3+2)(x_3+3)$$

are disjoint, where

$$\begin{cases} u_n = 178u_{n-1} - u_{n-2} + 264, & u_0 = 6,\ u_1 = 1326,\ u_2 = 236286; \\ v_n = 178v_{n-1} - v_{n-2}, & v_0 = 1,\ v_1 = 179,\ v_2 = 31861, \end{cases}$$

$$y_n = 660u_n(u_n+1)(u_n+2)(u_n+3)v_n,$$

and $n \geq 1$. This is not covered by Theorem 1.1 of [19].

2) For $r=3$, $d=1$, $k_1=3$, $k_2=4$, $k_3=5$, we can put $x_3=2$, which is the third case in Theorem 1.1 of [19]. In fact, our result is motivated by this case.

3) For $r=3$, $d=1$, $k_1=3$, $k_2=4$, $k_3=6$, this is a new case. Set $x_3=1$, then

$$1 \cdot (1+1) \cdot (1+2) \cdot (1+3) \cdot (1+4) \cdot (1+5) = 5 \cdot 12^2.$$

Note that $(U_0, V_0) = (5, 2)$ is a positive integer solution of the Pell equation $U^2 - 5V^2 = 5$ satisfying the condition

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2.$$

Hence, (1.1) has infinitely many positive integer solutions

$$(u_n(u_n + 3), u_n, 1, y_n),$$

such that

$$x_1(x_1 + 1)(x_1 + 2), \quad x_2(x_2 + 1)(x_2 + 2)(x_2 + 3),$$
$$x_3(x_3 + 1)(x_3 + 2)(x_3 + 3)(x_3 + 4)(x_3 + 5)$$

are disjoint, where

$$\begin{cases} u_n = 18u_{n-1} - u_{n-2} + 24, & u_0 = 1,\ u_1 = 41,\ u_2 = 761; \\ v_n = 18v_{n-1} - v_{n-2}, & v_0 = 1,\ v_1 = 19,\ v_2 = 341, \end{cases}$$

$$y_n = 60u_n(u_n + 1)(u_n + 2)(u_n + 3)v_n,$$

and $n \geq 1$.

Proof of Theorem 1.4. For $r \geq 4$, $d = 1$, $k_1 = 4$, $k_2 = 6$, $k_i \geq 4$, $i = 3, \dots, r$, let

$$\prod_{i=3}^{r} x_i(x_i + 1) \cdots (x_i + k_i - 1) = Aw^2.$$

Choose $x_i \in \mathbb{Z}^+$, $k_i \geq 4$, $i = 3, \dots, r$ such that $x_i(x_i + 1) \cdots (x_i + k_i - 1)$ are disjoint, A is not a perfect square, and the Pell equation $U^2 - AV^2 = 17$ has a positive integer solution (U_0, V_0) satisfying the condition

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2.$$

By the transformation

$$x_1 = \frac{u(u + 5)}{2}, \quad x_2 = u,$$

(1.1) leads to

$$A(u^2 + 5u + 2)w^2\Big(\frac{u(u + 1)(u + 2)(u + 3)(u + 4)(u + 5)}{4}\Big)^2 = y^2.$$

Let $u^2 + 5u + 2 = Av^2$, then $U^2 - AV^2 = 17$, where $U = 2u + 5$, $V = 2v$.

The rest of the proof is similar as Theorem 1.3, so we omit it. □

EXAMPLE 2.4. For $r = 4$, $d = 1$, $k_1 = 4$, $k_2 = 6$, $k_3 = 5$, $k_4 = 7$. Set $x_3 = 12$, $x_4 = 1$, then

$$12 \cdot (12+1) \cdot (12+2) \cdot (12+3) \cdot (12+4) \cdot (1+5)$$
$$\cdot 1 \cdot (1+1) \cdot (1+2) \cdot (1+3) \cdot (1+4) \cdot (1+5) \cdot (1+6) = 26 \cdot 10080^2.$$

Note that $(U_0, V_0) = (11, 2)$ is a positive integer solution of the Pell equation $U^2 - 26V^2 = 17$ satisfying the condition

$$U_0 \equiv 1 \pmod 2, \quad V_0 \equiv 0 \pmod 2.$$

Hence, (1.1) has infinitely many positive integer solutions

$$\Big(\frac{u_n(u_n+5)}{2}, u_n, 12, 1, y_n\Big),$$

such that

$$x_1(x_1+1)(x_1+2)(x_1+3), \quad x_2(x_2+1)(x_2+2)(x_2+3)(x_2+4)(x_2+5),$$
$$x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4),$$
$$x_4(x_4+1)(x_4+2)(x_4+3)(x_4+4)(x_4+5)(x_4+6)$$

are disjoint, where

$$\begin{cases} u_n = 102u_{n-1} - u_{n-2} + 250, & u_0 = 3,\ u_1 = 538,\ u_2 = 55123; \\ v_n = 102v_{n-1} - v_{n-2}, & v_0 = 1,\ v_1 = 106,\ v_2 = 10811, \end{cases}$$

$$y_n = 65520u_n(u_n+1)(u_n+2)(u_n+3)(u_n+4)(u_n+5)v_n,$$

and $n \geq 1$.

PROOF OF THEOREM 1.5. For even number $d \geq 2$, let $z = 2x + 3d$ and $w = 2y - d$. From (1.2), we have

$$x(x+d)(2x+3d)(2x+5d)(y+2d)(y+3d)(2y-d)(2y+d) \times 16y^2(y+d)^2(x+2d)^2(x+3d)^2 = t^2. \tag{2.3}$$

As in Theorem 1.1, if we let

$$x(x+5d) = \frac{3}{5}(y+3d)(2y-d), \tag{2.4}$$

then

$$(x+d)(2x+3d) = \frac{3}{5}(y+2d)(2y+d),$$

and (2.3) has positive integer solutions.

(2.4) is equivalent to the Pell equation

$$X^2 - 15Y^2 = -110d^2,$$

where $X = 20x + 25d$, $Y = 4y + 5d$. An infinity of positive integer solutions of $X^2 - 15Y^2 = -110d^2$ are given by

$$X_n + Y_n\sqrt{15} = \left(5d + 3d\sqrt{15}\right)\left(4 + \sqrt{15}\right)^n, \quad n \geq 0.$$

Thus,

$$\begin{cases} X_n = 8X_{n-1} - X_{n-2}, & X_0 = 5d,\ X_1 = 65d,\ X_2 = 515d; \\ Y_n = 8Y_{n-1} - Y_{n-2}, & Y_0 = 3d,\ Y_1 = 17d,\ Y_2 = 133d. \end{cases}$$

From

$$x = \frac{X - 25d}{20}, \quad y = \frac{Y - 5d}{4},$$

we have

$$\begin{cases} x_n = 8x_{n-1} - x_{n-2} + \frac{15d}{2}, & x_0 = -d,\ x_1 = 2d,\ x_2 = \frac{49d}{2}; \\ y_n = 8y_{n-1} - y_{n-2} + \frac{15d}{2}, & y_0 = -\frac{d}{2}\ , y_1 = 3d,\ y_2 = 32d. \end{cases}$$

For even number $d \geq 2$, we have

$$x_n \in \mathbb{Z}^+, \quad y_n \in \mathbb{Z}^+, \quad n \geq 1.$$

It is easy to prove that

$$d \nmid x_{4n+2}, \quad d \nmid x_{4n+3}, \quad n \geq 0,$$

and

$$d \nmid y_{4n+3}, \quad d \nmid y_{4n+4}, \quad n \geq 0.$$

We only consider the $4n+2$-th term of x_n and y_n, then

$$z_{4n+2} = 2x_{4n+2} + 3d \in \mathbb{Z}^+, \quad w_{4n+2} = 2y_{4n+2} - d \in \mathbb{Z}^+, \quad n \geq 0.$$

Thus, we have

$$t_{4n+2} = 4y_{4n+2}(y_{4n+2} + d)(y_{4n+2} + 3d)(2y_{4n+2} - d)$$
$$\times (x_{4n+2} + d)(x_{4n+2} + 2d)(x_{4n+2} + 3d)(2x_{4n+2} + 3d) \in \mathbb{Z}^+, \quad n \geq 0.$$

From the recurrence relations of x_n and y_n, we can obtain

$$x_{4n+2} + 3d < y_{4n+2},\ y_{4n+2} + 3d < 2x_{4n+2} + 3d,\ 2x_{4n+2} + 6d < 2y_{4n+2} - d,$$

for $n \geq 0$.

Therefore, for even number $d \geq 2$, (1.2) has infinitely many nontrivial positive integer solutions

$$(x_{4n+2}, y_{4n+2}, 2x_{4n+2} + 3d, 2y_{4n+2} - d, t_{4n+2}),$$

where $n \geq 0$. □

EXAMPLE 2.5. For $d = 2$, (1.2) has infinitely many nontrivial positive integer solutions

$$(x_{4n+2}, y_{4n+2}, 2x_{4n+2} + 6, 2y_{4n+2} - 2, t_{4n+2}),$$

such that $2 \nmid x_{4n+2}$ and $x_i(x_i + 2)(x_i + 4)(x_i + 6)$, $i = 1, \ldots, 4$ are disjoint, where

$$\begin{cases} x_n = 8x_{n-1} - x_{n-2} + 15, & x_0 = -2,\ x_1 = 4,\ x_2 = 49; \\ y_n = 8y_{n-1} - y_{n-2} + 15, & y_0 = -1,\ y_1 = 6,\ y_2 = 64, \end{cases}$$

$$\begin{aligned} t_{4n+2} &= 4y_{4n+2}(y_{4n+2} + 2)(y_{4n+2} + 6)(2y_{4n+2} - 2) \\ &\times (x_{4n+2} + 2)(x_{4n+2} + 4)(x_{4n+2} + 6)(2x_{4n+2} + 6), \end{aligned}$$

and $n \geq 0$.

PROOF OF THEOREM 1.6. For positive integer d with $3 \mid d$, let $z = 3x + 3d$ and $w = 2y + 4d$. From Eq. (1.2), we have

$$\begin{aligned} x(x + 3d)(3x + 4d)(3x + 5d)y(y + d)(2y + 5d)(2y + 7d) \\ \times 36(x + d)^2(x + 2d)^2(y + 2d)^2(y + 3d)^2 = t^2. \end{aligned} \tag{2.5}$$

As in Theorem 1.1, take

$$x(x + 3d) = \frac{4}{9}y(2y + 7d), \tag{2.6}$$

then

$$(3x + 4d)(3x + 5d) = 4(y + d)(2y + 5d),$$

and (2.5) has positive integer solutions.

(2.6) is equivalent to the Pell equation

$$X^2 - 2Y^2 = -17d^2,$$

where $X = 6x + 9d$, $Y = 4y + 7d$. An infinity of positive integer solutions of $X^2 - 2Y^2 = -17d^2$ is given by

$$X_n + Y_n\sqrt{2} = \left(d + 3d\sqrt{2}\right)\left(3 + 2\sqrt{2}\right)^n, \quad n \geq 0.$$

Thus,

$$\begin{cases} X_n = 6X_{n-1} - X_{n-2}, & X_0 = d,\ X_1 = 15d,\ X_2 = 89d; \\ Y_n = 6Y_{n-1} - Y_{n-2}, & Y_0 = 3d,\ Y_1 = 11d,\ Y_2 = 63d. \end{cases}$$

From

$$x = \frac{X - 9d}{6}, \quad y = \frac{Y - 7d}{4},$$

we have

$$\begin{cases} x_n = 6x_{n-1} - x_{n-2} + 6d, & x_0 = -\frac{4d}{3},\ x_1 = d,\ x_2 = \frac{40d}{3}; \\ y_n = 6y_{n-1} - y_{n-2} + 7d, & y_0 = -d,\ y_1 = d,\ y_2 = 14d. \end{cases}$$

For positive integer d with $3 \mid d$, we have

$$x_n \in \mathbb{Z}^+, \quad y_n \in \mathbb{Z}^+,\ n \geq 1.$$

It is easy to prove that $d \nmid x_{2n}$, $n \geq 1$. Then

$$z_{2n} = 2x_{2n} + 3d \in \mathbb{Z}^+, \quad w_{2n} = 2y_{2n} + 4d \in \mathbb{Z}^+, \quad n \geq 1.$$

Thus, we have

$$t_{2n} = 18x_{2n}(x_{2n} + d)(x_{2n} + 2d)(x_{2n} + 3d)$$
$$\times (y_{2n} + d)(y_{2n} + 2d)(y_{2n} + 3d)(2y_{2n} + 5d) \in \mathbb{Z}^+,\ n \geq 1.$$

From the recurrence relations of x_n and y_n, we can check the following inequalities

$$x_{2n} + 3d < y_{2n}, \quad y_{2n} + 3d < 2y_{2n} + 4d, \quad 2y_{2n} + 7d < 3x_{2n} + 3d,$$

for $n \geq 2$.

Therefore, for positive integer d with $3 \mid d$, (1.2) has infinitely many nontrivial positive integer solutions

$$(x_{2n}, y_{2n}, 3x_{2n} + 3d, 2y_{2n} + 4d, t_{2n}),$$

where $n \geq 2$. □

Example 2.6. For $d = 3$, (1.2) has infinitely many nontrivial positive integer solutions

$$(x_{2n}, y_{2n}, 3x_{2n} + 9, 2y_{2n} + 12, t_{2n}),$$

such that $3 \nmid x_{2n}$ and $x_i(x_i+3)(x_i+6)(x_i+9)$, $i = 1, \dots, 4$ are disjoint, where

$$\begin{cases} x_n = 6x_{n-1} - x_{n-2} + 18, & x_0 = -4,\ x_1 = 3,\ x_2 = 40; \\ y_n = 6y_{n-1} - y_{n-2} + 21, & y_0 = -3,\ y_1 = 3,\ y_2 = 42, \end{cases}$$

$$t_{2n} = 18x_{2n}(x_{2n}+3)(x_{2n}+6)(x_{2n}+9) \times (y_{2n}+3)(y_{2n}+6)(y_{2n}+9)(2y_{2n}+15),$$

and $n \ge 2$.

Proof of Theorem 1.8. Take

$$z = 2x + 4d, \quad w = \frac{y - 2d}{2}.$$

From (1.3), we have

$$\text{(2.7)} \qquad x(x+d)(2x+5d)(2x+7d)(y-2d)(y+d)(y+3d)(y+6d) \times \frac{y^2(y+2d)^2(y+4d)^2(x+2d)^2(x+3d)^2(x+4d)^2}{4} = t^2.$$

As Theorem 1.1, if we put

$$\text{(2.8)} \qquad x(2x+7d) = \frac{1}{3}(y-d)(y+6d),$$

then

$$(x+d)(2x+5d) = \frac{1}{3}(y+d)(y+3d),$$

and (2.7) has positive integer solutions.

(2.8) is equivalent to the Pell equation

$$X^2 - 6Y^2 = 57d^2,$$

where $X = 12x + 21d$, $Y = 2y + 4d$. An infinity of positive integer solutions of $X^2 - 6Y^2 = 57d^2$ are given by

$$X_n + Y_n\sqrt{6} = \left(9d + 2d\sqrt{6}\right)\left(5 + 2\sqrt{6}\right)^n, \quad n \ge 0.$$

Thus,

$$\begin{cases} X_n = 10X_{n-1} - X_{n-2}, & X_0 = 9d,\ X_1 = 69d,\ X_2 = 681d; \\ Y_n = 10Y_{n-1} - Y_{n-2}, & Y_0 = 2d,\ Y_1 = 28d,\ Y_2 = 278d. \end{cases}$$

From

$$x = \frac{X - 21d}{12}, \quad y = \frac{Y - 4d}{2},$$

we have

$$\begin{cases} x_n = 10x_{n-1} - x_{n-2} + 14d, & x_0 = -d,\ x_1 = 4d,\ x_2 = 55d; \\ y_n = 10y_{n-1} - y_{n-2} + 16d, & y_0 = -d,\ y_1 = 12d,\ y_2 = 137d. \end{cases}$$

Then

$$z_n = 2x_n + 4d, \quad w_n = \frac{y_n - 2d}{2}.$$

For even number $d \geq 2$, the x_n, y_n, z_n and w_n are positive integers for $n \geq 1$. It is easy to prove that $d \nmid w_{2n}$, $n \geq 1$. Thus, we have

$$t_{2n} = \frac{1}{2} y_{2n}(y_{2n} - d)(y_{2n} + 2d)(y_{2n} + 4d)(y_{2n} + 6d)$$
$$\times (x_{2n} + d)(x_{2n} + 2d)(x_{2n} + 3d)(x_{2n} + 4d)(2x_{2n} + 5d) \in \mathbb{Z}^+, \quad n \geq 1.$$

From the recurrence relations of x_n and y_n, we can obtain

$$x_{2n} + 4d < \frac{y_{2n} - 2d}{2}, \quad \frac{y_{2n} - 2d}{2} + 4d < 2x_{2n} + 4d, \quad 2x_{2n} + 8d < y_{2n},$$

for $n \geq 1$.

Therefore, for even number $d \geq 2$, (1.3) has infinitely many nontrivial positive integer solutions

$$\left(x_{2n}, y_{2n}, 2x_{2n} + 4d, \frac{y_{2n} - 2d}{2}, t_{2n}\right),$$

where $n \geq 1$. $\square$

Example 2.7. For $d = 2$, (1.3) has infinitely many nontrivial positive integer solutions

$$\left(x_{2n}, y_{2n}, 2x_{2n} + 8, \frac{y_{2n} - 4}{2}, t_{2n}\right),$$

such that $2 \nmid w_{2n} = \frac{y_{2n}-4}{2}$ and $x_i(x_i + 2)(x_i + 4)(x_i + 6)(x_i + 8)$, $i = 1, \ldots, 4$ are disjoint, where

$$\begin{cases} x_n = 10x_{n-1} - x_{n-2} + 28, & x_0 = -2,\ x_1 = 8,\ x_2 = 110; \\ y_n = 10y_{n-1} - y_{n-2} + 32, & y_0 = -2,\ y_1 = 24,\ y_2 = 274, \end{cases}$$

$$t_{2n} = \frac{1}{2} y_{2n}(y_{2n} - 2)(y_{2n} + 4)(y_{2n} + 8)(y_{2n} + 12)$$
$$\times (x_{2n} + 2)(x_{2n} + 4)(x_{2n} + 6)(x_{2n} + 8)(2x_{2n} + 10),$$

and $n \geq 1$.

3. Some related questions

As we know, for $r \geq 3$, $d = 1$, $k_1 = 3$, $k_2 = 3, 4, 5$, $k_i \geq 3$, $i = 3, \dots, r$, (1.1) has infinitely many positive integer solutions. Hence, we have

QUESTION 3.1. *For* $r \geq 3$, $d = 1$, $k_1 = 3$, $k_i \geq 6$, $i = 2, \dots, r$, *does* (1.1) *have infinitely many positive integer solutions?*

Bauer and Bennett [1] conjectured that if $r = 2$, $d = 1$, $k_1 \geq 4$, then (1.1) has at most finitely many positive integer solutions. However, the case is different for $r = 3$, $d = 1$, $k_1 \geq 4$. In Theorem 1.2, we have got infinitely many positive integer solutions of (1.1) for $r = 3$, $d = 1$, $k_1 = k_2 = 4$, $k_3 = 5, 6, 7, 8, 9$. For $r \geq 3$, $d = 1$, $k_1 \geq 4$, there are some simple cases which we cannot give infinitely many positive integer solutions, such as

QUESTION 3.2. *For* $r = 3$, $d = 1$, $k_1 = k_2 = 4$, $k_3 \geq 10$, *are there infinitely many positive integer solutions of* (1.1)*?*

QUESTION 3.3. *For* $r = 3$, $d = 1$, $k_1 = 4$, $k_2 = 5$, $k_3 \geq 5$, *does* (1.1) *have infinitely many positive integer solutions?*

QUESTION 3.4. *For* $r \geq 3$, $d = 1$, $k_1 = 4$, $k_i \geq 5$, $i = 2, \dots, r$, *except* $k_2 = 6$, *are there infinitely many positive integer solutions of* (1.1)*?*

For $r \geq 3$, $d = 1$, $k_1 \geq 5$, or $r \geq 3$, $d \geq 2$, $k_1 \geq 3$, we can study similar questions for (1.1).

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