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THE ARCSINE LAW ON DIVISORS IN ARITHMETIC PROGRESSIONS MODULO PRIME POWERS

B. FENG[1,†] and J. WU[2,*]

[1]School of Mathematics and Statistics, Yangtze Normal University, Fuling, Chongqing 408100, China
e-mail: binfengcq@163.com

[2]CNRS LAMA 8050 Université Paris-Est Créteil, 94010 Créteil cedex, France
e-mail: jie.wu@math.cnrs.fr

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Abstract. Let $x \to \infty$ be a parameter. Feng [5] proved that the Deshouillers–Dress–Tenenbaum's arcsine law on divisors of the integers less than x also holds in arithmetic progressions for "non-exceptional moduli" $q \leqslant \exp\{(\frac{1}{4} - \varepsilon)(\log_2 x)^2\}$, where ε is an arbitrarily small positive number. We show that in the case of a prime-power modulus ($q := \mathfrak{p}^{\varpi}$ with $\mathfrak{p}$ a fixed odd prime and $\varpi \in \mathbb{N}$) the arcsine law on divisors holds in arithmetic progressions for $q \le x^{15/52-\varepsilon}$.

1. Introduction

For each positive integer n, denote by $\tau(n)$ the number of divisors of n and define the random variable D_n to take the value $(\log d)/\log n$, as d runs through the set of the divisors of n, with the uniform probability $1/\tau(n)$. The distribution function F_n of D_n is given by

$$F_n(t) := \mathrm{Prob}(D_n \leqslant t) = \frac{1}{\tau(n)} \sum_{d|n,\, d\leqslant n^t} 1 \quad (0 \leqslant t \leqslant 1). \tag{1.1}$$

Deshouillers, Dress and Tenenbaum ([4] or [10, Theorem II.6.7]) proved that the Cesàro means of F_n converge uniformly to the arcsine law. More pre-

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cisely, the asymptotic formula

$$\frac{1}{x}\sum_{n\leqslant x} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\Big(\frac{1}{\sqrt{\log x}}\Big) \tag{1.2}$$

holds uniformly for $x \geqslant 2$ and $0 \leqslant t \leqslant 1$ and the error term in (1.2) is optimal. Various cases of (1.2) have been investigated by different authors. In particular, Cui and Wu [3] and Cui, Lü and Wu [2] considered generalizations of (1.2) to the short interval case; and Feng and Wu [6] showed that the average distribution of divisors over integers representable as the sum of two squares converges to the beta law. Based on Cui and Wu's method [3], Feng [5] studied the analogue of (1.2) for arithmetic progressions. His result can be stated as follows: Let a and q be integers such that $(a, q) = 1$, and suppose that q is not an "exceptional modulus". Then for any $\varepsilon \in (0, \frac{1}{4})$ we have

$$\frac{1}{(x/q)}\sum_{\substack{n\leqslant x\\ n\equiv a(\mathrm{mod}\, q)}} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O_\varepsilon\bigg(\frac{\mathrm{e}^{\sqrt{\log q}}}{\sqrt{\log x}}\bigg) \tag{1.3}$$

uniformly for $0 \leqslant t \leqslant 1$, $x \geqslant 2$ and $q \leqslant \exp\{(\frac{1}{4} - \varepsilon)(\log_2 x)^2\}$, where $\log_2 := \log\log$.

The aim of this paper is to improve the result above in the case of prime power modulus. Our result is as follows.

THEOREM 1.1. *Let $q := \mathfrak{p}^{\varpi}$ with $\mathfrak{p}$ an odd prime and $\varpi \in \mathbb{N}$. Then for any $\varepsilon > 0$, we have*

$$\frac{1}{(x/q)}\sum_{\substack{n\leqslant x\\ n\equiv a(\mathrm{mod}\, q)}} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O_{\mathfrak{p},\varepsilon}\Big(\frac{1}{\sqrt{\log x}}\Big) \tag{1.4}$$

uniformly for $0 \leqslant t \leqslant 1$, $x \geqslant 2$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{N}$ such that $(a, q) = 1$, where the implied constant depends only on $\mathfrak{p}$ and ε.

Our improvement is double. Firstly, with $q = \mathfrak{p}^{\varpi}$ any Siegel zero must occur for $L(s, \chi)$ where χ is a real character modulo $\mathfrak{p}$. Since the implied constant in Theorem 1 is allowed to depend on $\mathfrak{p}$, there is no Siegel zero for the modulus $q = \mathfrak{p}^{\varpi}$. These considerations allow to remove the assumption of Siegel zero in Feng's result for $q = \mathfrak{p}^{\varpi}$ with an implied constant in the error term depending on $\mathfrak{p}$. Alternatively, this follows from Feng's result and Corollary 3.4 of Banks and Shparlinski's paper [1] (cf. Lemma 2.3 below). Secondly the domain of q is extended significantly.

2. Preliminary

Our first lemma is an effective Perron formula (cf. [10, Corollary II.2.2.1]).

LEMMA 2.1. *Let $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_a. Suppose that there exist some real number $\alpha > 0$ and a non-decreasing function $B(n)$ such that*
(a) $\sum_{n=1}^{\infty} |a_n| n^{-\varsigma} \ll (\varsigma - \sigma_a)^{-\alpha}$ $(\varsigma > \sigma_a)$,
(b) $|a_n| \leqslant B(n)$ $(n \geqslant 1)$.
Then for $x \geqslant 2$, $T \geqslant 2$, $\sigma \leqslant \sigma_a$ and $\kappa := \sigma_a - \sigma + 1/\log x$, we have

$$\sum_{n \leqslant x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa+iT}^{\kappa-iT} F(s+w) x^w \frac{\mathrm{d}w}{w}$$
$$+ O\Big(x^{\sigma_a - \sigma} \frac{(\log x)^{\alpha}}{T} + \frac{B(2x)}{x^{\sigma}}\Big(1 + x\frac{\log T}{T}\Big)\Big).$$

LEMMA 2.2. *Let $q > 2$ be an integer.*
(i) *If χ is a Dirichlet character modulo q, then we have*

$$L(\sigma + i\tau, \chi) \ll q^{1-\sigma} (|\tau| + 1)^{1/6} \log(|\tau| + 1).$$

(ii) *If χ is a non-principal Dirichlet character modulo q, then for $0 < \varepsilon < \frac{1}{2}$, $\varepsilon \leqslant \sigma \leqslant 1$, $|\tau| + 2 \leqslant T$, we have*

$$L(\sigma + i\tau, \chi) \ll_{\varepsilon} (q^{1/2} T)^{1-\sigma+\varepsilon}.$$

PROOF. See [9, p. 485, Theorem 1] and [11, Exercise 241]. □

The next lemma is due to Banks and Shparlinski [1, Corollary 3.4.] and plays a key role in the proof of Theorem 1.1.

LEMMA 2.3. *Let $q = \mathfrak{p}^{\varpi}$ with $\mathfrak{p}$ an odd prime and $\varpi \in \mathbb{N}$. For each constant $A > 0$, there is a constant $c_0 = c_0(A, \mathfrak{p}) > 0$ depending only on A and $\mathfrak{p}$ such that for any character χ modulo q, the Dirichlet L-function has no zero in the region*

$$\sigma > 1 - \frac{c_0}{(\log q)^{2/3} (\log_2 q)^{1/3}} \quad \textit{and} \quad |\tau| \le q^A. \tag{2.1}$$

The following lemma is a key for the proof of Theorem 1.1.

LEMMA 2.4. *Let $q := \mathfrak{p}^{\varpi}$ with $\mathfrak{p}$ an odd prime and $\varpi \in \mathbb{N}$ and let χ_0 be the principal character to the modulus q. Then we have*

$$\sum_{n \leqslant x} \frac{\chi_0(n)}{\tau(nd)} = \frac{hx}{\sqrt{\pi \log x}} \Big\{ g(d) + O\Big(\frac{(3/4)^{\omega(d)}}{\log x}\Big) \Big\} \tag{2.2}$$

uniformly for $x \geqslant 2$, $1 \leqslant d \leqslant x$ and $\varpi \geqslant 1$, where the implied constant is absolute, $\omega(d)$ is the number of all distinct prime factors of d,

$$h := \sqrt{1-\mathfrak{p}^{-1}} \prod_{(p,\mathfrak{p})=1} \sqrt{1-p^{-1}}\, \frac{\log(1-p^{-1})}{-p^{-1}} \tag{2.3}$$

and

$$g(d) := \prod_{p^\alpha \| d} \biggl(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-1})^j}{j+\alpha+1} \biggr) \frac{-\chi_0(p)p^{-1}}{\log(1-\chi_0(p)p^{-1})}. \tag{2.4}$$

Proof. As usual, denote by $v_p(n)$ the p-adic valuation of n. By using the formula

$$\tau(dn) = \prod_p (v_p(n) + v_p(d) + 1), \tag{2.5}$$

we write for $\operatorname{Re} s > 1$

$$\begin{aligned} f_d(s,\chi_0) &:= \sum_{n=1}^{\infty} \frac{\chi_0(n)}{\tau(dn)} n^{-s} = \prod_p \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+v_p(d)+1} \\ &= \prod_{(p,d)=1} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+1} \times \prod_{p^\alpha \| d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+\alpha+1} \\ &= \prod_p \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+1} \times \prod_{p^\alpha \| d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+\alpha+1} \biggl(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+1} \biggr)^{-1} \\ &= L(s,\chi_0)^{1/2} G_d(s,\chi_0), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} G_d(s,\chi_0) &:= \prod_p \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+1} \sqrt{1-\chi_0(p)/p^s} \\ &\times \prod_{p^\alpha \| d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+\alpha+1} \biggl(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+1} \biggr)^{-1} \end{aligned}$$

is a Dirichlet series that converges absolutely for $\operatorname{Re} s > \frac{1}{2}$.

We easily see that

$$\prod_{p^\alpha \| d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+\alpha+1} \biggl(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j+1} \biggr)^{-1} = \frac{1}{\alpha+1} + O\Bigl(\frac{1}{\sqrt{p}}\Bigr).$$

for $\operatorname{Re} s \geqslant \frac{1}{2}$, where the implied constant is absolute. This implies that for any $\varepsilon > 0$,

$$G_d(s,\chi_0) \ll \prod_{p^\alpha \| d} \Big\{\frac{1}{\alpha+1} + O\Big(\frac{1}{\sqrt{p}}\Big)\Big\} \leqslant C_\varepsilon \Big(\frac{3}{4}\Big)^{\omega(d)} \tag{2.7}$$

for $\operatorname{Re} s \geqslant \frac{1}{2} + \varepsilon$, where $C_\varepsilon > 0$ is a constant depending on ε only.

We can apply Lemma 2.1 with the choice of parameters $\sigma_a = 1$, $B(n) = 1$, $\alpha = \frac{1}{2}$ and $\sigma = 0$ to write

$$\sum_{n \leqslant x} \frac{\chi_0(n)}{\tau(nd)} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f_d(s,\chi_0) \frac{x^s}{s} \,\mathrm{d}s + O_\varepsilon\Big(\frac{x \log x}{T}\Big),$$

where $b = 1 + 2/\log x$ and $100 \leqslant T \leqslant x$ such that $\zeta(\sigma + iT) \neq 0$ for $0 < \sigma < 1$.

Let $\mathscr{M}_T$ be the boundary of the modified rectangle with vertices $(\frac{1}{2} + \varepsilon) \pm iT$ and $b \pm iT$ as follows (see Fig. 1):

- $\varepsilon > 0$ is a small constant chosen such that $\zeta(\frac{1}{2} + \varepsilon + i\gamma) \neq 0$ for $|\gamma| < T$;
- the zeros of $\zeta(s)$ of the form $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2} + \varepsilon$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + i\gamma$;
- the pole of $\zeta(s)$ at the points $s = 1$ is avoided by the truncated Hankel contour Γ (its upper part is made up of an arc surrounding the point $s = 1$ with radius $r := 1/\log x$ and a line segment joining $1 - r$ to $(\frac{1}{2} + \varepsilon)$.

Clearly the function $f_d(s,\chi_0)$ is analytic inside $\mathscr{M}_T$. By the residue theorem, we can write

$$\sum_{n \leqslant x} \frac{\chi_0(n)}{\tau(nd)} = I + \frac{1}{2\pi i}\Big(I_1 + \cdots + I_4 + \sum_{\beta > \frac{1}{2}+\varepsilon,\, |\gamma| < T} I_\rho\Big) + O_\varepsilon\Big(\frac{x \log x}{T}\Big), \tag{2.8}$$

where

$$I := \frac{1}{2\pi i} \int_\Gamma f_d(s,\chi_0) \frac{x^s}{s} \,\mathrm{d}s,\ I_\rho := \int_{\Gamma_\rho} f_d(s,\chi_0) \frac{x^s}{s} \,\mathrm{d}s,\ I_j := \int_{\mathscr{L}_j} f_d(s,\chi_0) \frac{x^s}{s} \,\mathrm{d}s.$$

A. *Evaluation of I.* Let $0 < c < \frac{1}{10}$ be a small constant. Since

$$G_d(s,\chi_0)((s-1)\zeta(s))^{1/2}(1 - \mathfrak{p}^{-s})^{1/2}$$

is holomorphic and $O((3/4)^{\omega(d)})$ in the disc $|s-1| \leqslant c$ thanks to (2.7), the Taylor formula allows us to write

$$\begin{aligned} &G_d(s,\chi_0)((s-1)\zeta(s))^{1/2}(1 - \mathfrak{p}^{-s})^{1/2} \\ &= G_d(1,\chi_0)(1 - \mathfrak{p}^{-1})^{1/2} + O((3/4)^{\omega(d)}|s-1|) \end{aligned}$$

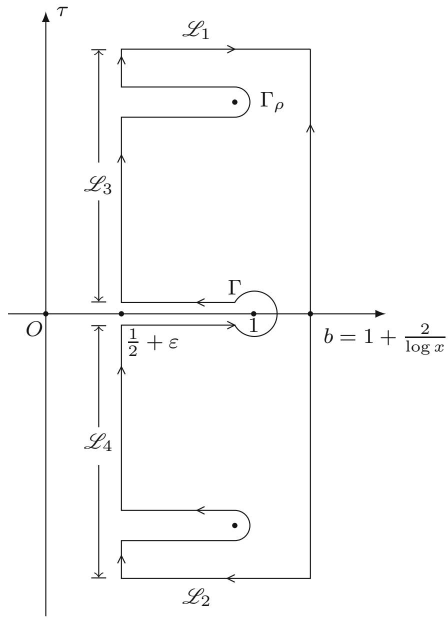

Fig. 1: Contour $\mathscr{M}_T$

for $|s-1| \leqslant \frac{1}{2}c$. In view of

$$L(s,\chi_0) = \zeta(s)(1-\mathfrak{p}^{-s}) \quad \text{and} \quad G_d(1,\chi_0)(1-\mathfrak{p}^{-1})^{1/2} = hg(d),$$

it follows that

$$f_d(s,\chi_0) = hg(d)(s-1)^{-1/2} + O((3/4)^{\omega(d)}|s-1|^{1/2})$$

for $|s-1| \leqslant \frac{1}{2}c$. So we have

$$I = hg(d)M(x) + O((3/4)^{\omega(d)}E_0(x)), \tag{2.9}$$

where

$$M(x) := \frac{1}{2\pi i}\int_\Gamma (s-1)^{-1/2}x^s \,\mathrm{d}s, \quad E_0(x) := \int_\Gamma |(s-1)^{1/2}x^s||\mathrm{d}s|.$$

Firstly we evaluate $M(x)$. By using [10, Corollary II.5.2.1], we have

$$M(x) := \frac{x}{\sqrt{\log x}}\Big\{\frac{1}{\Gamma(\frac{1}{2})} + O(x^{-c/2})\Big\}. \tag{2.10}$$

Next we deduce that

$$E_0(x) \ll \int_{1/2+\varepsilon}^{1-1/\log x} (1-\sigma)^{1/2} x^\sigma \,\mathrm{d}\sigma + \frac{x}{(\log x)^{3/2}} \tag{2.11}$$
$$\ll \frac{x}{(\log x)^{3/2}} \Big(\int_1^\infty t^{1/2} \mathrm{e}^{-t} \,\mathrm{d}t + 1 \Big) \ll \frac{x}{(\log x)^{3/2}}.$$

Inserting (2.10) and (2.11) into (2.9) and noticing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we find that

$$I = \frac{x}{\sqrt{\pi \log x}} \Big\{ hg(d) + O_\varepsilon\Big(\frac{(3/4)^{\omega(d)}}{\log x} \Big) \Big\}. \tag{2.12}$$

B. *Estimations of* I_1 *and* I_2. It is well known that (cf. [10, Corollary II.3.5.2])

$$|\zeta(\sigma + i\tau)| \ll |\tau|^{(1-\sigma)/3} \log|\tau| \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1 + \log^{-1}|\tau|,\ |\tau| \geqslant 3). \tag{2.13}$$

Noticing that $q := \mathfrak{p}^\varpi$, it follows that

$$L(s, \chi_0) = \zeta(s)(1 - \mathfrak{p}^{-s}) \ll |\tau|^{(1-\sigma)/3} \log|\tau| \tag{2.14}$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \log^{-1}(|\tau| + 3)$ and $|\tau| \geqslant 3$. From (2.6), (2.7) and (2.14), we derive that

$$|I_1| + |I_2| \ll_\varepsilon (3/4)^{\omega(d)} \int_{1/2+\varepsilon}^{1+2/\log x} T^{(1-\sigma)/6} (\log T) \frac{x^\sigma}{T} \,\mathrm{d}\sigma \tag{2.15}$$
$$\ll_\varepsilon (3/4)^{\omega(d)} \frac{x}{T} \log T.$$

C. *Estimations of* I_3 *and* I_4. As before, (2.6) and (2.14) allow us to deduce

$$|I_3| + |I_4| \ll_\varepsilon (3/4)^{\omega(d)} \int_1^T (|\tau|+1)^{1/12} \log(|\tau|+1) \frac{x^{1/2+\varepsilon}}{|(\frac{1}{2}+\varepsilon) + i\tau)|} \,\mathrm{d}\tau \tag{2.16}$$
$$\ll_\varepsilon (3/4)^{\omega(d)} x^{1/2+\varepsilon} \int_1^T (\tau+1)^{-1+1/12} \,\mathrm{d}\tau \ll_\varepsilon (3/4)^{\omega(d)} x^{1/2+\varepsilon} T^{1/12}.$$

D. *Estimation of* I_ρ. With the help of (2.14) and (2.7), we can derive that for $s = \sigma + i\gamma$ with

$$I_\rho \ll_\varepsilon (3/4)^{\omega(d)} \int_{1/2+\varepsilon}^{\beta} |\gamma|^{(1-\sigma)/6} (\log|\gamma|)^{1/2} \frac{x^\sigma}{|\sigma + i\gamma|} \,\mathrm{d}\sigma. \tag{2.17}$$

Denote by $N(\alpha, T)$ the number of zeros of $\zeta(s)$ in the region $\operatorname{Re} s \geqslant \alpha$ and $|\operatorname{Im} s| \leqslant T$ and define $\sigma(\tau) := c \log^{-2/3}(|\tau|+3)\log_2^{-1/3}(|\tau|+3)$ ($c>0$ absolute constant). Summing (2.17) over $|\gamma|<T$ and interchanging the summations and noticing that $\beta < 1-\sigma(T_1)$ (the Korobov–Vinogradov zero free region), we have

$$\sum_{\beta>\frac12+\varepsilon,\,|\gamma|<T} |I_\rho| \ll (3/4)^{\omega(d)}(\log T) \max_{T_1\leqslant T} \sum_{\beta>\frac12+\varepsilon,\,T_1/2<|\gamma|<T_1} |I_\rho|$$

$$\ll_\varepsilon (3/4)^{\omega(d)}(\log T) \max_{T_1\leqslant T} \int_{1/2+\varepsilon}^{1-\sigma(T_1)} T_1^{(1-\sigma)/6}\cdot \frac{x^\sigma}{T_1}\cdot N(\sigma,T_1)\,\mathrm{d}\sigma.$$

According to [7], it is well known that

$$N(\sigma,T) \ll T^{(12/5)(1-\sigma)}(\log T)^{44} \tag{2.18}$$

for $\frac12+\varepsilon \leqslant \sigma \leqslant 1$, and $T \geqslant 2$. Thus

$$\begin{aligned}\sum_{\beta>\frac12+\varepsilon,\,|\gamma|<T} |I_\rho| &\ll (3/4)^{\omega(d)}(\log T)^{45} \max_{T_1\leqslant T} \int_{1/2+\varepsilon}^{1-\sigma(T_1)} T_1^{(1-\sigma)/6}\frac{x^\sigma}{T_1}T_1^{(12/5)(1-\sigma)}\,\mathrm{d}\sigma \\ &\ll x(\log T)^{45}\max_{T_1\leqslant T}\int_{1/2+\varepsilon}^{1-\sigma(T_1)}\Big(\frac{T_1^{17/30}}{x}\Big)^{1-\sigma}\mathrm{d}\sigma \\ &\ll x(\log T)^{45}\max_{T_1\leqslant T}\Big(\frac{T_1^{17/30}}{x}\Big)^{\sigma(T_1)} \ll x(\log T)^{45}\Big(\frac{T^{17/30}}{x}\Big)^{\sigma(T)}.\end{aligned}\tag{2.19}$$

Inserting (2.12), (2.15), (2.16) and (2.19) into (2.8), we find that

$$\sum_{n\leqslant x}\frac{\chi_0(n)}{\tau(nd)} = \frac{x}{\sqrt{\pi\log x}}\Big\{hg(d) + O_\varepsilon\Big(\frac{(3/4)^{\omega(d)}}{\log x}\Big)\Big\} + O_\varepsilon(R_{d,T}(x)),$$

where

$$\begin{aligned}R_{d,T}(x) := \Big(\frac34\Big)^{\omega(d)}\Big\{&\frac{x}{T}\log T + x^{1/2+\varepsilon}T^{1/12} \\ &+ x(\log T)^{45}\Big(\frac{T^{17/30}}{x}\Big)^{\sigma(T)}\Big\} + \frac{x\log x}{T}.\end{aligned}$$

Taking $T = x$ and $\varepsilon = 10^{-3}$ and noticing that

$$\omega(d) \ll (\log x)/\log_2 x \quad \text{for } d \le x,$$

it is easy to verify that

$$R_{d,T}(x) \ll (3/4)^{\omega(d)} x/(\log x)^{3/2} \quad \text{for } d \le x.$$

This completes the proof. $\square$

LEMMA 2.5. *Under the notation in Lemma* 2.4, *we have*

$$h \sum_{d \leqslant x} \chi_0(d) g(d) = \frac{(\varphi(q)/q)x}{\sqrt{\pi \log x}} \Big\{ 1 + O\Big(\frac{1}{\log x}\Big) \Big\}, \tag{2.20}$$

where the implied constant is absolute.

PROOF. According to (2.4), it is easy to see that $g(d)$ is a multiplicative function and

$$\begin{aligned} g(p^\nu) &= \sum_{j \ge 0} \frac{(\chi_0(p)p^{-1})^j}{j+\nu+1} \bigg(\sum_{k \ge 0} \frac{(\chi_0(p)p^{-1})^k}{k+1} \bigg)^{-1} \\ &= \frac{-\chi_0(p)p^{-1}}{\log(1-\chi_0(p)p^{-1})} \sum_{j \ge 0} \frac{(\chi_0(p)p^{-1})^j}{j+\nu+1}. \end{aligned} \tag{2.21}$$

For $\sigma > 1$, we can write

$$\begin{aligned} \sum_{n \geqslant 1} \chi_0(n) g(n) n^{-s} &= L(s, \chi_0)^{1/2} \sum_{n \geqslant 1} \beta(n) n^{-s} \\ &= \zeta(s)^{1/2} (1 - \mathfrak{p}^{-s})^{1/2} \sum_{n \geqslant 1} \beta(n) n^{-s}, \end{aligned}$$

where $\beta(n)$ is a multiplicative function determined by

$$\sum_{\nu \geqslant 1} \beta(p^\nu) \xi^\nu = (1 - \chi_0(p)\xi)^{1/2} \sum_{\nu \ge 0} \chi_0(p) g(p^\nu) \xi^\nu \quad (|\xi| < 1). \tag{2.22}$$

Since $|g(p^\nu)| \leqslant 1$, the right-hand side is holomorphic for $|\xi| < 1$ and we have $\beta(p^\nu) \ll \big(\frac{3}{2}\big)^\nu$ $(\nu = 1, 2, \ldots)$. In addition,

$$\beta(p) = \chi_0(p)(g(p) - 1/2) = O(1/p).$$

These imply the absolute convergence of $\sum \beta(n) n^{-s}$ for $\sigma > \frac{1}{2}$ and $\sum \beta(n) n^{-s} \ll_\varepsilon 1$ for $\sigma \ge \frac{1}{2} + \varepsilon$.

Applying [10, Theorem II.5.3], we have

$$\sum_{n \leqslant x} \chi_0(n) g(n) = \frac{x}{\sqrt{\log x}} \Big\{ \lambda_0(\tfrac{1}{2}) + O\Big(\frac{1}{\log x}\Big) \Big\},$$

where we have

$$\lambda_0(\tfrac{1}{2}) := \frac{(1-\mathfrak{p}^{-1})^{1/2}}{\Gamma(\frac{1}{2})}\prod_p (1-\chi_0(p)p^{-1})^{1/2}\sum_{\nu\geq 0}\frac{\chi_0(p)^\nu g(p^\nu)}{p^\nu},$$

thanks to (2.21) and (2.22). In view of (2.21), it follows, with the notation $\xi = \chi_0(p)p^{-1}$,

$$\sum_{\nu\geq 0}\frac{\chi_0(p)^\nu g(p^\nu)}{p^\nu}(1-\chi_0(p)p^{-1}) = \Big(\sum_{j\geq 0}\frac{\xi^j}{j+1}\Big)^{-1}(1-\xi)\sum_{\nu\geq 0}\sum_{j\geq 0}\frac{\xi^{j+\nu}}{j+\nu+1}$$
$$= \Big(\sum_{j\geq 0}\frac{\xi^j}{j+1}\Big)^{-1}(1-\xi)\sum_{k\geq 0}\xi^k = \frac{-\chi_0(p)p^{-1}}{\log(1-\chi_0(p)p^{-1})}.$$

Thus

$$\lambda_0(\tfrac{1}{2}) = \frac{(1-\mathfrak{p}^{-1})^{1/2}}{\sqrt{\pi}}\prod_p (1-\chi_0(p)p^{-1})^{-1/2}\frac{-\chi_0(p)p^{-1}}{\log(1-\chi_0(p)p^{-1})}$$

and $h\lambda_0(\frac{1}{2}) = (1-\mathfrak{p}^{-1})/\sqrt{\pi} = (\varphi(q)/q)/\sqrt{\pi}$, which concludes the proof of (2.20). □

LEMMA 2.6. *Let $q = \mathfrak{p}^\varpi$ with $\mathfrak{p}$ an odd prime and $\varpi \in \mathbb{N}$. For any $\varepsilon > 0$, there is a positive constant $c_1(\varepsilon) > 0$ depending on ε such that we have*

$$\sum_{\chi\neq\chi_0}\overline{\chi}(a)\chi(d)\sum_{n\leqslant x}\frac{\chi(n)}{\tau(nd)} \ll x\mathrm{e}^{-c_1(\varepsilon)(\log x)^{1/3}(\log_2 x)^{-1/3}} \tag{2.23}$$

uniformly for $d \geqslant 1$, $x \geqslant 2$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a,q) = 1$.

PROOF. Since the proof is rather close to that of Lemma 2.4, we only mention the principal points. As before, by (2.5), we can write for $\sigma := \operatorname{Re} s > 1$

$$f_d(s,\chi) := \sum_{n=1}^{\infty}\chi(n)\tau(dn)^{-1}n^{-s} = L(s,\chi)^{1/2}G_d(s,\chi), \tag{2.24}$$

where

$$G_d(s,\chi) := \prod_p\sum_{j=0}^{\infty}\frac{(\chi(p)p^{-s})^j}{j+1}(1-\chi(p)p^{-s})^{1/2}$$
$$\times\prod_{p^\alpha\|d}\sum_{j=0}^{\infty}\frac{(\chi(p)p^{-s})^j}{j+\alpha+1}\Big(\sum_{j=0}^{\infty}\frac{(\chi(p)p^{-s})^j}{j+1}\Big)^{-1}$$

is a Dirichlet series that converges absolutely for $\sigma > \frac{1}{2}$ and verifies $|G_d(s,\chi)| \leqslant C_\varepsilon(\frac{3}{4})^{\omega(d)}$ for $\sigma \geqslant \frac{1}{2}+\varepsilon$ and $d \geq 1$, where ε is an arbitrarily small positive constant and $C_\varepsilon > 0$ is a constant depending only on ε.

We apply Lemma 2.1 with $\sigma_a = 1$, $B(n) = 1$, $\alpha = \frac{1}{2}$ and $\sigma = 0$ to write

$$\sum_{n\leqslant x} \frac{\chi(n)}{\tau(nd)} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f_d(s,\chi)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{x\log x}{T}\Big),$$

where $b = 1 + 2/\log x$ and $100 \leqslant T \leqslant x$ such that $L(\sigma+iT,\chi) \neq 0$ for $0 < \sigma < 1$.

Let $\mathcal{M}_T$ be the boundary of the modified rectangle with vertices $(\frac{1}{2}+\varepsilon) \pm iT$ and $b \pm iT$ as follows:

• $\varepsilon > 0$ is a small constant chosen such that $L(\frac{1}{2}+\varepsilon+i\gamma,\chi) \neq 0$ for $|\gamma| < T$;

• the zeros of $L(s,\chi)$ of the form $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + i\gamma$.

Clearly the function $f_d(s,\chi)$ is analytic inside $\mathscr{M}_T$. By the Cauchy residue theorem, we can write

$$\sum_{n\leqslant x} \frac{\chi(n)}{\tau(nd)} = I_1 + \cdots + I_4 + \sum_{\beta>\frac{1}{2}+\varepsilon,\, |\gamma|<T} I_\rho + O\Big(\frac{x\log x}{T}\Big), \tag{2.25}$$

where

$$I_j := \frac{1}{2\pi i}\int_{\mathcal{L}_j} f_d(s,\chi)\frac{x^s}{s}\,\mathrm{d}s, \quad I_\rho := \frac{1}{2\pi i}\int_{\Gamma_\rho} f_d(s,\chi)\frac{x^s}{s}\,\mathrm{d}s$$

and $\mathcal{L}_j$ and Γ_ρ are as in Fig. 1.

A. *Estimations of I_1 and I_2.* In view of (2.24) and Lemma 2.2, we have

$$\begin{aligned} |I_1| + |I_2| &\ll \int_{1/2+\varepsilon}^{1+2/\log x} (q^{1/2}T)^{\frac{1}{2}(1-\sigma)+\varepsilon} \cdot \frac{x^\sigma}{T}\,\mathrm{d}\sigma \\ &\ll \frac{x}{T}\int_{1/2+\varepsilon}^{1+2/\log x} \Big(\frac{q^{1/4}T^{1/2}}{x}\Big)^{1-\sigma}\mathrm{d}\sigma \ll \frac{x}{T}. \end{aligned} \tag{2.26}$$

B. *Estimations of I_3 and I_4.* By (2.24) and Lemma 2.2, we have

$$|I_3|+|I_4| \ll \int_1^T q^{1/4}(|\tau|+1)^{1/12}\frac{x^{1/2+\varepsilon}}{|(\frac{1}{2}+\varepsilon)+i\tau)|}\mathrm{d}\tau \ll x^{1/2+\varepsilon}q^{1/4}T^{1/12}. \tag{2.27}$$

C. *Estimation of* I_ρ. With the help of (2.24) and Lemma 2.5, we have

$$I_\rho \ll \int_{1/2+\varepsilon}^{\beta} \big(q^{\frac{1-\sigma}{2}}|\gamma|^{1/12+\varepsilon}\big)\,\frac{x^\sigma}{|\sigma+i\gamma|}\,\mathrm{d}\sigma. \tag{2.28}$$

Denote by $N(\sigma,T,\chi)$ the number of zeros of $L(s,\chi)$ in the region $\operatorname{Re} s \geqslant \sigma$ and $|\operatorname{Im} s| \leqslant T$. Summing (2.28) over $|\gamma|<T$ and interchanging the summations, we have

$$\sum_{\beta>\frac{1}{2}+\varepsilon,\,|\gamma|<T} |I_\rho| \ll (\log T)\max_{T_1\le T}\int_{1/2+\varepsilon}^{1-\sigma(T_1;q)} q^{\frac{1}{2}(1-\sigma)}T_1^{1/12+\varepsilon}\frac{x^\sigma}{T_1}N(\sigma,T_1,\chi)\,\mathrm{d}\sigma.$$

where

$$\sigma(\tau;q) := C\log^{-2/3}(q|\tau|+3q)\log_2^{-1/3}(q|\tau|+3q),$$

$C=C(\mathfrak{p}$ is a positive constant depending on $\mathfrak{p})$ and we have used Lemma 2.3.

It is well-known that (cf. [8, Theorem 12.1] and [7])

$$N(\sigma,T,q) := \sum_{\chi \,(\mathrm{mod}\, q)} N(\sigma,T,\chi) \ll (qT)^{\frac{12}{5}(1-\sigma)}\log^9(qT).$$

Thus

$$\sum_{\chi\neq\chi_0}\ \sum_{\beta>\frac{1}{2}+\varepsilon,\,|\gamma|<T} |I_\rho| \tag{2.29}$$

$$\ll \log^{10}(qT)\max_{T_1\le T}\int_{1/2+\varepsilon}^{1-\sigma(T_1;q)} q^{\frac{1-\sigma}{2}}T_1^{1/12+\varepsilon}\frac{x^\sigma}{T_1}(qT_1)^{\frac{12}{5}(1-\sigma)}\,\mathrm{d}\sigma$$

$$\ll x\log^{10}(qT)\max_{T_1\le T}\int_{1/2+\varepsilon}^{1-\sigma(T_1;q)} \Big(\frac{q^{87/30}T_1^{17/30}}{x}\Big)^{1-\sigma}\mathrm{d}\sigma$$

$$\ll x\log^{10}(qT)\max_{T_1\le T}\Big(\frac{q^{87/30}T_1^{17/30}}{x}\Big)^{\sigma(T_1;q)} \ll x\log^{10}(qT)\Big(\frac{q^{87/30}T^{17/30}}{x}\Big)^{\sigma(T;q)}.$$

provided $q^{87/30}T^{17/30}\le x$. Inserting (2.26), (2.27) and (2.29) into (2.25), we find that

$$\sum_{\chi\neq\chi_0}\overline{\chi}(a)\chi(d)\sum_{n\leqslant x}\frac{\chi(n)}{\tau(nd)}$$

$$\ll \frac{qx\log x}{T} + x^{1/2}q^{5/4}T^{1/12+\varepsilon} + x\log^{10}(qT)\Big(\frac{q^{87/30}T^{17/30}}{x}\Big)^{\sigma(T;q)}$$

$$\ll (x^{-13}q^{104})^{1/17+\varepsilon} + (x^{33}q^{42})^{1/51+\varepsilon} + x(\log x)^{10}x^{-\varepsilon\sigma(T;q)/195}$$

thanks to the choice of $T=(x^{30(1-\varepsilon)}q^{-87})^{1/17}$. This implies the required result. $\square$

3. Proof of Theorem 1

Firstly we write

$$S(x,t;q,a):=\frac{1}{(x/q)}\sum_{\substack{n\leqslant x\\ n\equiv a(\operatorname{mod} q)}} F_n(t). \tag{3.1}$$

In view of the symmetry of the divisors of n about $\sqrt{n}$, it follows that

$$F_n(t)=\operatorname{Prob}(D_n\geqslant 1-t)=1-\operatorname{Prob}(D_n<1-t)=1-F_n(1-t)+O(\tau(n)^{-1}).$$

Summing over $n\leqslant x$ with $n\equiv a\,(\operatorname{mod} q)$, we have

$$\begin{aligned}&S(x,t;q,a)+S(x,1-t;q,a)\\ &=\frac{1}{(x/q)}\sum_{\substack{n\leqslant x\\ n\equiv a(\operatorname{mod} q)}}\{1+O(\tau(n)^{-1})\}=1+O\Big(\frac{1}{\sqrt{\log x}}\Big)\end{aligned}$$

uniformly for $x\geq 3$, $q\leq x^{15/52-\varepsilon}$ and $a\in\mathbb{Z}^*$ such that $(a,q)=1$, where we have used the orthogonality and Lemmas 2.4 and 2.6 with $d=1$ to deduce that

$$\begin{aligned}\frac{1}{(x/q)}\sum_{\substack{n\leqslant x\\ n\equiv a(\operatorname{mod} q)}}\frac{1}{\tau(n)}&=\frac{q}{x\varphi(q)}\sum_{\chi(\operatorname{mod} q)}\overline{\chi}(a)\sum_{n\leqslant x}\frac{\chi(n)}{\tau(n)}\\ &\ll\frac{(q/\varphi(q))}{\mathrm{e}^{c_1(\varepsilon)(\log x)^{1/3}(\log_2 x)^{-1/3}}}\ll\frac{1}{\sqrt{\log x}}.\end{aligned}$$

On the other hand, we have the identity

$$\frac{2}{\pi}\arcsin\sqrt{t}+\frac{2}{\pi}\arcsin\sqrt{1-t}=1\quad(0\leqslant t\leqslant 1).$$

Therefore it is sufficient to prove (1.3) for $0\leqslant t\leqslant\frac12$.

For $0\leqslant t\leqslant\frac12$, we can write

$$S(x,t;q,a)=\frac{q}{x\varphi(q)}\sum_{n\leqslant x}\sum_{\chi(\operatorname{mod} q)}\frac{\overline{\chi}(a)\chi(n)}{\tau(n)}\sum_{d\mid n,\,d\leqslant n^t}1\quad(n=dm) \tag{3.2}$$

$$= \frac{q}{x\varphi(q)} \sum_{d\leqslant x^t} \sum_{\chi (\operatorname{mod} q)} \overline{\chi}(a)\chi(d) \sum_{d^{1/t-1}\leq m\leqslant x/d} \frac{\chi(m)}{\tau(md)}$$
$$= \frac{q}{x\varphi(q)}(S_1 - S_2 + S_3 - S_4),$$

where

$$S_1 := \sum_{d\leqslant x^t} \overline{\chi}_0(a)\chi_0(d) \sum_{m\leqslant x/d} \frac{\chi_0(m)}{\tau(md)}, \ S_2 := \sum_{d\leqslant x^t} \overline{\chi}_0(a)\chi_0(d) \sum_{m\leqslant d^{1/t-1}} \frac{\chi_0(m)}{\tau(md)},$$
$$S_3 := \sum_{d\leqslant x^t} \sum_{\chi\neq\chi_0} \overline{\chi}(a)\chi(d) \sum_{m\leqslant x/d} \frac{\chi(m)}{\tau(md)},$$
$$S_4 := \sum_{d\leqslant x^t} \sum_{\chi\neq\chi_0} \overline{\chi}(a)\chi(d) \sum_{m\leqslant d^{1/t-1}} \frac{\chi(m)}{\tau(md)}.$$

For S_1, we apply Lemmas 2.4 and 2.5 to write

$$S_1 = \frac{h}{\sqrt{\pi}} \sum_{d\leqslant x^t} \frac{\chi_0(d)}{d\sqrt{\log(x/d)}} \Big\{ g(d) + O\Big(\frac{(3/4)^{\omega(d)}}{\log x} \Big) \Big\} \tag{3.3}$$
$$= \frac{\varphi(q)}{q} x \Big\{ \frac{2}{\pi} \arcsin\sqrt{t} + O\Big(\frac{1}{\sqrt{\log x}} \Big) \Big\}.$$

For S_2, we have

$$S_2 \leqslant \sum_{d\leqslant x^t} \sum_{m<d^{1/t-1}} \frac{1}{\tau(m)} \ll \sum_{d\leqslant x^t} \frac{d^{1/t-1}}{\sqrt{1+\log d^{1/t-1}}} \tag{3.4}$$
$$\ll \frac{x}{\sqrt{1+\log x^{1-t}}} \ll \frac{x}{\sqrt{\log x}}.$$

By Lemma 2.6, we have

$$S_3 \ll \sum_{d\leqslant x^t} \sum_{\chi\neq\chi_0} \overline{\chi}(a)\chi(d) \sum_{m\leqslant x/d} \frac{\chi(m)}{\tau(md)} \ll x\mathrm{e}^{-c_2(\varepsilon)\sqrt[3]{(\log x)/\log_2 x}}, \tag{3.5}$$

$$S_4 \ll \sum_{d\leqslant x^t} \sum_{\chi\neq\chi_0} \overline{\chi}(a)\chi(d) \sum_{m\leqslant d^{1/t-1}} \frac{\chi(m)}{\tau(md)} \ll x\mathrm{e}^{-c_2(\varepsilon)\sqrt[3]{(\log x)\log_2 x}} \tag{3.6}$$

uniformly for $x \geq 3$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a, q) = 1$.

Inserting (3.3)–(3.6) into (3.2), we find that

$$S(x,t;q,a) = \frac{2}{\pi}\arcsin\sqrt{t} + O_{\mathfrak{p},\varepsilon}\Big(\frac{1}{\sqrt{\log x}}\Big)$$

uniformly for $0 \le t \le \frac{1}{2}$, $x \ge 3$, $q \le x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a,q)=1$.

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