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POWER COMPARISON THEOREMS FOR OSCILLATION PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH $p(t)$-LAPLACIAN

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Abstract. This paper deals with the nonlinear differential equation

$$(r(t)|x'|^{p(t)-2}x')' + c(t)|x|^{p(t)-2}x = 0,$$

where $r(t) > 0$ and $c(t)$ are continuous functions, and $p(t) > 1$ is a smooth function. We establish a comparison theorem for the oscillation problem for this equation with respect to the power $p(t)$. Using our result, we can utilize oscillation criteria given for half-linear differential equations to equations with $p(t)$-Laplacian.

1. Introduction

We consider the second order nonlinear differential equation

$$(r(t)|x'|^{p(t)-2}x')' + c(t)|x|^{p(t)-2}x = 0, \tag{1.1}$$

where $r(t)$ and $c(t)$ are positive continuous functions satisfying

$$0 < \liminf_{t\to\infty} r(t) \quad \text{and} \quad \limsup_{t\to\infty} r(t) < \infty, \tag{1.2}$$

and $p(t) > 1$ is a smooth function defined on $(0, \infty)$.

A nontrivial solution $x(t)$ of (1.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory*, that is, it is eventually positive (or eventually negative). For simplicity, we call it a *positive solution* (or *negative solution*).

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The differential operator in (1.1) is called $p(t)$-Laplacian. Such operator appears in mathematical models in the study of image processing and electrorheological fluids (see [1,10,11]). In recent years, increasing attention has been paid to the study of oscillation problems for nonlinear differential equations with $p(t)$-Laplacian. For example, those results can be found in [9,13–15,18–20].

If $p(t) \equiv p > 1$, then $p(t)$-Laplacian is the well-known p-Laplacian, and (1.1) becomes the *half-linear* differential equation

$$\big(r(t)|x'|^{p-2}x'\big)' + c(t)|x|^{p-2}x = 0, \tag{1.3}$$

whose solution space has just one half of the properties which characterize linearity, namely homogeneity. Numerous papers have been devoted to the study of oscillation problems for half-linear differential equations; we can refer to [2–8,12,16,17] and the references cited therein. For example, the following Leighton–Wintner type oscillation criterion is well-known (see [7]).

THEOREM A. *All nontrivial solutions of* (1.3) *are oscillatory provided*

$$\int_1^\infty (r(t))^{1-\tilde{p}}\,dt = \infty \quad \textit{and} \quad \int_1^\infty c(t)\,dt = \infty,$$

where $\tilde{p} = p/(p-1)$.

Moreover, according to [7], we see that the classical linear Sturmian comparison theorem extends verbatim to (1.3). From the Sturmian comparison theorem, the comparison theorems with respect to the power p were established in [2,4,7,12,16]. In particular, Sugie and Yamaoka [16] gave the following result, which serves as a main motivation for our research in this paper.

THEOREM B. *Assume* $r(t) \equiv 1$ *and* $c(t) > 0$. *Consider the equation*

$$(|x'|^{q-2}x')' + c(t)|x|^{q-2}x = 0, \tag{1.4}$$

where $q > 1$ *is a constant. Suppose that* $p > q$. *If all nontrivial solutions of* (1.4) *are oscillatory, then those of* (1.3) *are also oscillatory.*

For the elliptic partial differential equations involving higher-dimensional $p(t)$-Laplacian, comparison theorems were given in [18,19] as applications of Picone identities. However, the structure of solutions of this equation differs from one of (1.1). Hence, unfortunately, we cannot expect to obtain the same kind of results. Here, a natural question now arises. Does (1.1) have any comparison properties? The purpose of this paper is to answer the question. To be precise, we extend Theorem B for (1.1), in order to utilize the oscillation criteria which have been given for (1.3) to (1.1). Consider the pair of equations (1.1) and

$$(r(t)|x'|^{q(t)-2}x')' + c(t)|x|^{q(t)-2}x = 0, \tag{1.5}$$

where $q(t) > 1$ is a smooth function defined on $(0, \infty)$. Our main result is following.

THEOREM 1.1. *Assume* (1.2) *and the hypotheses*
(H_1) *$p(t)$ is nondecreasing and tends to $p_* > 1$ as $t \to \infty$,*
(H_2) *$q(t)$ is nonincreasing and tends to $q_* > 1$ as $t \to \infty$,*
(H_3) *$p_* > q_*$,*
(H_4) *either $p(t) \equiv p_*$ for t sufficiently large or*

$$\int_1^\infty c(s)\,ds < \infty \quad \text{and} \quad \int_1^\infty \left(\int_t^\infty c(s)\,ds\right)^{1/(p(t)-1)} dt = \infty \tag{1.6}$$

hold.

If all solutions of (1.5) *are oscillatory, then those of* (1.1) *are also oscillatory.*

A prototype of (1.1) is the Euler type differential equation

$$(|x'|^{p(t)-2}x')' + \frac{\lambda}{t^{p(t)}}|x|^{p(t)-2}x = 0. \tag{1.7}$$

Assume (H_1). Suppose that there exists $0 < M < e$ satisfying the log-Hölder decay condition

$$t^{|p_*-p(t)|} < M \tag{1.8}$$

for t sufficiently large. Then, (1.6) holds obviously. In [9], it has been proved that (1.7) is *conditionally oscillatory*, that is, if $\lambda > \gamma(p_*)$ then all nontrivial solutions of (1.7) are oscillatory, and if $\lambda < \gamma(p_*)$ then (1.7) has a nonoscillatory solution, where $\gamma(p_*) = ((p_*-1)/p_*)^{p_*}$. Hence, together with Theorem A, (1.6) is optimal in a certain sense.

In addition, we get the following oscillation criterion, which is an analogue of Theorem A.

THEOREM 1.2. *Assume* (1.2) *and* (H_1). *All nontrivial solutions of* (1.1) *are oscillatory provided*

$$\int_1^\infty (r(t))^{1-\tilde{p}(t)}\,dt = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\int_1^t c(s)\,ds = \infty, \tag{1.9}$$

where $\tilde{p}(t) = p(t)/(p(t)-1)$.

This paper is organized as follows. In Section 2, we prepare two lemmas to prove our result. In Section 3, we give the proofs of our main results Theorems 1.1 and 1.2. In the final section, we illustrate our result by corollaries and simple examples. Moreover, we propose some open problems.

2. Preliminaries

LEMMA 2.1. *Assume*

$$\limsup_{t\to\infty} r(t) < \infty, \tag{2.1}$$

and

$$1 < \liminf_{t\to\infty} p(t) \quad \textit{and} \quad \limsup_{t\to\infty} p(t) < \infty. \tag{2.2}$$

Suppose that (1.1) *has a positive solution* $x(t)$*. Then* $x(t)$ *is strictly increasing for* t *sufficiently large. In addition*, *if* (1.6) *holds then* $x(t)$ *tends to infinity as* $t \to \infty$.

PROOF. There exists $t_0 \geq 1$ such that $x(t) > 0$ for $t \geq t_0$. From (1.1), we have

$$(r(t)|x'(t)|^{p(t)-2}x'(t))' = -c(t)(x(t))^{p(t)-1} < 0 \tag{2.3}$$

for $t \geq t_0$. Suppose that there exists $t_1 \geq t_0$ such that $x'(t_1) \leq 0$. Without loss of generality, we may assume $x'(t_1) < 0$. In fact, (2.3) shows that

$$r(t)|x'(t)|^{p(t)-2}x'(t) < r(t_1)|x'(t_1)|^{p(t_1)-2}x'(t_1) \leq 0,$$

which implies $x'(t) < 0$ for $t > t_1$ even if $x'(t_1) = 0$. Hence we get

$$-r(t)|x'(t)|^{p(t)-1} \leq -r(t_1)|x'(t_1)|^{p(t_1)-1} < 0$$

for $t \geq t_1$. Together with (2.1) and (2.2), we can find $c_1 > 0$ such that

$$x'(t) \leq -\Big(\frac{r(t_1)}{r(t)}\Big)^{1/(p(t)-1)}|x'(t_1)|^{(p(t_1)-1)/(p(t)-1)} \leq -c_1$$

for $t \geq t_1$. Integrating both sides of this inequality from t_1 to t, we get $x(t) \leq x(t_1) - c_1(t - t_1) \to -\infty$ as $t \to \infty$, which is a contradiction. Thus, we see that $x'(t) > 0$ for $t \geq t_0$.

We next assume (1.6). From (2.2) and $x'(t) > 0$ for t sufficiently large, there exist $t_2 \geq t_0$ and $c_2 > 0$ such that

$$(r(t)(x'(t))^{p(t)-1})' = -c(t)(x(t))^{p(t)-1} \leq -c_2c(t)$$

for $t \geq t_2$. Integrating both sides of this inequality from $t \geq t_2$ to $T \geq t$, we get

$$r(t)(x'(t))^{p(t)-1} \geq r(T)(x'(T))^{p(T)-1} + c_2 \int_t^T c(s)\,ds \geq c_2 \int_t^T c(s)\,ds,$$

which implies

$$x'(t) \geq \Big(\frac{c_2}{r(t)}\Big)^{1/(p(t)-1)} \Big(\int_t^T c(s)\,ds\Big)^{1/(p(t)-1)}.$$

Using (2.1) and (2.2), and taking limit $T \to \infty$, we can choose $t_3 \geq t_2$ and $c_3 > 0$ such that

$$x'(t) \geq c_3 \Big(\int_t^\infty c(s)\,ds\Big)^{1/(p(t)-1)}$$

for $t \geq t_3$. Hence we obtain

$$x(t) \geq x(t_3) + c_3 \int_{t_3}^t \Big(\int_s^\infty c(\tau)\,d\tau\Big)^{1/(p(s)-1)} ds,$$

which implies that $x(t) \to \infty$ as $t \to \infty$ because of (1.6). □

LEMMA 2.2. *Assume* $\liminf_{t\to\infty} q(t) > 1$. *Suppose that there exist* $c_0 \geq 0$ *and an eventually positive smooth function* $y(t)$ *satisfying*

$$r(t)(y'(t))^{q(t)-1} \geq c_0 + \int_t^\infty c(s)(y(s))^{q(s)-1}\,ds \tag{2.4}$$

for t *sufficiently large. Then* (1.5) *has a positive solution.*

PROOF. There exists $t_0 > 1$ such that $y(t)$ is positive and (2.4) holds for $t \geq t_0$. Let $\{w_n\}$ and $\{x_n\}$ be sequences of continuous functions satisfying

$$w_1(t) = y'(t), \quad x_1(t) = y(t),$$

$$w_{n+1}(t) = \bigg\{\frac{1}{r(t)}\Big(c_0 + \int_t^\infty c(s)(x_n(s))^{q(s)-1}\,ds\Big)\bigg\}^{\tilde{q}(t)-1}, \tag{2.5}$$

$$x_{n+1}(t) = \int_{t_0}^t w_{n+1}(s)\,ds + y(t_0) \tag{2.6}$$

for $t \geq t_0$, where $\tilde{q}(t) = q(t)/(q(t)-1)$. Then $w_1(t)$ and $x_1(t)$ are well-defined.

By mathematical induction, we will show that

$$0 < w_{n+1}(t) \leq w_n(t) \quad \text{and} \quad y(t_0) \leq x_{n+1}(t) \leq x_n(t) \tag{2.7}$$

hold for any $n \in \mathbb{N}$ and $t \ge t_0$. In the case of $n = 1$, using (2.4) and (2.5), we get

$$0 < (w_2(t))^{q(t)-1} = \frac{1}{r(t)}\left(c_0 + \int_t^\infty c(s)(y(s))^{q(s)-1}\,ds\right)$$
$$\le (y'(t))^{q(t)-1} = (w_1(t))^{q(t)-1}$$

for $t \ge t_0$. Hence we have $0 < w_2(t) \le w_1(t)$ and

$$y(t_0) \le x_2(t) = \int_{t_0}^t w_2(s)\,ds + y(t_0) \le \int_{t_0}^t w_1(s)\,ds + y(t_0) = x_1(t)$$

for $t \ge t_0$, which imply (2.7) with $n = 1$. Suppose that (2.7) with $n = k$ holds. Then we get

$$0 < (w_{k+2}(t))^{q(t)-1} = \frac{1}{r(t)}\left(c_0 + \int_t^\infty c(s)(x_{k+1}(s))^{q(s)-1}\,ds\right)$$
$$\le \frac{1}{r(t)}\left(c_0 + \int_t^\infty c(s)(x_k(s))^{q(s)-1}\,ds\right) = (w_{k+1}(t))^{q(t)-1}$$

for $t \ge t_0$. Hence we have $0 < w_{k+2}(t) \le w_{k+1}(t)$ and

$$y(t_0) \le x_{k+2}(t) = \int_{t_0}^t w_{k+2}(s)\,ds + y(t_0) \le x_{k+1}(t)$$

for $t \ge t_0$, which imply (2.7) with $n = k + 1$.

Let $x(t) = \lim_{n\to\infty} x_n(t)$ and $w(t) = \lim_{n\to\infty} w_n(t)$ for $t \ge t_0$. Then, using (2.7) and applying the Lebesgue dominated convergence theorem to (2.5) and (2.6), we have

$$r(t)(w(t))^{q(t)-1} = c_0 + \int_t^\infty c(s)(x(s))^{q(s)-1}\,ds \text{ and } x(t) = \int_{t_0}^t w(s)\,ds + y(t_0)$$

for $t \ge t_0$. Since $x_n'(t) = w_n(t) > 0$, we get

$$x(t) = \lim_{n\to\infty} x_n(t) \ge \lim_{n\to\infty} x_n(t_0) = y(t_0) > 0$$

for $t \ge t_0$. Thus, we see that $x(t)$ is a positive solution of (1.5). □

3. Proofs of the main theorems

PROOF OF THEOREM 1.1. In order to prove Theorem 1.1 by contradiction, we suppose that (1.1) has a positive solution $x(t)$. We consider the case when (1.6) holds. Using Lemma 2.1, we see that there exists $t_0 > 1$ such that $x(t) > 1$ and $x'(t) > 0$ for $t \geq t_0$. Let

$$w(t) = r(t)\Big(\frac{x'(t)}{x(t)}\Big)^{p(t)-1} > 0$$

for $t \geq t_0$. Then, we have

$$\begin{aligned} w'(t) &= -c(t) - (p(t)-1)r(t)\Big(\frac{w(t)}{r(t)}\Big)^{p(t)/(p(t)-1)} - p'(t)(\log x(t))w(t) \\ &\leq -c(t) - (p(t)-1)r(t)\Big(\frac{w(t)}{r(t)}\Big)^{p(t)/(p(t)-1)} < 0 \end{aligned} \tag{3.1}$$

for $t \geq t_0$. Hence $w(t) > 0$ is strictly decreasing for $t \geq t_0$. We note that, in the case when $p(t) \equiv p_*$, we can show this inequality without $x(t) > 1$ for t sufficiently large because of $p'(t) = 0$.

Suppose that there exists $w_* > 0$ such that $w(t) \to w_*$ as $t \to \infty$, and let $r_* = \limsup_{t\to\infty} r(t) < \infty$. Then we can find $0 < \eta < 1$ satisfying $\eta < w_*/r_*$. Since $c(t)$ is positive and $p(t)$ is nondecreasing, there exist $t_1 \geq t_0$ and $c_1 > 0$ such that

$$\begin{aligned} w'(t) &\leq -c(t) - (p(t)-1)r(t)\eta^{p(t)/(p(t)-1)} \\ &\leq -c_1(p(t_1)-1)\eta^{p(t_1)/(p(t_1)-1)} \end{aligned} \tag{3.2}$$

for $t \geq t_1$. Here, we note that the function

$$f(x) = \frac{x}{x-1} = 1 + \frac{1}{x-1}$$

is strictly decreasing for $x > 1$. Integrating both sides of (3.2) from t_1 to t, we get

$$w(t) \leq w(t_1) - c_1(p(t_1)-1)\eta^{p(t_1)/(p(t_1)-1)}(t-t_1).$$

This implies $w(t) \to -\infty$ as $t \to \infty$, which is a contradiction to the positivity of $w(t)$. Hence we obtain $w(t) \to 0$ as $t \to \infty$.

From (1.2) and $p_* > q_*$, there exists $t_2 \geq t_0$ such that

$$(q(t)-1)\Big(\frac{w(t)}{r(t)}\Big)^{q(t)/(q(t)-1)} \leq (p(t)-1)\Big(\frac{w(t)}{r(t)}\Big)^{p(t)/(p(t)-1)}$$

for $t \geq t_2$. By using (3.1), we have

$$w'(t) \leq -c(t) - (q(t)-1)r(t)\Big(\frac{w(t)}{r(t)}\Big)^{q(t)/(q(t)-1)} \tag{3.3}$$

for $t \geq t_2$. Let

$$y(t) = \exp\biggl(\int_{t_2}^{t}\Bigl(\frac{w(s)}{r(s)}\Bigr)^{1/(q(s)-1)}ds\biggr) > 1.$$

Then we have

$$y'(t) = \Bigl(\frac{w(t)}{r(t)}\Bigr)^{1/(q(t)-1)}y(t) > 0.$$

Since $q(t)$ is nonincreasing, we obtain

$$\begin{aligned}&(r(t)(y'(t))^{q(t)-1})' \\ &= \Bigl\{w'(t) + (q(t)-1)r(t)\Bigl(\frac{w(t)}{r(t)}\Bigr)^{q(t)/(q(t)-1)} + q'(t)(\log y(t))w(t)\Bigr\}(y(t))^{q(t)-1} \\ &\leq \Bigl\{w'(t) + (q(t)-1)r(t)\Bigl(\frac{w(t)}{r(t)}\Bigr)^{q(t)/(q(t)-1)}\Bigr\}(y(t))^{q(t)-1}.\end{aligned}$$

Thus, from (3.3), we get

$$(r(t)(y'(t))^{q(t)-1})' \leq -c(t)(y(t))^{q(t)-1} < 0,$$

and therefore, $r(t)(y'(t))^{q(t)-1}$ is positive decreasing. Integrating both sides of this inequality from $t \geq t_2$ to $T \geq t$, we have

$$r(t)(y'(t))^{q(t)-1} \geq r(T)(y'(T))^{q(T)-1} + \int_t^T c(s)(y(s))^{q(s)-1}\,ds.$$

Taking limit $T \to \infty$, we can find $c_0 \geq 0$ such that

$$r(t)(y'(t))^{q(t)-1} \geq c_0 + \int_t^\infty c(s)(y(s))^{q(s)-1}\,ds.$$

Thus, from Lemma 2.2, (1.5) has a positive solution, which is a contradiction. □

Proof of Theorem 1.2. Suppose that (1.1) has a positive solution $x(t)$. Then, from Lemma 2.1, there exists $t_0 > 1$ such that $x(t) > 0$ and $x'(t) > 0$ for $t \geq t_0$. Let

$$w(t) = r(t)\Bigl(\frac{x'(t)}{x(t)}\Bigr)^{p(t)-1} > 0.$$

Then we have

$$w'(t) = -c(t) - (p(t)-1)r(t)\Bigl(\frac{w(t)}{r(t)}\Bigr)^{p(t)/(p(t)-1)} - p'(t)(\log x(t))w(t). \tag{3.4}$$

We first consider the case when there exists $t_1 \geq t_0$ such that $x(t) \geq 1$ for $t \geq t_1$. In this case, we have

$$w'(t) \leq -c(t) - (p(t)-1)r(t)\Big(\frac{w(t)}{r(t)}\Big)^{p(t)/(p(t)-1)}$$

because $p(t)$ is nondecreasing. Integrating both sides of this inequality from $T \geq t_0$ to $t \geq T$, we obtain

$$w(t) \leq w(T) - \int_T^t c(s)\,ds - \int_T^t (p(s)-1)r(s)\Big(\frac{w(s)}{r(s)}\Big)^{p(s)/(p(s)-1)}\,ds.$$

From (1.9), there exists $t_2 \geq t_1$ such that

$$w(t) \leq -\int_T^t (p(s)-1)r(s)\Big(\frac{w(s)}{r(s)}\Big)^{p(s)/(p(s)-1)}\,ds < 0$$

for $t \geq t_2$. This is a contradiction to the positivity of $w(t)$.

We next consider the case when $x(t) < 1$ for t sufficiently large. In this case, we get $w(t) \to 0$ as $t \to \infty$ because $x(t)$ is strictly increasing, $p(t) > 1$ is nondecreasing, and $x(t)$, $p(t)$, and $r(t)$ are bounded above. Since $\log x(t) < 0$ for t sufficiently large, there exist $t_3 \geq t_0$ and $c_1 > 0$ such that

$$-c_1 \leq p'(t)(\log x(t))w(t) \leq 0$$

for $t \geq t_3$. Integrating both sides of (3.4) from $T \geq t_3$ to $t \geq T$, we get

$$\begin{aligned} w(t) &\leq w(T) - \int_T^t c(s)\,ds - \int_T^t (p(s)-1)r(s)\Big(\frac{w(s)}{r(s)}\Big)^{p(s)/(p(s)-1)}\,ds + \int_T^t c_1\,ds \\ &= w(T) - c_1T + t\Big(c_1 - \frac{1}{t}\int_T^t c(s)\,ds\Big) \\ &\quad - \int_T^t (p(s)-1)r(s)\Big(\frac{w(s)}{r(s)}\Big)^{p(s)/(p(s)-1)}\,ds. \end{aligned}$$

From (1.9), there exists $t_4 \geq t_3$ such that

$$w(t) \leq -\int_T^t (p(s)-1)r(s)\Big(\frac{w(s)}{r(s)}\Big)^{p(s)/(p(s)-1)}\,ds < 0$$

for $t \geq t_4$. This is a contradiction to the positivity of $w(t)$. □

4. Examples and discussion

From Theorem 1.1 and [9, Theorem 1.2], we have the following example.

Example 4.1. Assume (H_1), (H_2), and (H_3). We see that if $\lambda < \gamma(p_*)$ then equation

$$(|x'|^{q(t)-2}x')' + \frac{\lambda}{t^{p(t)}}|x|^{q(t)-2}x = 0$$

has a nonoscillatory solution, where $\gamma(p_*) = ((p_* - 1)/p_*)^{p_*}$. We note that, by Lemma 2.1, it is a positive increasing solution. In addition, if there exists $0 < M < e$ such that (1.8) holds for t sufficiently large, then it tends to infinity as $t \to \infty$.

In the case when $p(t) \equiv p > 1$, in view of Lemma 2.1, we can easily get the following corollary.

Corollary 4.1. *Assume* (1.2), (H_2), *and* $q_* < p$. *If* (1.3) *has a nonoscillatory solution, then* (1.5) *has a positive increasing solution.*

Let us consider the equation

$$\left(\left(a_1 + \frac{a_2}{\log^2 t}\right)^{1-p}|x'|^{p-2}x'\right)' + \left(b_1 + \sin t^2 + \frac{b_2}{\log^2 t}\right)\frac{1}{t^p}|x|^{p-2}x = 0, \tag{4.1}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ satisfy $a_1 > 0$, $b_1 > 0$. According to [5], we see that all nontrivial solutions of (4.1) are oscillatory if $a_1^{p-1}b_1 > \tilde{p}^{-p}$, and are nonoscillatory if $a_1^{p-1}b_1 < \tilde{p}^{-p}$, where $\tilde{p} = p/(p-1)$. Furthermore, in the case when $a_1^{p-1}b_1 = \tilde{p}^{-p}$, all nontrivial solutions of (4.1) are oscillatory if

$$p\frac{a_2}{a_1} + \tilde{p}\frac{b_2}{b_1} > \frac{\tilde{p}^2}{2},$$

and are nonoscillatory if

$$p\frac{a_2}{a_1} + \tilde{p}\frac{b_2}{b_1} < \frac{\tilde{p}^2}{2}. \tag{4.2}$$

Hence, using Corollary 4.1, we obtain the following example.

Example 4.2. Assume (1.2), (H_2), and $b_1 > 1$. If either $a_1^{p-1}b_1 < \tilde{p}^{-p}$ holds for some $p > q_*$, or $a_1^{p-1}b_1 = \tilde{p}^{-p}$ with (4.2) holds for some $p > q_*$, then the equation

$$\left(\left(a_1 + \frac{a_2}{\log^2 t}\right)^{1-p}|x'|^{q(t)-2}x'\right)' + \left(b_1 + \sin t^2 + \frac{b_2}{\log^2 t}\right)\frac{1}{t^p}|x|^{q(t)-2}x = 0$$

has a positive increasing solution.

Moreover, from Corollary 4.1, we can give the following application.

COROLLARY 4.2. *Assume* (H_2). *If*

$$\int_1^\infty s^{p-1} c(s)\, ds < \infty \tag{4.3}$$

holds for some $p > q_*$, *then the equation*

$$(|x'|^{q(t)-2}x')' + c(t)|x|^{q(t)-2}x = 0 \tag{4.4}$$

has a positive increasing solution.

PROOF. From [7, Theorem 2.2.8], all nontrivial solutions of (1.3) with (4.3) are nonoscillatory. Hence, together with Corollary 4.1, we see that (4.4) has a positive increasing solution. □

EXAMPLE 4.3. Assume (H_2). Consider the equation

$$(|x'|^{p(t)-2}x')' + \frac{\lambda}{t^{p+1}}|x|^{p(t)-2}x = 0, \tag{4.5}$$

where $p > q_*$ and $\lambda > 0$. From Corollary 4.2, for any $\lambda > 0$, (4.5) has a positive increasing solution.

Using Theorem 1.2, we also have the following example.

EXAMPLE 4.4. Assume (1.2) and (H_1). Consider the equation

$$(|x'|^{p(t)-2}x')' + (\log t)|x|^{p(t)-2}x = 0. \tag{4.6}$$

From Theorem 1.2, all nontrivial solutions of (4.6) are oscillatory.

Finally, we propose the following open problems.

(1) Extend Theorem 1.1 to the case of $p_* \geq q_*$.

(2) Does Theorem 1.1 hold when $p(t)$ is nonincreasing and $q(t)$ is nondecreasing?

(3) Yoshida [18, Theorem 3.1] studied the equation of the form

$$(r(t)|x'|^{p(t)-2}x')' - p'(t)r(t)|x'|^{p(t)-2}x' \log|x| + c(t)|x|^{p(t)-2}x = 0.$$

Picone identities have been applied to a comparison theorem. However, due to the expression $\log|x|$, the structure of solutions of this equation and one of (1.1) differs considerably. It is an open problem to give the Sturmian comparison theorem for (1.1).

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