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SUMMABILITY OF FOURIER SERIES IN PERIODIC HARDY SPACES WITH VARIABLE EXPONENT

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Abstract. Let $p(\cdot)\colon \mathbb{T}^n \to (0,\infty)$ be a variable exponent function satisfying the globally log-Hölder condition and $0 < q \leq \infty$. We introduce the periodic variable Hardy and Hardy–Lorentz spaces $H_{p(\cdot)}(\mathbb{T}^d)$ and $H_{p(\cdot),q}(\mathbb{T}^d)$ and prove their atomic decompositions. A general summability method, the so called θ-summability is considered for multi-dimensional Fourier series. Under some conditions on θ, it is proved that the maximal operator of the θ-means is bounded from $H_{p(\cdot)}(\mathbb{T}^d)$ to $L_{p(\cdot)}(\mathbb{T}^d)$ and from $H_{p(\cdot),q}(\mathbb{T}^d)$ to $L_{p(\cdot),q}(\mathbb{T}^d)$. This implies some norm and almost everywhere convergence results for the summability means. The Riesz, Bochner–Riesz, Weierstrass, Picard and Bessel summations are investigated as special cases.

1. Introduction

It was proved by Lebesgue [24] that the Fejér means [11] of the trigonometric Fourier series of a one-dimensional integrable function $f \in L_1(\mathbb{T})$ converge almost everywhere to the function. This result was generalized for several summability methods, such as for the Riesz, Weierstrass, Abel, etc. summations in Zygmund [46], Butzer and Nessel [2], Stein and Weiss [38] or Trigub and Belinsky [39].

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes all summations just mentioned, is studied intensively in the literature (see e.g. Butzer and Nessel [2], Trigub and Belinsky [39], Gát [12–14], Goginava [15–17], Persson, Tephnadze and Wall [31], Simon [33–35] and Feichtinger and Weisz [9,10,

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41–43]). For multi-dimensional Fourier series, the summability means are defined by

$$\sigma_n^\theta f(x) := \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_d\in\mathbb{Z}} \theta\Big(\frac{|k|}{n}\Big) \widehat{f}(k) e^{2\pi\imath k\cdot x},$$

where $|\cdot|$ denotes the Euclidean norm and $\widehat{f}(k)$ is the kth Fourier coefficient of f. The choice $\theta(u) = \max(1-|u|,0)$ yields the Fejér summation.

Stein, Taibleson and Weiss [37] proved for the Bochner–Riesz summability that the maximal operator σ_*^θ of the θ-means is bounded from the Hardy space $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ if $p > p_0$ (see also Grafakos [18] and Lu [28]). Recently, the author [44] generalized this result and verified for multi-dimensional Fourier transforms that σ_*^θ is bounded from $H_{p(\cdot)}(\mathbb{R}^d)$ to $L_{p(\cdot)}(\mathbb{R}^d)$ and from $H_{p(\cdot),q}(\mathbb{R}^d)$ to $L_{p(\cdot),q}(\mathbb{R}^d)$ for all $p(\cdot) > p_0$, where $p(\cdot)$: $\mathbb{R}^d \to (0,\infty)$ is a variable exponent function satisfying the globally log-Hölder condition and $0 < q \le \infty$.

In this paper, we prove similar results for the periodic spaces and for multi-dimensional Fourier series. We consider a variable exponent defined on $\mathbb{T}^d$ and the periodic Lebesgue, Lorentz, Hardy and Hardy–Lorentz spaces $L_{p(\cdot)}(\mathbb{T}^d)$, $L_{p(\cdot),q}(\mathbb{T}^d)$, $H_{p(\cdot)}(\mathbb{T}^d)$ and $H_{p(\cdot),q}(\mathbb{T}^d)$ defined by $p(\cdot)$. If $p(\cdot)$ is a constant, then we get back the classical spaces. We will prove the atomic decompositions of $H_{p(\cdot)}(\mathbb{T}^d)$ and $H_{p(\cdot),q}(\mathbb{T}^d)$. Moreover, we show that σ_*^θ is bounded from $H_{p(\cdot)}(\mathbb{T}^d)$ to $L_{p(\cdot)}(\mathbb{T}^d)$ and from $H_{p(\cdot),q}(\mathbb{T}^d)$ to $L_{p(\cdot),q}(\mathbb{T}^d)$. As a consequence of these results, we obtain some norm and almost everywhere convergence results for the summability means. Our results can be applied to the Riesz, Bochner–Riesz, Weierstrass, Picard and Bessel summations.

2. Variable Lebesgue and Lorentz spaces

In this section, we recall some basic notations on variable Lebesgue spaces and variable Lorentz spaces and give some elementary and necessary facts about these spaces. Our main references are Cruz-Uribe and Fiorenza [5], Diening et al. [6] and Kempka and Vybíral [22].

For a constant p, the $L_p(\mathbb{T}^d)$ space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p} \quad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure λ on the torus $\mathbb{T}^d$, that can be identified with $[0,1)^d$. The Lebesgue measure of a set H will be denoted also by $|H|$.

We are going to generalize these spaces. A measurable function $p(\cdot)$: $\mathbb{T}^d \to (0,\infty]$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{T}^d} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{T}^d} p(x).$$

Denote by $\mathcal{P}(\mathbb{T}^d)$ the *collection of all variable exponents* $p(\cdot)$ satisfying

$$0 < p_- \le p_+ < \infty.$$

In what follows, we use the symbol

$$\underline{p} = \min\{p_-, 1\}.$$

For $p(\cdot) \in \mathcal{P}(\mathbb{T}^d)$ and a measurable function f, the *modular functional* $\varrho_{p(\cdot)}$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{T}^d} |f(x)|^{p(x)}\, dx$$

and the *Luxemburg quasi-norm* is given by setting

$$\|f\|_{L_{p(\cdot)}(\mathbb{T}^d)} := \inf\left\{\rho \in (0,\infty) :\ \varrho_{p(\cdot)}(f/\rho) \le 1\right\}.$$

The *variable Lebesgue space* $L_{p(\cdot)}(\mathbb{T}^d)$ is defined to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f) < \infty$ and equipped with the quasi-norm $\|\cdot\|_{L_{p(\cdot)}(\mathbb{T}^d)}$. It is easy to see that if $p(\cdot)$ is a constant, then we get back the $L_p(\mathbb{T}^d)$ spaces. It is known that $\|\rho f\|_{L_{p(\cdot)}(\mathbb{T}^d)} = |\rho|\, \|f\|_{L_{p(\cdot)}(\mathbb{T}^d)}$,

$$\big\| |f|^s \big\|_{L_{p(\cdot)}(\mathbb{T}^d)} = \|f\|^s_{L_{sp(\cdot)}(\mathbb{T}^d)} \tag{1}$$

and

$$\big\| f+g \big\|^{\underline{p}}_{L_{p(\cdot)}(\mathbb{T}^d)} \le \big\| f \big\|^{\underline{p}}_{L_{p(\cdot)}(\mathbb{T}^d)} + \big\| g \big\|^{\underline{p}}_{L_{p(\cdot)}(\mathbb{T}^d)},$$

where $p(\cdot) \in \mathcal{P}(\mathbb{T}^d)$, $s \in (0,\infty)$, $\rho \in \mathbb{C}$ and $f, g \in L_{p(\cdot)}(\mathbb{T}^d)$.

We denote by $C^{\log}(\mathbb{T}^d)$ the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{T}^d)$ satisfying the so-called *log-Hölder continuous condition*, namely, there exists a positive constants $C_{\log}(p)$ such that, for any $x, y \in \mathbb{T}^d$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/|x-y|)}.$$

Given an integrable function f, the *Hardy–Littlewood maximal operator* M is defined by

$$Mf(x) := \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)|\, dy \quad (x \in \mathbb{T}^d),$$

where the supremum is taken over all balls B of $\mathbb{T}^d$ containing x. It is known that M is bounded on $L_p(\mathbb{T}^d)$ if $1<p<\infty$ and is of weak type $(1,1)$. This is extended to the variable Lebesgue spaces in the following lemma (see Cruz-Uribe et al. [4], Nekvinda [30] or Cruz-Uribe and Fiorenza [5, Theorem 3.16]).

LEMMA 1. *Suppose that* $p(\cdot)\in C^{\log}(\mathbb{T}^d)$ *and* $f\in L_{p(\cdot)}(\mathbb{T}^d)$. *If* $p_-\geq 1$, *then*

$$\sup_{\rho\in(0,\infty)}\Big(\rho\big\|\chi_{\{x\in\mathbb{T}^d:\ Mf(x)>\rho\}}\big\|_{L_{p(\cdot)}(\mathbb{T}^d)}\Big)\leq \|f\|_{L_{p(\cdot)}(\mathbb{T}^d)}. \tag{2}$$

If in addition $p_->1$, *then*

$$\|Mf\|_{L_{p(\cdot)}(\mathbb{T}^d)}\leq C\|f\|_{L_{p(\cdot)}(\mathbb{T}^d)}. \tag{3}$$

The *variable Lorentz spaces* were introduced and investigated by Kempka and Vybíral [22]. $L_{p(\cdot),q}(\mathbb{T}^d)$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L_{p(\cdot),q}(\mathbb{T}^d)}:=\begin{cases}\Big(\int_0^\infty \rho^q\big\|\chi_{\{x\in\mathbb{T}^d:\ |f(x)|>\rho\}}\big\|^q_{L_{p(\cdot)}(\mathbb{T}^d)}\frac{d\rho}{\rho}\Big)^{1/q}, & \text{if } 0<q<\infty;\\ \sup_{\rho\in(0,\infty)}\rho\big\|\chi_{\{x\in\mathbb{T}^d:\ |f(x)|>\rho\}}\big\|_{L_{p(\cdot)}(\mathbb{T}^d)}, & \text{if } q=\infty\end{cases}$$

is finite. If $p(\cdot)$ is a constant, we get back the classical Lorentz spaces (see Lorentz [27] or Bergh and Löfström [1]).

Now we give an equivalent discrete characterization of $\|\cdot\|_{L_{p(\cdot),q}(\mathbb{T}^d)}$. Later it allows us to do the calculation in a more convenient way. For the proof, we refer to Kempka and Vybríal [22, Lemma 2.4].

LEMMA 2. *Assume that* $p(\cdot)\in\mathcal{P}(\mathbb{T}^d)$ *and* $0<q\leq\infty$. *Then, for any measurable function* f,

$$\|f\|_{L_{p(\cdot),q}(\mathbb{T}^d)}\sim\begin{cases}\Big(\sum_{i\in\mathbb{Z}}2^{iq}\big\|\chi_{\{x\in\mathbb{T}^d:|f(x)|>2^i\}}\big\|^q_{L_{p(\cdot)}(\mathbb{T}^d)}\Big)^{1/q}, & \textit{if } 0<q<\infty;\\ \sup_{i\in\mathbb{Z}}2^i\big\|\chi_{\{x\in\mathbb{T}^d:|f(x)|>2^i\}}\big\|_{L_{p(\cdot)}(\mathbb{T}^d)}, & \textit{if } q=\infty.\end{cases}$$

Now we generalize inequality (3) and recall the Fefferman–Stein vector-valued inequality on variable Lebesgue spaces, whose proof is contained in Cruz-Uribe et al. [3, Corollary 2.1] and Jiao [21, Theorem 3.4].

LEMMA 3. *If* $p(\cdot)\in C^{\log}(\mathbb{T}^d)$ *with* $p_->1$, $0<q\leq\infty$ *and* $1<r<\infty$, *then*

$$\Big\|\Big(\sum_{j=1}^\infty(Mf_j)^r\Big)^{1/r}\Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}\leq C\Big\|\Big(\sum_{j=1}^\infty|f_j|^r\Big)^{1/r}\Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}$$

and

$$\Big\| \Big(\sum_{j=1}^{\infty} (Mf_j)^r \Big)^{1/r} \Big\|_{L_{p(\cdot),q}(\mathbb{T}^d)} \le C \Big\| \Big(\sum_{j=1}^{\infty} |f_j|^r \Big)^{1/r} \Big\|_{L_{p(\cdot),,q}(\mathbb{T}^d)}.$$

3. Variable Hardy and Hardy–Lorentz spaces

Now we introduce the *variable Hardy* and *Hardy–Lorentz spaces* and give the atomic decompositions. Denote by $S(\mathbb{R}^d)$ the set of all Schwartz functions, by $S'(\mathbb{R}^d)$ the set of all tempered distributions and by $D(\mathbb{T}^d)$ the set of all distributions. For a distribution $f \in D(\mathbb{T}^d)$, the nth *Fourier coefficient* is defined by $\widehat{f}(n) := f(e_n)$, where $e_n(x) := e^{-2\pi \imath n\cdot x}$ $(n \in \mathbb{Z}^d,\ x = (x_1, \dots, x_d) \in \mathbb{R}^d)$, $u \cdot x := \sum_{k=1}^d u_k x_k$ $(x, u \in \mathbb{R}^d)$ and $\imath = \sqrt{-1}$. In special case, if $f \in L_1(\mathbb{T}^d)$, then

$$\widehat{f}(n) = \int_{\mathbb{T}^d} f(x) e^{-2\pi \imath n\cdot x}\, dx \quad (n \in \mathbb{Z}^d).$$

For $f \in D(\mathbb{T}^d)$,

$$f = \sum_{n\in\mathbb{N}^d} \widehat{f}(n) e_n \quad \text{in } D(\mathbb{T}^d)$$

and $\widehat{f}(n) = O(|n|^k)$, where $k \in \mathbb{N}$ is the order of f (see Edwards [7, p. 68]). Conversely, if $c_n = O(|n|^k)$, then $f = \sum_{n\in\mathbb{N}^d} c_n e_n$ in $D(\mathbb{T}^d)$. We define the convolution of $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$ by

$$f * \psi := \sum_{n\in\mathbb{N}^d} \widehat{f}(n) \widehat{\psi}(n) e_n \quad \text{in } D(\mathbb{T}^d),$$

where $\widehat{\psi}$ denotes the Fourier transform of $\psi \in L_1(\mathbb{R}^d)$,

$$\widehat{\psi}(x) := \int_{\mathbb{R}^d} \psi(t) e^{-2\pi \imath x\cdot t}\, dt \quad (x \in \mathbb{R}^d).$$

Note that this is the usual convolution if $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{T}^d)$. For $t \in (0, \infty)$ and $\xi \in \mathbb{T}^d$, let

$$\psi_t(\xi) := t^{-d} \psi(\xi/t).$$

For $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$, we have

$$f * \psi_t = \sum_{n\in\mathbb{N}^d} \widehat{f}(n) \widehat{\psi}(tn) e_n \quad \text{in } D(\mathbb{T}^d). \tag{4}$$

The convergence in (4) does exist because $\widehat{\psi} \in L_\infty(\mathbb{R}^d)$. Moreover, if $\psi \in S(\mathbb{R}^d)$, then (4) converges absolutely in each point as well. It is easy to see that

$$f * \psi_t(x) = \int_{\mathbb{R}^d} f(x-u)\psi_t(u)\,du \tag{5}$$

for $f \in L_1(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$.

Fix $\psi \in S(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx \neq 0$. We define the radial maximal function and the non-tangential maximal function of $f \in S'(\mathbb{R}^n)$ associated to ψ by

$$\psi_+^*(f)(x) := \sup_{t\in(0,\infty)} |f * \psi_t(x)|$$

and

$$\psi_\triangledown^*(f)(x) := \sup_{t\in(0,\infty),|y-x|<t} |f * \psi_t(y)|,$$

respectively. For $N \in \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^d) := \Big\{\psi \in S(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d,\|\beta\|_1\le N} (1+|x|)^{N+d}|\partial^\beta\psi(x)| \le 1\Big\},$$

where $\|\beta\|_1 = \beta_1 + \cdots + \beta_d$. For any $N \in \mathbb{N}$, the radial grand maximal function and the non-tangential grand maximal function of $f \in S'(\mathbb{R}^d)$ are defined by

$$f_+^*(x) := \sup_{\psi\in\mathcal{F}_N(\mathbb{R}^d)} \sup_{t\in(0,\infty)} |f * \psi_t(y)|$$

and

$$f_\triangledown^*(x) := \sup_{\psi\in\mathcal{F}_N(\mathbb{R}^d)} \sup_{t\in(0,\infty),|y-x|<t} |f * \psi_t(y)|,$$

respectively. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < q \le \infty$. We introduce the number $d_{p(\cdot)} := \lfloor d(1/p_- - 1)\rfloor$ and fix a positive integer $N > d_{p(\cdot)}$, where $\lfloor x\rfloor$ denotes the integer part of $x \in \mathbb{R}$. The variable Hardy and Hardy–Lorentz spaces $H_{p(\cdot)}(\mathbb{T}^d)$ and $H_{p(\cdot),q}(\mathbb{T}^d)$ are defined to be the sets of all $f \in D(\mathbb{R}^d)$ such that $f_\triangledown^* \in L_{p(\cdot)}(\mathbb{T}^d)$ and $f_\triangledown^* \in L_{p(\cdot),q}(\mathbb{T}^d)$ equipped with the quasi-norms

$$\|f\|_{H_{p(\cdot)}(\mathbb{T}^d)} := \|\psi_+^*(f)\|_{L_{p(\cdot)}(\mathbb{T}^d)} \text{ and } \|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} := \|\psi_+^*(f)\|_{L_{p(\cdot),q}(\mathbb{T}^d)},$$

respectively. Let us denote by $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$ the closure of the step functions in $H_{p(\cdot),\infty}(\mathbb{T}^d)$. We will see in the next theorem that the Hardy spaces are independent of N, more exactly, different integers N give the same space with equivalent norms.

THEOREM 1. *Let $p(\cdot) \in C^{\log}(\mathbb{T}^d)$ and $0 < q \le \infty$. Fix $\psi \in S(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx \neq 0$ and fix a positive integer $N > d_{p(\cdot)}$. Then $f \in H_{p(\cdot)}(\mathbb{T}^d)$ if and only if $f_\triangledown^* \in L_{p(\cdot)}(\mathbb{T}^d)$ or $\psi_\triangledown^*(f) \in L_{p(\cdot)}(\mathbb{T}^d)$ or $\psi_+^*(f) \in L_{p(\cdot)}(\mathbb{T}^d)$. Moreover, $f \in H_{p(\cdot),q}(\mathbb{T}^d)$ if and only if $f_\triangledown^* \in L_{p(\cdot),q}(\mathbb{T}^d)$ or $\psi_\triangledown^*(f) \in L_{p(\cdot),q}(\mathbb{T}^d)$ or $\psi_+^*(f) \in L_{p(\cdot),q}(\mathbb{T}^d)$. We have the following equivalences of norms*:

$$\|f\|_{H_{p(\cdot)}(\mathbb{T}^d)} \sim \|f_+^*\|_{L_{p(\cdot)}(\mathbb{T}^d)} \sim \|f_\triangledown^*\|_{L_{p(\cdot)}(\mathbb{T}^d)} \sim \|\psi_\triangledown^*(f)\|_{L_{p(\cdot)}(\mathbb{T}^d)}$$

and

$$\|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} \sim \|f_+^*\|_{L_{p(\cdot),q}(\mathbb{T}^d)} \sim \|f_\triangledown^*\|_{L_{p(\cdot),q}(\mathbb{T}^d)} \sim \|\psi_\triangledown^*(f)\|_{L_{p(\cdot),q}(\mathbb{T}^d)}.$$

THEOREM 2. *Let $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, $1 < p_- < \infty$ and $0 < q \le \infty$. Then*

$$H_{p(\cdot)}(\mathbb{T}^d) \sim L_{p(\cdot)}(\mathbb{T}^d), \quad H_{p(\cdot),q}(\mathbb{T}^d) \sim L_{p(\cdot),q}(\mathbb{T}^d).$$

We omit the proofs of these theorems because they are very similar to the proofs of the corresponding theorems for $H_{p(\cdot)}(\mathbb{R}^d)$ and $H_{p(\cdot),q}(\mathbb{R}^d)$ (see e.g. Nakai and Sawano [29,32], Yan et al. [45], Liu et al. [25,26] and Jiao et al. [21]). If $p(\cdot)$ is a constant, then we get back the classical Hardy and Hardy–Lorentz spaces $H_p(\mathbb{T}^d)$ and $H_{p,q}(\mathbb{T}^d)$ investigated in Fefferman, Stein and Weiss [8,36,38], Gundy [19], Lu [28], Uchiyama [40] and Weisz [43].

The atomic decomposition is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, inequalities and interpolation results can be proved.

Let $p(\cdot) \in \mathcal{P}(\mathbb{T}^d)$ and fix a nonnegative integer $d_{p(\cdot)} \le s < \infty$. A measurable function a is called a $(p(\cdot), r)$-atom if there exists a ball $B \subset \mathbb{T}^d$ such that

(i) supp $a \subset B$,

(ii) $\|a\|_{L_r(\mathbb{T}^d)} \le \frac{\lambda(B)^{1/r}}{\|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}}$,

(iii) $\int_{\mathbb{T}^d} a(x)x^\alpha dx = 0$ for all multi-indices α with $|\alpha| \le s$.

Note that supp a denotes the support of a. The atomic decomposition of variable Hardy spaces $H_{p(\cdot)}(\mathbb{R}^d)$ was proved in Nakai and Sawano [29,32, Theorem 4.5, Theorem 1.1] and Liu et al. [25] (in the classical case see e.g. Latter [23], Lu [28] or Weisz [43]).

Before proving the atomic decomposition, we present the next theorem about the atoms. For a ball B with center c and radius ρ, let τB denote the ball with the same center and with radius $\tau\rho$ ($\tau > 0$). Set $\mathbb{Z}_0 := \{l \in \mathbb{Z}^d : l_i \in \{0, 1, -1\}, i = 1, \ldots, d\}$.

THEOREM 3. *Let $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, $N = d_{p(\cdot)} + 1$, $1 < r \le \infty$ and $d_{p(\cdot)} \le s < \infty$. If a is a $(p(\cdot), r)$-atom and $\psi \in \mathcal{F}_N(\mathbb{R}^d)$, then*

$$\left|\psi_+^*(a)(x)\right| \le C\left\|\chi_B\right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_B(x)|^{(N+d)/d} \tag{6}$$

for all $x \in \mathbb{T}^d \setminus 2B$, *where the ball* B *is the support of the atom.*

Proof. Suppose that B is a ball with center c and radius ρ. We use Taylor's formula for a fixed $l \in \mathbb{Z}^d$ and for $g(u) = \psi\left(\frac{x-u}{t}\right)$:

$$g(u) = \sum_{k=0}^{N-1} \sum_{\|i\|_1=k} \partial_1^{i_1} \cdots \partial_d^{i_d} g(c+l) \prod_{j=1}^{d} \frac{(u_j - c_j - l_j)^{i_j}}{i_j!}$$

$$+ \sum_{\|i\|_1=N} \partial_1^{i_1} \cdots \partial_d^{i_d} g(u^l) \prod_{j=1}^{d} \frac{(u_j - c_j - l_j)^{i_j}}{i_j!}$$

for some $u^l \in B + l$. By the definition of the atom and by (5),

$$a * \psi_t(x) = \sum_{l \in \mathbb{Z}^d} t^{-d} \int_{B+l} a(u) \psi\Big(\frac{x-u}{t}\Big)\, du = \sum_{l \in \mathbb{Z}^d} t^{-d} \int_{B+l} a(u)$$

$$\times \left(\psi\Big(x - ut\Big) - \sum_{k=0}^{N-1} \sum_{\|i\|_1=k} \partial_1^{i_1} \cdots \partial_d^{i_d} g(c+l) \prod_{j=1}^{d} \frac{(u_j - c_j - l_j)^{i_j}}{i_j!} \right) du.$$

Note that $s \geq N-1$, where s is given in the definition of the $(p(\cdot), r)$-atoms. Since

$$\partial_1^{i_1} \cdots \partial_d^{i_d} g(u) = (-1)^{\|i\|_1} t^{-\|i\|_1} \partial_1^{i_1} \cdots \partial_d^{i_d} \psi\Big(\frac{x-u}{t}\Big),$$

we conclude

$$|a * \psi_t(x)| \leq \sum_{l \in \mathbb{Z}^d} t^{-d} \int_{B+l} |a(u)| \sum_{\|i\|_1=N} t^{-\|i\|_1} \tag{7}$$

$$\times \left| \partial_1^{i_1} \cdots \partial_d^{i_d} \psi\Big(\frac{x-u^l}{t}\Big) \right| \prod_{j=1}^{d} \frac{|u_j - c_j - l_j|^{i_j}}{i_j!}\, du$$

$$\leq C \sum_{l \in \mathbb{Z}^d} t^{-N-d} \int_{B+l} |a(u)| \left| \frac{x-u^l}{t} \right|^{-N-d} |u-c-l|^N \, du$$

$$\leq C \sum_{l \in \mathbb{Z}_0} \rho^N \int_{B+l} |a(u)| |x-u^l|^{-N-d}\, du$$

$$+ C \sum_{l \in \mathbb{Z}^d \setminus \mathbb{Z}_0} \rho^N \int_{B+l} |a(u)| |x-u^l|^{-N-d}\, du =: A + B.$$

Since $x \notin \bigcup_{l\in\mathbb{Z}_0} 2(B+l)$,

$$|x-u^l| \ge |x-c-l| - |u^l-c-l| \ge |x-c-l|/2 \quad (l\in\mathbb{Z}_0).$$

By the definition of the atom,

$$\begin{aligned}
A &\le C\sum_{l\in\mathbb{Z}_0} \rho^N |x-c-l|^{-N-d} \int_{B+l} |a(u)|\,du \\
&\le C\sum_{l\in\mathbb{Z}_0} \rho^N |x-c-l|^{-N-d} \|a\|_{L_r(\mathbb{T}^d)} \lambda(B)^{1/r'} \\
&\le C\sum_{l\in\mathbb{Z}_0} \rho^{N+d} \|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |x-c-l|^{-N-d} \\
&\le C\sum_{l\in\mathbb{Z}_0} \|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |M\chi_B(x-l)|^{(N+d)/d} \\
&\le C \|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |M\chi_B(x)|^{(N+d)/d}.
\end{aligned}$$

If $l\notin\mathbb{Z}_0$ and $x\in\mathbb{T}^d$, then $|x-u^l|\ge |l|/2 \quad (l\notin\mathbb{Z}_0)$ and

$$B \le C\sum_{l\in\mathbb{Z}^d\setminus\mathbb{Z}_0} \rho^{N+d}\|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |l|^{-N-d} \le C\rho^{N+d}\|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)}$$

$$\le C\rho^{N+d}\|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |x-c|^{-N-d} \le C\|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |M\chi_B(x)|^{(N+d)/d},$$

which completes the proof of the theorem. □

THEOREM 4. *Let $p(\cdot)\in C^{\log}(\mathbb{T}^d)$, $\max(p_+,1)<r\le\infty$ and $d_{p(\cdot)}\le s<\infty$. A distribution $f\in D(\mathbb{T}^d)$ is in $H_{p(\cdot)}(\mathbb{T}^d)$ if and only if there exist a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $(p(\cdot),r)$-atoms with support $\{B_i\}_{i\in\mathbb{N}}$ and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive numbers such that*

$$f=\sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } D(\mathbb{T}^d). \tag{8}$$

Moreover, for every $0<t\le \underline{p}$,

$$\|f\|_{H_{p(\cdot)}(\mathbb{T}^d)} \sim \inf \left\| \left(\sum_{i\in\mathbb{N}} \left(\frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \right)^t \right)^{1/t} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}, \tag{9}$$

where the infimum is taken over all decompositions of f as above.

PROOF. Let $N=d_{p(\cdot)}+1$ and choose $\psi\in\mathcal{F}_N(\mathbb{R}^d)$ such that the support of ψ is a subset of T^d and $\int_{\mathbb{R}^d}\psi(x)dx\ne 0$. Suppose that $f\in D(\mathbb{T}^d)$ has an atomic decomposition (8) such that the right hand side of (9) is finite. Since the sum of (8) converges in $D(\mathbb{T}^d)$, we have

$$f * \psi_t = \sum_{i\in\mathbb{N}} \lambda_i a_i * \psi_t \quad \text{(a.e.)}$$

Then

$$\psi_+^*(f) \le \sum_{i\in\mathbb{N}} \lambda_i \psi_+^*(a_i) \tag{10}$$

and so

$$\|\psi_+^*(f)\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim \Big\| \sum_{i\in\mathbb{N}} \lambda_i \psi_+^*(a_i) \chi_{2B_i} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)} + \Big\| \sum_{i\in\mathbb{N}} \lambda_i \psi_+^*(a_i) \chi_{(2B_i)^c} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)} =: A_1 + A_2.$$

Let us choose $0 < t < \underline{p} \le 1$ and apply (1) to obtain

$$A_1 \le \Big\| \sum_{i\in\mathbb{N}} \lambda_i^t \psi_+^*(a_i)^t \chi_{2B_i} \Big\|_{L_{p(\cdot)/t}(\mathbb{T}^d)}^{1/t}.$$

We use $p'(\cdot)$ to denote the conjugate variable exponent, namely, $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. Choose $g \in L_{(p(\cdot)/t)'}(\mathbb{R}^d)$ with $\|g\|_{L_{(p(\cdot)/t)'}(\mathbb{R}^d)} \le 1$ such that

$$\Big\| \sum_{i\in\mathbb{N}} \lambda_i^t \psi_+^*(a_i)^t \chi_{2B_i} \Big\|_{L_{p(\cdot)/t}(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \sum_{i\in\mathbb{N}} \lambda_i^t \psi_+^*(a_i)^t \chi_{2B_i} g \, d\lambda.$$

Choosing $p_+/t < u < r < \infty$ and applying Hölder's inequality, we deduce

$$A_1^t \le \int_{\mathbb{T}^d} \sum_{i\in\mathbb{N}} \lambda_i^t \psi_+^*(a_i)^t \chi_{2B_i} g \, d\lambda \le \sum_{i\in\mathbb{N}} \lambda_i^t \big\| \psi_+^*(a_i)^t \chi_{2B_i} \big\|_{L_u(\mathbb{T}^d)} \big\| \chi_{2B_i} g \big\|_{L_{u'}(\mathbb{T}^d)}$$
$$\lesssim \sum_{i\in\mathbb{N}} \lambda_i^t \big\| \psi_+^*(a_i) \big\|_{L_r(\mathbb{T}^d)}^t \lambda(2B_i)^{1/u - t/r} \big\| \chi_{2B_i} g \big\|_{L_{u'}(\mathbb{T}^d)}.$$

By the definition of the $p(\cdot)$-atom and the boundedness of $\psi_+^*(a_i)$, we conclude

$$A_1^t \lesssim \sum_{i\in\mathbb{N}} \lambda_i^t \|a_i\|_{L_r(\mathbb{T}^d)}^t \lambda(2B_i)^{1/u - t/r} \|\chi_{2B_i} g\|_{L_{u'}(\mathbb{T}^d)}$$

$$\leq \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \lambda(2B_i)^{1/u} \|\chi_{2B_i} g\|_{L_{u'}(\mathbb{T}^d)}$$

$$\leq \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \lambda(2B_i) \Big(\frac{1}{\lambda(2B_i)} \int_{2B_i} g^{u'} \, d\lambda \Big)^{1/u'}$$

$$\leq 2 \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \int_{\mathbb{T}^d} \chi_{B_i} \big(M(g^{u'}) \big)^{1/u'} d\lambda.$$

Again by Hölder's inequality,

$$A_1^t \leq 2 \int_{\mathbb{T}^d} \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \chi_{B_i} \big(M(g^{u'}) \big)^{1/u'} d\lambda$$

$$\leq 2 \Big\| \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \chi_{B_i} \Big\|_{L_{p(\cdot)/t}(\mathbb{T}^d)} \big\| \big(M(g^{u'}) \big)^{1/u'} \big\|_{L_{(p(\cdot)/t)'}(\mathbb{T}^d)}.$$

Since $(p(\cdot)/t)' < \infty$ and $p_+/t < u$ imply that $(p(\cdot)/t)' > u'$, we get by (1) and (3) that

$$A_1 \leq \Big\| \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \chi_{B_i} \Big\|_{L_{p(\cdot)/t}(\mathbb{T}^d)}^{1/t} \big\| M(g^{u'}) \big\|_{L_{((p(\cdot)/t)')/u'}(\mathbb{T}^d)}^{1/tu'}$$

$$\lesssim \Big\| \sum_{i\in\mathbb{N}} \lambda_i^t \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-t} \chi_{B_i} \Big\|_{L_{p(\cdot)/t}(\mathbb{T}^d)}^{1/t} \|g\|_{L_{(p(\cdot)/t)'}(\mathbb{T}^d)}^{1/t}$$

$$\lesssim \Big\| \Big(\sum_{i\in\mathbb{N}} \Big(\frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big)^t \Big)^{1/t} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}.$$

By Theorem 3,

$$A_2 \lesssim \Big\| \sum_{i\in\mathbb{N}} \lambda_i \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_{B_i}|^{(N+d)/d} \chi_{(2B_i)^c} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}$$

$$\lesssim \Big\| \sum_{i\in\mathbb{N}} \Big(\lambda_i^{d/(N+d)} \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-d/(N+d)} |M\chi_{B_i}| \Big)^{(N+d)/d} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}$$

$$\leq \Big\| \Big(\sum_{i\in\mathbb{N}} \Big(\lambda_i^{d/(N+d)} \|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-d/(N+d)} |M\chi_{B_i}| \Big)^{(N+d)/d} \Big)^{d/(N+d)} \Big\|_{L_{(N+d)p(\cdot)/d}(\mathbb{T}^d)}^{(N+d)/d}.$$

Since $N = d_{p(\cdot)} + 1$ implies $p_- > d/(N+d)$, we can apply Lemma 3 to conclude

$$A_2 \le \Big\| \Big(\sum_{i\in\mathbb{N}} \lambda_i \|\chi_{B_i}\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |\chi_{B_i}| \Big)^{d/(N+d)} \Big\|^{(N+d)/d}_{L_{(N+d)p(\cdot)/d}(\mathbb{T}^d)}$$
$$\lesssim \Big\| \sum_{i\in\mathbb{N}} \frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)},$$

which shows that $f \in H_{p(\cdot)}(\mathbb{T}^d)$. The other part of the theorem can be shown as for $H_{p(\cdot)}(\mathbb{R}^d)$ (see e.g. Nakai and Sawano [29,32] and Liu et al. [25] or Weisz [43]). $\square$

The next result can be found for $H_{p(\cdot),q}(\mathbb{R}^d)$ in Yan et al. [45, Theorem 4.4], Liu et al. [26] and Jiao et al. [21, Theorem 5.4].

THEOREM 5. *Let $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, $0 < q \le \infty$, $\max(p_+, 1) < r \le \infty$ and $d_{p(\cdot)} \le s < \infty$. A tempered distribution $f \in D(\mathbb{T}^d)$ is in $H_{p(\cdot),q}(\mathbb{T}^d)$ if and only if there exists a sequence $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ of $(p(\cdot), r)$-atoms with support $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that*

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} \quad \text{in } D(\mathbb{T}^d), \tag{11}$$

where

$$\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \le A$$

for all $x \in \mathbb{T}^d$ and $i \in \mathbb{Z}$ and $\lambda_{i,j} := C2^i \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}$ $(i \in \mathbb{Z}, j \in \mathbb{N})$ with A and C being positive constants. Moreover,

$$\|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} \sim \inf \Big(\sum_{i\in\mathbb{Z}} \Big\| \Big(\sum_{j\in\mathbb{N}} \Big(\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big)^{\underline{p}} \Big)^{1/\underline{p}} \Big\|^q_{L_{p(\cdot)}(\mathbb{T}^d)} \Big)^{1/q}, \tag{12}$$

where the infimum is taken over all decompositions of f as above and with the usual modification for $q = \infty$.

PROOF. Suppose that $f \in D(\mathbb{T}^d)$ has an atomic decomposition (11) such that the right hand side of (12) is finite. As in (10),

$$\psi_+^*(f) \le \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} \psi_+^*(a_{i,j})$$
$$= \sum_{i<k_0} \sum_{j\in\mathbb{N}} \lambda_{i,j} \psi_+^*(a_{i,j}) \chi_{2B_{i,j}} + \sum_{i<k_0} \sum_{j\in\mathbb{N}} \lambda_{i,j} \psi_+^*(a_{i,j}) \chi_{(2B_{i,j})^c}$$

$$+\sum_{i\ge k_0}\sum_{j\in\mathbb{N}}\lambda_{i,j}\psi_+^*(a_{i,j})\chi_{2B_{i,j}}+\sum_{i\ge k_0}\sum_{j\in\mathbb{N}}\lambda_{i,j}\psi_+^*(a_{i,j})\chi_{(2B_{i,j})^c}$$
$$=: A_1+A_2+A_3+A_4,$$

where $k_0\in\mathbb{Z}$ is arbitrary. Then

$$\|\chi_{\{\psi_+^*(f)>2^{k_0}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}\lesssim\sum_{i=1}^4\|\chi_{\{A_i>2^{k_0-2}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}. \tag{13}$$

Let us choose the numbers ε,δ,ν,u such that $0<\varepsilon<\min(\underline{p},q)$, $1<\delta<\nu$, $\max(p_+,1)<u\varepsilon<u\varepsilon\nu<r$ and $\nu\varepsilon<1$. Then

$$\|\chi_{\{A_1>2^{k_0-2}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}\lesssim\Big\|\frac{A_1^\nu}{2^{k_0\nu}}\Big\|_{L_{p(\cdot)}(\mathbb{T}^d)} \tag{14}$$
$$\lesssim 2^{-k_0\nu}\Big\|\Big(\sum_{i<k_0}\sum_{j\in\mathbb{N}}\lambda_{i,j}\psi_+^*(a_{i,j})\chi_{2B_{i,j}}\Big)^{\nu\varepsilon}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}^{1/\varepsilon}$$
$$\lesssim 2^{-k_0\nu}\Big\|\sum_{i<k_0}2^{i\nu\varepsilon}\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon}\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{2B_{i,j}}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}^{1/\varepsilon}$$
$$\lesssim 2^{-k_0\nu}\Big(\sum_{i<k_0}2^{i\nu\varepsilon}\Big\|\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon}\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{2B_{i,j}}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}\Big)^{1/\varepsilon}.$$

To compute the norm in the last expression let us choose $g\in L_{(p(\cdot)/\varepsilon)'}(\mathbb{T}^d)$ with $\|g\|_{L_{(p(\cdot)/\varepsilon)'}(\mathbb{T}^d)}\le 1$ such that

$$\Big\|\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon}\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{2B_{i,j}}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}$$
$$=\int_{\mathbb{T}^d}\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon}\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{2B_{i,j}}g.$$

By Hölder's inequality,

$$\Big\|\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon}\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{2B_{i,j}}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}$$
$$\le\sum_{j\in\mathbb{N}}\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon}\big\|(\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{2B_{i,j}}\big\|_{L_u(\mathbb{T}^d)}\big\|\chi_{2B_{i,j}}g\big\|_{L_{u'}(\mathbb{T}^d)}$$

$$\lesssim \sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon} \|\psi_+^*(a_{i,j})\|_{L_r(\mathbb{T}^d)}^{\nu\varepsilon} \lambda(2B_{i,j})^{1/u-\nu\varepsilon/r} \|\chi_{2B_{i,j}} g\|_{L_{u'}(\mathbb{T}^d)}.$$

By Theorem 2 and the definition of the atom,

$$\Big\| \sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon} \psi_+^*(a_{i,j})^{\nu\varepsilon} \chi_{2B_{i,j}} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}$$
$$\lesssim \sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon} \|a_{i,j}\|_{L_r(\mathbb{T}^d)}^{\nu\varepsilon} \lambda(2B_{i,j})^{1/u-\nu\varepsilon/r} \|\chi_{2B_{i,j}} g\|_{L_{u'}(\mathbb{T}^d)}$$
$$\le \sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-\nu\varepsilon} \lambda(2B_{i,j})^{1/u} \|\chi_{2B_{i,j}} g\|_{L_{u'}(\mathbb{T}^d)}$$
$$\le \sum_{j\in\mathbb{N}} \lambda(2B_{i,j}) \Big(\frac{1}{\lambda(2B_{i,j})} \int_{2B_{i,j}} g^{u'}\, d\lambda \Big)^{1/u'} \le \sum_{j\in\mathbb{N}} \int_{\mathbb{T}^d} \chi_{B_{i,j}} \big(M(g^{u'}) \big)^{1/u'} d\lambda$$
$$\le \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)} \Big\| \big(M(g^{u'}) \big)^{1/u'} \Big\|_{L_{(p(\cdot)/\varepsilon)'}(\mathbb{T}^d)}.$$

Since $\varepsilon < p_-$ and $p_+/\varepsilon < u$, we have that $((p(\cdot)/\varepsilon)')_+ < \infty$ and $(p(\cdot)/\varepsilon)' > u'$. Thus

$$\Big\| \sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\nu\varepsilon} \psi_+^*(a_{i,j})^{\nu\varepsilon} \chi_{2B_{i,j}} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}$$
$$\lesssim \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)} \|g\|_{L_{(p(\cdot)/\varepsilon)'}(\mathbb{T}^d)}.$$

As $\|g\|_{L_{(p(\cdot)/\varepsilon)'}(\mathbb{T}^d)} \le 1$ and $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \le A$ $(i \in \mathbb{Z})$, we conclude

$$\|\chi_{\{A_1 > 2^{k_0-2}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim 2^{-k_0\nu} \Big(\sum_{i<k_0} 2^{i\nu\varepsilon} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)} \Big)^{1/\varepsilon} \tag{15}$$
$$\lesssim 2^{-k_0\nu} \Big(\sum_{i<k_0} 2^{i\nu\varepsilon} \Big\| \Big(\sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big)^{1/\varepsilon} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\varepsilon} \Big)^{1/\varepsilon}$$
$$\le 2^{-k_0\nu} \Big(\sum_{i<k_0} 2^{i\nu\varepsilon} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\varepsilon} \Big)^{1/\varepsilon}.$$

For a finite q, we suppose that $\varepsilon < q$ and apply Hölder's inequality for $\frac{q-\varepsilon}{q} + \frac{\varepsilon}{q} = 1$ to obtain

$$\begin{aligned} &\left\| \chi_{\{A_1 > 2^{k_0-2}\}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)} \\ &\le 2^{-k_0\nu} \left(\sum_{i<k_0} 2^{i(\nu-\delta)\varepsilon \frac{q}{q-\varepsilon}} \right)^{\frac{q-\varepsilon}{\varepsilon q}} \left(\sum_{i<k_0} 2^{i\delta q} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \right)^{1/q} \\ &\lesssim 2^{-k_0\delta} \left(\sum_{i<k_0} 2^{i\delta q} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \right)^{1/q}. \end{aligned}$$

This implies that

$$\begin{aligned} &\sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \|\chi_{\{A_1 > 2^{k_0}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \lesssim \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q} \sum_{i=-\infty}^{k_0-1} 2^{i\delta q} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \\ &= \sum_{i=-\infty}^{\infty} 2^{i\delta q} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \sum_{k_0=i+1}^{\infty} 2^{k_0(1-\delta)q} \lesssim \sum_{i=-\infty}^{\infty} 2^{iq} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q. \end{aligned}$$

Hence, for all $0 < q < \infty$,

$$\begin{aligned} &\left(\sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \|\chi_{\{A_1 > 2^{k_0}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \right)^{1/q} \\ &\lesssim \left(\sum_{i\in\mathbb{Z}} \left\| \sum_{j\in\mathbb{N}} \frac{\lambda_{i,j} \chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \right)^{1/q}. \end{aligned} \tag{16}$$

For $q = \infty$, (15) implies that

$$\begin{aligned} &\left\| \chi_{\{A_1 > 2^{k_0-2}\}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)} \le 2^{-k_0\nu} \left(\sum_{i<k_0} 2^{i\varepsilon(\nu-1)} 2^{i\varepsilon} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{\varepsilon} \right)^{1/\varepsilon} \\ &\le \left(\sup_{i\in\mathbb{Z}} 2^i \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)} \right) 2^{-k_0\nu} \left(\sum_{i<k_0} 2^{i\varepsilon(\nu-1)} \right)^{1/\varepsilon} \\ &\le 2^{-k_0} \left(\sup_{i\in\mathbb{Z}} 2^i \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L_{p(\cdot)}(\mathbb{T}^d)} \right). \end{aligned}$$

Henceforth,

$$\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{A_1>2^{k_0-2}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim \sup_{i\in\mathbb{Z}} \Big\| \sum_{j\in\mathbb{N}} \frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}.$$

For A_2, we choose ε,δ,ν such that $0<\varepsilon<\min(\underline{p},q)$, $1<\delta<\nu$, $\nu\varepsilon<1$ and $\nu\varepsilon(N+d)/d>1$. Note that $N=d_{p(\cdot)}+1$ implies $p_->d/(N+d)$. Using (6), we get as in (14) that

$$\begin{aligned}
&\big\|\chi_{\{A_2>2^{k_0-2}\}}\big\|_{L_{p(\cdot)}(\mathbb{T}^d)} \\
&\lesssim 2^{-k_0\nu}\Big(\sum_{i<k_0} 2^{i\nu\varepsilon}\Big\|\sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|^{\nu\varepsilon}_{L_{p(\cdot)}(\mathbb{T}^d)}\psi_+^*(a_{i,j})^{\nu\varepsilon}\chi_{(2B_{i,j})^c}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}\Big)^{1/\varepsilon} \\
&\lesssim 2^{-k_0\nu}\Big(\sum_{i<k_0} 2^{i\nu\varepsilon}\Big\|\sum_{j\in\mathbb{N}} |M\chi_{B_{i,j}}|^{\nu\varepsilon(N+d)/d}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}\Big)^{1/\varepsilon} \\
&\lesssim 2^{-k_0\nu}\Big(\sum_{i<k_0} 2^{i\nu\varepsilon}\Big\|\Big(\sum_{j\in\mathbb{N}} |M\chi_{B_{i,j}}|^{\nu\varepsilon(N+d)/d}\Big)^{d/\nu\varepsilon(N+d)}\Big\|^{\nu\varepsilon(N+d)/d}_{L_{p(\cdot)\nu(N+d)/d}(\mathbb{T}^d)}\Big)^{1/\varepsilon}.
\end{aligned}$$

Since $\nu\varepsilon(N+d)/d>1$ and $p_-\nu(N+d)/d>1$, we can apply Lemma 3 to get

$$\|\chi_{\{A_2>2^{k_0-2}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim 2^{-k_0\nu}\Big(\sum_{i<k_0} 2^{i\nu\varepsilon}\Big\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}\Big)^{1/\varepsilon}.$$

Then we can prove as after (15) that

$$\begin{aligned}
(17)\qquad &\Big(\sum_{k_0=-\infty}^{\infty} 2^{k_0q}\|\chi_{\{A_2>2^{k_0}\}}\|^q_{L_{p(\cdot)}(\mathbb{T}^d)}\Big)^{1/q} \\
&\lesssim \Big(\sum_{i\in\mathbb{Z}}\Big\|\sum_{j\in\mathbb{N}} \frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}}\Big\|^q_{L_{p(\cdot)}(\mathbb{T}^d)}\Big)^{1/q}
\end{aligned}$$

for $0<q\le\infty$.

For A_3 let us choose the numbers ε,δ,ν,u such that $0<\varepsilon<\min(\underline{p},q)$, $\nu<\delta<1$ and $\max(p_+,1)<u\varepsilon<r$. Then, for $0<q\le\infty$,

$$(18)\qquad \Big(\sum_{k_0=-\infty}^{\infty} 2^{k_0q}\|\chi_{\{A_3>2^{k_0}\}}\|^q_{L_{p(\cdot)}(\mathbb{T}^d)}\Big)^{1/q}$$

$$\lesssim \Big(\sum_{i\in\mathbb{Z}} \Big\| \sum_{j\in\mathbb{N}} \frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \Big)^{1/q}$$

can be proved exactly as (16).

For A_4, we choose ε, δ, ν such that $0<\varepsilon<\min(\underline{p},q)$, $\nu<\delta<1$, $\nu(N+d)/d>1$ and $p_-\nu(N+d)/d>1$. Similarly to (14),

$$\big\|\chi_{\{A_4>2^{k_0-2}\}}\big\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim \Big\| \frac{A_1^\nu}{2^{k_0\nu}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}$$

$$\lesssim 2^{-k_0\nu} \Big\| \Big(\sum_{i\ge k_0} \sum_{j\in\mathbb{N}} \lambda_{i,j} \psi_+^*(a_{i,j}) \chi_{(2B_{i,j})^c} \Big)^{\nu\varepsilon} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}^{1/\varepsilon}$$

$$\lesssim 2^{-k_0\nu} \Big\| \sum_{i\ge k_0} 2^{i\nu\varepsilon} \Big(\sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^\nu \psi_+^*(a_{i,j})^\nu \chi_{(2B_{i,j})^c} \Big)^{\varepsilon} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)}^{1/\varepsilon}$$

$$\lesssim 2^{-k_0\nu} \Big(\sum_{i\ge k_0} 2^{i\nu\varepsilon} \Big\| \Big(\sum_{j\in\mathbb{N}} \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^\nu \psi_+^*(a_{i,j})^\nu \chi_{(2B_{i,j})^c} \Big)^{\varepsilon} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)} \Big)^{1/\varepsilon}.$$

By Theorem 3 and Lemma 3,

$$\big\|\chi_{\{A_4>2^{k_0-2}\}}\big\|_{L_{p(\cdot)}(\mathbb{T}^d)}$$

$$\lesssim 2^{-k_0\nu} \Big(\sum_{i\ge k_0} 2^{i\nu\varepsilon} \Big\| \Big(\sum_{j\in\mathbb{N}} |M\chi_{B_{i,j}}|^{\nu(N+d)/d} \Big)^{\varepsilon} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)} \Big)^{1/\varepsilon}$$

$$\lesssim 2^{-k_0\nu} \Big(\sum_{i\ge k_0} 2^{i\nu\varepsilon} \Big\| \Big(\sum_{j\in\mathbb{N}} |M\chi_{B_{i,j}}|^{\nu(N+d)/d} \Big)^{d/\nu(N+d)} \Big\|_{L_{p(\cdot)\nu(N+d)/d}(\mathbb{T}^d)}^{\nu\varepsilon(N+d)/d} \Big)^{1/\varepsilon}$$

$$\lesssim 2^{-k_0\nu} \Big(\sum_{i\ge k_0} 2^{i\nu\varepsilon} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|_{L_{p(\cdot)/\varepsilon}(\mathbb{T}^d)} \Big)^{1/\varepsilon}$$

and so

$$\Big(\sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \|\chi_{\{A_4>2^{k_0}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \Big)^{1/q} \tag{19}$$

$$\lesssim \Big(\sum_{i\in\mathbb{Z}} \Big\| \sum_{j\in\mathbb{N}} \frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \Big)^{1/q}$$

for $0 < q \leq \infty$. Taking into account (13), (16), (17), (18) and (19), we get

$$\|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} \lesssim \Big(\sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \|\chi_{\{\psi_+^*(f)>2^{k_0}\}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \Big)^{1/q}$$

$$\lesssim \Big(\sum_{i\in\mathbb{Z}} \Big\| \sum_{j\in\mathbb{N}} \frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{T}^d)}} \Big\|_{L_{p(\cdot)}(\mathbb{T}^d)}^q \Big)^{1/q}.$$

The proof can be finished as for $H_{p(\cdot),q}(\mathbb{R}^d)$ (see e.g. Yan et al. [45], Liu et al. [26] and Jiao et al. [21] or Weisz [43]). □

4. θ-summability of Fourier transforms

The θ-summability is a general summation generated by a single function θ. This summation was considered in a great number of papers and books, see e.g. Butzer and Nessel [2], Grafakos [18], Trigub and Belinsky [39] and Feichtinger and Weisz [10,41,42] and the references therein. Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be even and $\theta_0(x) := \theta(|x|)$, where $|\cdot|$ denotes the Euclidean norm. We suppose always that

$$\theta(0) = 1 \quad \text{and} \quad \widehat{\theta_0} \in L^1(\mathbb{R}^d). \tag{20}$$

For $n \in \mathbb{N}_+$, the nth θ-mean of a distribution $f \in D(\mathbb{T}^d)$ is defined by

$$\sigma_n^\theta f := \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_d\in\mathbb{Z}} \theta_0\Big(\frac{-k}{n}\Big) \widehat{f}(k) e_n.$$

Similarly to (5), this can be rewritten as

$$\sigma_n^\theta f(x) = n^d \int_{\mathbb{R}^d} f(x-u)\widehat{\theta_0}(nu)\,du$$

if $f \in L_1(\mathbb{T}^d)$ and $\widehat{\theta_0} \in L_1(\mathbb{R}^d)$. In this case, we can also write

$$\sigma_n^\theta f(x) = \int_{\mathbb{T}^d} f(x-u) K_n^\theta(u)\,du,$$

where

$$K_n^\theta(u) = n^d \sum_{k\in\mathbb{Z}^d} \widehat{\theta_0}(n(u+k)) \quad (u \in \mathbb{T}^d)$$

is the nth θ-kernel. Thus $K_n^\theta \in L_1(\mathbb{T}^d)$.

First, we estimate the maximal θ-operator defined by

$$\sigma_*^\theta f := \sup_{n>0} |\sigma_n^\theta f|.$$

THEOREM 6. *Let* (20) *be satisfied. Assume that* $\widehat{\theta_0}$ *is* $(N+1)$*-times differentiable for some* $N \in \mathbb{N}$ *and there exists* $d+N < \beta \leq d+N+1$ *such that*

$$\left| \partial_1^{i_1} \cdots \partial_d^{i_d} \widehat{\theta_0}(x) \right| \leq C|x|^{-\beta} \quad (x \neq 0) \tag{21}$$

whenever $i_1 + \cdots + i_d = N$ *or* $i_1 + \cdots + i_d = N+1$. *If* $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, *then*

$$\left| \sigma_*^\theta a(x) \right| \leq C\|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_B(x)|^{\beta/d}$$

for all $(p(\cdot),\infty)$*-atoms* a *and all* $x \in \mathbb{T}^d \setminus 2B$, *where the ball* B *is the support of the atom.*

PROOF. Suppose that B is a ball with center c and radius ρ. We may assume that $s \geq N+1$, where s is given in the definition of the $(p(\cdot),\infty)$-atoms. First let $n \geq 1/\rho$. Similarly to (7),

$$\begin{aligned}
|\sigma_n^\theta a(x)| &\leq \sum_{l \in \mathbb{Z}^d} n^d \int_{B+l} |a(u)| \sum_{\|i\|_1 = N} n^{\|i\|_1} \\
&\times \left| \partial_1^{i_1} \cdots \partial_d^{i_d} \widehat{\theta_0}(n(x-u^l)) \right| \prod_{j=1}^d \frac{|u_j - c_j - l_j|^{i_j}}{i_j!} \, du \\
&\leq C \sum_{l \in \mathbb{Z}^d} n^{N+d} \int_{B+l} |a(u)| \left| n(x-u^l) \right|^{-\beta} |u-c-l|^N \, du \\
&\leq C \sum_{l \in \mathbb{Z}_0} n^{N+d-\beta} \rho^N \int_{B+l} |a(u)| \, |x-u^l|^{-\beta} \, dt \\
&+ C \sum_{l \in \mathbb{Z}^d \setminus \mathbb{Z}_0} n^{N+d-\beta} \rho^N \int_{B+l} |a(u)| \, |x-u^l|^{-\beta} \, du =: A + B.
\end{aligned}$$

Moreover,

$$\begin{aligned}
A &\leq C \sum_{l \in \mathbb{Z}_0} \rho^{\beta-d} |x-c-l|^{-\beta} \int_{B+l} |a(u)| \, du \\
&\leq C \sum_{l \in \mathbb{Z}_0} \rho^\beta \|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |x-c-l|^{-\beta} \leq C\|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_B(x)|^{\beta/d}
\end{aligned}$$

and

$$B \le C \sum_{l\in\mathbb{Z}^d\setminus\mathbb{Z}_0} \rho^\beta \|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |l|^{-\beta}$$

$$\le C\rho^\beta \|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |x-c|^{-\beta} \le C\|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_B(x)|^{\beta/d}.$$

Now suppose that $n < 1/\rho$. We can see as above that

$$|\sigma_n^\theta a(x)| \le \sum_{l\in\mathbb{Z}^d} n^d \int_{B+l} |a(u)| \sum_{\|i\|_1=N+1} n^{\|i\|_1}$$

$$\times \left|\partial_1^{i_1}\cdots\partial_d^{i_d}\widehat{\theta_0}(n(x-u^l))\right| \prod_{j=1}^d \frac{|u_j-c_j-l_j|^{i_j}}{i_j!}\,du$$

$$\le C\sum_{l\in\mathbb{Z}^d} n^{N+1+d}\rho^{N+1} \int_{B+l} |a(u)||n(x-u^l)|^{-\beta}\,du$$

$$\le C\sum_{l\in\mathbb{Z}_0} \|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_{B+l}(x)|^{\beta/d}. \qquad \square$$

Note that if $\beta = d+N+1$ in (21), then it is enough to suppose that

$$\left|\partial_1^{i_1}\cdots\partial_d^{i_d}\widehat{\theta_0}(x)\right| \le C|x|^{-d-N-1} \quad (x\ne 0) \tag{22}$$

for $i_1+\cdots+i_d = N+1$.

THEOREM 7. *Let* (20) *and* (22) *be satisfied. If* $p(\cdot)\in C^{\log}(\mathbb{T}^d)$, *then*

$$\left|\sigma_*^\theta a(x)\right| \le C\|\chi_B\|_{L_{p(\cdot)}(\mathbb{T}^d)}^{-1} |M\chi_B(x)|^{(d+N+1)/d}$$

for all $(p(\cdot),\infty)$*-atoms* a *and all* $x\in\mathbb{T}^d\setminus 2B$, *where the ball* B *contains the support of the atom.*

5. Summability in $H_{p(\cdot)}(\mathbb{T}^d)$

In this section, we investigate the boundedness of the maximal operator V_* from $H_{p(\cdot)}(\mathbb{T}^d)$ to $L_{p(\cdot)}(\mathbb{T}^d)$, where V_n is defined on $D(\mathbb{T}^d)$ and

$$V_*f := \sup_{n\in\mathbb{N}} |V_nf|.$$

THEOREM 8. *Let $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, $\gamma > 1$ and $p_- > 1/\gamma$. For each $n \in \mathbb{N}$, let the linear operator V_n be defined on $D(\mathbb{T}^d)$ and be bounded on $L_1(\mathbb{T}^d)$. Suppose that*

$$|V_* a(x)| \le C \|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{T}^d)} |M\chi_B(x)|^\gamma \quad (x \notin 2B)$$

for all $(p(\cdot), \infty)$-atoms a, where the ball B is the support of the atom. If V_* is bounded from $L_\infty(\mathbb{T}^d)$ to $L_\infty(\mathbb{T}^d)$, then

$$\|V_* f\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{T}^d)} \quad (f \in H_{p(\cdot)}(\mathbb{T}^d) \cap H_1(\mathbb{T}^d)). \tag{23}$$

If $\lim_{k\to\infty} f_k = f$ in the $H_{p(\cdot)}(\mathbb{T}^d)$-norm implies that $\lim_{k\to\infty} V_n f_k = V_n f$ in $D(\mathbb{T}^d)$ for all $n \in \mathbb{N}$, then (23) holds for all $f \in H_{p(\cdot)}(\mathbb{T}^d)$.

PROOF. If $f \in H_{p(\cdot)}(\mathbb{T}^d)$, then, by Theorem 4, f can be written as

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } D(\mathbb{T}^d).$$

It is known (see e.g. Weisz [42]) that the series converge in the $H_1(\mathbb{T}^d)$-norm as well as in the $L_1(\mathbb{T}^d)$-norm if $f \in H_{p(\cdot)}(\mathbb{T}^d) \cap H_1(\mathbb{T}^d)$. Since V_n is a bounded linear operator on the $L_1(\mathbb{T}^d)$ space,

$$V_n(f) = \sum_{i\in\mathbb{N}} \lambda_i V_n(a_i)$$

and so

$$V_*(f) \le \sum_{i\in\mathbb{N}} \lambda_i V_*(a_i).$$

Then inequality (23) can be proved exactly as Theorem 4. The density argument can be found in [44]. □

The following theorem is shown in [44].

THEOREM 9. *Suppose that $Vf := f * K$ for all $f \in D(\mathbb{T}^d)$, where $K \in L_1(\mathbb{T}^d)$. If $p(\cdot) \in C^{\log}(\mathbb{T}^d)$ and*

$$\lim_{k\to\infty} f_k = f \text{ in the } H_{p(\cdot)}(\mathbb{T}^d)\text{-norm, then} \quad \lim_{k\to\infty} V f_k = V f \text{ in } D(\mathbb{T}^d). \tag{24}$$

Since σ_*^θ is bounded from $L_\infty(\mathbb{T}^d)$ to $L_\infty(\mathbb{T}^d)$ (see e.g. Weisz [41]), Theorems 6, 8 and 9 imply

COROLLARY 1. *Let (20) and (21) be satisfied. If $p(\cdot) \in C^{\log}(\mathbb{T}^d)$ and $p_- > d/\beta$, then*

$$\|\sigma_*^\theta f\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{T}^d)} \quad (f \in H_{p(\cdot)}(\mathbb{T}^d)).$$

Note that if $p(\cdot)=p$ is a constant, then we get back the classical result (see Weisz [41,42]). The classical result was proved in a special case, for the Bochner–Riesz means in Stein, Taibleson and Weiss [37], Grafakos [18] and Lu [28]. For the same case [37] contains a counterexample which shows that the theorem is not true for $p \le d/\beta$. The corresponding result for $H_{p(\cdot)}(\mathbb{R}^d)$ was proved by the author in [44].

The following corollary comes from Theorems 7, 8 and 9.

COROLLARY 2. *Let* (20) *and* (22) *be satisfied. If* $p(\cdot) \in C^{\log}(\mathbb{T}^d)$ *and* $p_- > d/(d+N+1)$, *then*

$$\|\sigma_*^\theta f\|_{L_{p(\cdot)}(\mathbb{T}^d)} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{T}^d)} \quad (f \in H_{p(\cdot)}(\mathbb{T}^d)).$$

Using Corollaries 1 and 2 and a usual density argument, we obtain the next convergence results in the usual way.

COROLLARY 3. *Suppose the same conditions as in Corollary* 1 *or* 2. *If* $f \in H_{p(\cdot)}(\mathbb{T}^d)$, *then* $\sigma_T^\theta f$ *converges almost everywhere as well as in the* $L_{p(\cdot)}(\mathbb{T}^d)$*-norm as* $T \to \infty$.

COROLLARY 4. *Suppose the same conditions as in Corollary* 1 *or* 2. *If* $f \in H_{p(\cdot)}(\mathbb{T}^d)$ *and there exists an interval* $I \subset \mathbb{T}^d$ *such that the restriction* $f|_I \in L_{r(\cdot)}(I)$ *with* $r_- \ge 1$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \quad \textit{for a.e. } x \in I \textit{ as well as in the } L_{p(\cdot)}(I)\textit{-norm.}$$

The next consequence follows from Theorem 2.

COROLLARY 5. *Suppose the same conditions as in Corollary* 1 *or* 2. *If* $p_- > 1$ *and* $f \in L_{p(\cdot)}(\mathbb{T}^d)$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \quad \textit{for a.e. } x \in \mathbb{T}^d \textit{ as well as in the } L_{p(\cdot)}(\mathbb{T}^d)\textit{-norm.}$$

6. Summability in $H_{p(\cdot),q}(\mathbb{T}^d)$

The next theorem can be proved similarly to Theorems 5 and 8.

THEOREM 10. *Besides the conditions of Theorem* 8, *suppose that* $0 < q < \infty$. *Then*

$$\|V_* f\|_{L_{p(\cdot),q}(\mathbb{T}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} \quad (f \in H_{p(\cdot),q}(\mathbb{T}^d) \cap H_1(\mathbb{T}^d)). \tag{25}$$

If $\lim_{k\to\infty} f_k = f$ *in the* $H_{p(\cdot),q}(\mathbb{T}^d)$*-norm implies that* $\lim_{k\to\infty} V_t f_k = V_t f$ *in* $D(\mathbb{T}^d)$ *for all* $n \in \mathbb{N}$, *then* (25) *holds for all* $f \in H_{p(\cdot),q}(\mathbb{T}^d)$. *The theorem holds for* $q=\infty$ *as well if we change* $H_{p(\cdot),\infty}(\mathbb{T}^d)$ *by* $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$.

THEOREM 11. *Besides the conditions of Theorem* 9, *suppose that* $0 < q \leq \infty$. *Then* (24) *holds for* $H_{p(\cdot),q}(\mathbb{T}^d)$.

The following results follow from Theorems 6, 7, 10 and 11.

COROLLARY 6. *Let* (20) *and* (21) *be satisfied. If* $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, $0 < q < \infty$ *and* $p_- > d/\beta$, *then*

$$\|\sigma_*^\theta f\|_{L_{p(\cdot),q}(\mathbb{T}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} \quad (f \in H_{p(\cdot),q}(\mathbb{T}^d)).$$

The theorem holds for $q = \infty$ *as well if we change* $H_{p(\cdot),\infty}(\mathbb{T}^d)$ *by* $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$.

COROLLARY 7. *Let* (20) *and* (22) *be satisfied. If* $p(\cdot) \in C^{\log}(\mathbb{T}^d)$, $0 < q < \infty$ *and* $p_- > d/(d+N+1)$, *then*

$$\|\sigma_*^\theta f\|_{L_{p(\cdot),q}(\mathbb{T}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{T}^d)} \quad (f \in H_{p(\cdot),q}(\mathbb{T}^d)).$$

The theorem holds for $q = \infty$ *as well if we change* $H_{p(\cdot),\infty}(\mathbb{T}^d)$ *by* $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$.

Using Corollaries 6 and 7, the following consequences can be proved as in [9,10,20,26].

COROLLARY 8. *Suppose the same conditions as in Corollary* 6 *or* 7. *If* $f \in H_{p(\cdot),q}(\mathbb{T}^d)$ *with* $0 < q < \infty$, *then* $\sigma_T^\theta f$ *converges almost everywhere as well as in the* $L_{p(\cdot),q}(\mathbb{T}^d)$*-norm as* $T \to \infty$. *The theorem holds for* $q = \infty$ *as well if we change* $H_{p(\cdot),\infty}(\mathbb{T}^d)$ *by* $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$.

COROLLARY 9. *Suppose the same conditions as in Corollary* 6 *or* 7. *If* $f \in H_{p(\cdot),q}(\mathbb{T}^d)$ *with* $0 < q < \infty$ *and if there exists an interval* $I \subset \mathbb{T}^d$ *such that the restriction* $f|_I \in L_{r(\cdot),s}(I)$ *with* $r_- \geq 1$ *and* $1 \leq s \leq \infty$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \quad \textit{for a.e. } x \in I \textit{ as well as in the } L_{p(\cdot),q}(I)\textit{-norm.}$$

The theorem holds for $q = \infty$ *as well if we change* $H_{p(\cdot),\infty}(\mathbb{T}^d)$ *by* $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$.

By Theorem 2, we have

COROLLARY 10. *Suppose the same conditions as in Corollary* 6 *or* 7. *If* $p_- > 1$, $1 \leq q < \infty$ *and* $f \in L_{p(\cdot),q}(\mathbb{T}^d)$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \quad \textit{for a.e. } x \in I \textit{ as well as in the } L_{p(\cdot),q}(\mathbb{T}^d)\textit{-norm.}$$

The theorem holds for $q = \infty$ *as well if we change* $H_{p(\cdot),\infty}(\mathbb{T}^d)$ *by* $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T}^d)$.

We can verify the almost everywhere convergence for the spaces $L_{p(\cdot)}(\mathbb{T}^d)$ with $p_- \geq 1$ as well, which is an improvement of Corollary 5.

Corollary 11. *Suppose the same conditions as in Corollary 1 or 2. If $p_- \geq 1$ and $f \in L_{p(\cdot)}(\mathbb{T}^d)$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \quad \textit{for a.e. } x \in \mathbb{T}^d.$$

Proof. Since $f_\Box \leq M(f)$, inequality (2) implies

$$\|f\|_{H_{p(\cdot),\infty}(\mathbb{T}^d)} \leq C\|M(f)\|_{L_{p(\cdot),\infty}(\mathbb{T}^d)} \leq C\|f\|_{L_{p(\cdot)}(\mathbb{T}^d)} \quad (f \in L_{p(\cdot)}(\mathbb{T}^d)).$$

Now the result can be shown as Corollary 9. $\Box$

7. Some summability methods

As special cases, we consider some summability methods. The details of the necessary computations are left to the reader. All the examples satisfy condition (20).

Example 1. The function

$$\theta_0(t) = \begin{cases} (1-|t|^\gamma)^\alpha, & \text{if } |t| > 1; \\ 0, & \text{if } |t| \leq 1 \end{cases} \qquad (t \in \mathbb{R}^d)$$

defines the Riesz summation if $0 < \alpha < \infty$ and γ is a positive integer. It is called Bochner–Riesz summation if $\gamma = 2$. If $\alpha > \frac{d-1}{2}$, then (21) holds with $\beta = d/2 + \alpha + 1/2$ (see Weisz [43]) and the results of Sections 5 and 6 hold for

$$\alpha > \frac{d-1}{2}, \quad \frac{d}{d/2+\alpha+1/2} < p_- < \infty.$$

The results for constant p's can be found in Stein and Weiss [38], Lu [28, p. 132] and Weisz [43].

Example 2. The Weierstrass summation is defined by

$$\theta_0(t) = e^{-|t|^2/2} \quad (t \in \mathbb{R}^d)$$

or by

$$\theta_0(t) = e^{-|t|} \quad (t \in \mathbb{R}^d),$$

or, in the one-dimensional case, by

$$\theta_0(t) = e^{-|t|^\gamma} \quad (t \in \mathbb{R},\ 1 \leq \gamma < \infty).$$

It is called Abel summation if $\gamma = 1$. It is known that in the first case $\widehat{\theta_0}(x) = e^{-|x|^2/2}$ and in the second one $\widehat{\theta_0}(x) = c_d/(1+|x|^2)^{(d+1)/2}$ for some $c_d \in \mathbb{R}$ (see Stein and Weiss [38, p. 6]). Then (22) holds for all $N \in \mathbb{N}$ and the results of Sections 5 and 6 hold for all $p(\cdot) \in C^{\log}(\mathbb{T}^d)$.

Example 3. The Picard and Bessel summations are given by

$$\theta_0(t) = \frac{1}{(1+|t|^2)^{(d+1)/2}} \quad (t \in \mathbb{R}^d).$$

Then $\widehat{\theta_0}(x) = c_d e^{-|x|}$ for some $c_d \in \mathbb{R}$ and the same results hold as in Example 2.

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