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A NOTE ON PRODUCT SETS OF RANDOM SETS

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Abstract. Given two sets of positive integers A and B, let $AB := \{ab : a \in A,\, b \in B\}$ be their *product set* and put $A^k := A \cdots A$ (k times A) for any positive integer k. Moreover, for every positive integer n and every $\alpha = \alpha(n) \in [0,1]$, let $\mathcal{B}(n,\alpha)$ denote the probabilistic model in which a random set $A \subseteq \{1,\dots,n\}$ is constructed by choosing independently every element of $\{1,\dots,n\}$ with probability α. We prove that if $A_1, \dots, A_s$ are random sets in $\mathcal{B}(n_1,\alpha_1),\dots,\mathcal{B}(n_s,\alpha_s)$, respectively, $k_1,\dots,k_s$ are fixed positive integers, $\alpha_i n_i \to +\infty$, and $1/\alpha_i$ does not grow too fast in terms of a product of $\log n_j$; then $|A_1^{k_1} \cdots A_s^{k_s}| \sim \frac{|A_1|^{k_1}}{k_1!} \cdots \frac{|A_s|^{k_s}}{k_s!}$ with probability $1 - o(1)$. This is a generalization of a result of Cilleruelo, Ramana, and Ramaré [3], who considered the case $s = 1$ and $k_1 = 2$.

1. Introduction

Given two sets of positive integers A and B, let $AB := \{ab : a \in A,\, b \in B\}$ be their *product set* and put $A^k := A \cdots A$ (k times A) for any positive integer k.

Problems involving the cardinalities of product sets have been considered by many researchers. For example, the study of $M_n := |\{1,\dots,n\}^2|$ as $n \to +\infty$ is known as the "multiplicative table problem" and was started by Erdős [5,6]. The exact order of magnitude of M_n was determined by Ford [7] following an earlier work of Tenenbaum [12]. Furthermore, Koukoulopoulos [10] provided uniform bounds for $|\{1,\dots,n_1\}\cdots\{1,\dots,n_s\}|$ holding for a wide range of $n_1,\dots,n_s$. Cilleruelo, Ramana, and Ramaré [3] proved asymptotics or bounds for $|(A \cap \{1,\dots,n\})^2|$ when A is the set of shifted prime numbers, the set of sums of two squares, or the set of shifted sums of two squares.

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For every positive integer n and every $\alpha = \alpha(n) \in [0,1]$, let $\mathcal{B}(n,\alpha)$ denote the probabilistic model in which a random set $A \subseteq \{1,\dots,n\}$ is constructed by choosing independently every element of $\{1,\dots,n\}$ with probability α. Number-theoretic problems involving this probabilistic model have been considered by several authors [1–4,11]. In particular, Cilleruelo, Ramana, and Ramaré [3] proved the following:

THEOREM 1.1. *Let A be a random set in $\mathcal{B}(n,\alpha)$. If $\alpha n \to +\infty$ and $\alpha = o((\log n)^{-1/2})$, then $|A^2| \sim \frac{|A|^2}{2}$ with probability $1 - o(1)$.*

The contribution of this paper is the following generalization of Theorem 1.1.

THEOREM 1.2. *Let $A_1, \dots, A_s$ be random sets in $\mathcal{B}(n_1,\alpha_1), \dots, \mathcal{B}(n_s,\alpha_s)$, respectively; and let $k_1, \dots, k_s$ be fixed positive integers. If $\alpha_i n_i \to +\infty$ and*

$$\alpha_i = o\Bigg(\bigg((\log n_1)^{k_1-1}\prod_{i=2}^{s}(\log n_i)^{k_i}\bigg)^{-(k_1+\dots+k_s-1)/2}\Bigg),$$

for $i = 1,\dots,s$, then $|A_1^{k_1}\cdots A_s^{k_s}| \sim \frac{|A_1|^{k_1}}{k_1!}\cdots\frac{|A_s|^{k_s}}{k_s!}$ with probability $1 - o(1)$.

2. Notation

We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbol $\ll$, with their usual meanings. Any dependence of implied constants is explicitly stated or indicated with subscripts. For real random variables X and Y, we say that "$X = o(Y)$ with probability $1 - o(1)$" if $\mathbb{P}(|X| \geq \varepsilon|Y|) = o_\varepsilon(1)$ for every $\varepsilon > 0$, and that "$X \sim Y$ with probability $1 - o(1)$" if $X = Y + o(Y)$ with probability $1 - o(1)$.

3. Preliminaries

In this section we collect some preliminary results not directly related with product sets.

The next lemma is an upper bound on the number of matrices of positive integers with bounded products of rows and columns.

LEMMA 3.1. *Let m and n be positive integers. Then, for all $x_1,\dots,x_n$, $y_1,\dots,y_m \geq 2$, the number of $m \times n$ matrices $(c_{i,j})$ of positive integers satisfying $\prod_{i=1}^{m} c_{i,h} \leq x_h$ and $\prod_{j=1}^{n} c_{k,j} \leq y_k$, for $h = 1,\dots,n$ and $k = 1,\dots,m$, is at most*

$$O_{m,n}\Bigg(\bigg(\prod_{i=1}^{n} x_i \prod_{j=1}^{m} y_j\bigg)^{1/2}\bigg(\prod_{i=1}^{n-1}\log x_i\bigg)^{m-1}\Bigg) \tag{1}$$

Proof. We follow the same arguments of [8, p. 380], where the case $m = n$ and $x_1 = \cdots = x_n = y_1 = \cdots = y_m$ is proved.

The number of choices for $c_{m,n}$ is at most

$$\min\left(\frac{x_n}{\prod_{i=1}^{m-1} c_{i,n}}, \frac{y_m}{\prod_{j=1}^{n-1} c_{m,j}}\right) \leq \left(\frac{x_n y_m}{\prod_{i=1}^{m-1} c_{i,n} \prod_{j=1}^{n-1} c_{m,j}}\right)^{1/2}.$$

We shall sum this latter quantity over all the choices of $c_{i,n}$ and $c_{m,j}$, with $i = 1, \dots, m-1$ and $j = 1, \dots, n-1$. Since $c_{i,n} \leq y_i / \prod_{k=1}^{n-1} c_{i,k}$ and $c_{m,j} \leq x_j / \prod_{h=1}^{m-1} c_{h,j}$, we have

$$\sum_{c_{i,n}} \frac{1}{c_{i,n}^{1/2}} \ll \left(\frac{y_i}{\prod_{k=1}^{n-1} c_{i,k}}\right)^{1/2} \quad \text{and} \quad \sum_{c_{m,j}} \frac{1}{c_{m,j}^{1/2}} \ll \left(\frac{x_j}{\prod_{h=1}^{m-1} c_{h,j}}\right)^{1/2},$$

for $i = 1, \dots, m-1$ and $j = 1, \dots, n-1$. Consequently,

$$\begin{aligned} &\sum_{\substack{c_{1,n}, \dots, c_{m-1,n} \\ c_{m,1}, \dots, c_{m,n-1}}} \left(\frac{x_n y_m}{\prod_{i=1}^{m-1} c_{i,n} \prod_{j=1}^{n-1} c_{m,j}}\right)^{1/2} \\ &\leq (x_n y_m)^{1/2} \prod_{i=1}^{m-1} \left(\sum_{c_{i,n}} \frac{1}{c_{i,n}^{1/2}}\right) \prod_{j=1}^{n-1} \left(\sum_{c_{m,j}} \frac{1}{c_{m,j}^{1/2}}\right) \\ &\ll_{m,n} \left(\prod_{j=1}^{n} x_j \prod_{i=1}^{m} y_i\right)^{1/2} \left(\prod_{h=1}^{m-1} \prod_{k=1}^{n-1} c_{h,k}\right)^{-1}. \end{aligned}$$

It remains only to sum over all the possibilities for $c_{h,k}$, with $h = 1, \dots, m-1$ and $k = 1, \dots, n-1$. We have

$$\begin{aligned} \sum_{c_{h,k}} \left(\prod_{h=1}^{m-1} \prod_{k=1}^{n-1} c_{h,k}\right)^{-1} &\leq \prod_{k=1}^{n-1} \sum_{c_{1,k} \cdots c_{m-1,k} \leq x_k} \frac{1}{c_{1,k} \cdots c_{m-1,k}} \\ &\ll_{m,n} \left(\prod_{k=1}^{n-1} \log x_k\right)^{m-1}, \end{aligned}$$

and the desired result follows. $\square$

The next lemma is an upper bound for the number of solutions of a certain multiplicative equation with bounded factors.

Lemma 3.2. *Let m and n be positive integers. Then, for all $x_1, \dots, x_n$, $y_1, \dots, y_m \geq 2$, the number of solutions of the equation $a_1 \cdots a_n = b_1 \cdots b_m$,* *where* $a_1, \dots, a_n, b_1, \dots, b_m$ *are positive integers satisfying* $a_i \leq x_i$ *and* $b_j \leq y_j$, *for* $i = 1, \dots, n$ *and* $j = 1, \dots, m$, *is at most* (1).

PROOF. If $a_1 \cdots a_n = b_1 \cdots b_m$ then there exists an $m \times n$ matrix of positive integers $(c_{i,j})$ such that $a_h = \prod_{i=1}^m c_{i,h}$ and $b_k = \prod_{j=1}^n c_{k,j}$, for $h = 1, \dots, n$ and $k = 1, \dots, m$. Indeed, $a_1 \mid \prod_{i=1}^m b_i$ implies the existence of positive integers $c_{1,1}, \dots, c_{m,1}$ such that $a_1 = \prod_{i=1}^m c_{i,1}$ and $c_{i,1} \mid b_i$, for $i = 1, \dots, m$. Then $a_2 \mid \prod_{i=1}^m b_i / c_{i,1}$, which similary implies the existence of positive integers $c_{1,2}, \dots, c_{m,2}$ such that $a_2 = \prod_{i=1}^m c_{i,2}$ and $c_{i,1} c_{i,2} \mid b_i$, for $i = 1, \dots, m$. Then $a_3 \mid \prod_{i=1}^m b_i / (c_{i,1} c_{i,2})$, and so on, until $a_n = \prod_{i=1}^m b_i / (\prod_{j=1}^{n-1} c_{i,j})$, when we set $c_{i,n} := b_i / \prod_{j=1}^{n-1} c_{i,j}$ for $i = 1, \dots, m$. Applying Lemma 3.1 we get the desired result. $\square$

4. Proof of Theorem 1.2

The proof proceeds similarly to that of Theorem 1.1. The main idea is to give an asymptotic formula and an upper bound for the expectation and the variance of $|A_1^{k_1} \cdots A_s^{k_s}|$, respectively, and then conclude by an application of Markov's inequality. Some differences are that, in the computation of the expectation of $|A_1^{k_1} \cdots A_s^{k_s}|$, we have to replace [3, Lemma 2.1] with the more general Lemma 3.2, and some additional work is needed to deal with both the exponents k_i and the s factors of the product.

First, we need an asymptotic for the kth power of the size of a random set A in $\mathcal{B}(n, \alpha)$.

LEMMA 4.1. *Let* A *be a random set in* $\mathcal{B}(n, \alpha)$, *and fix an integer* $k \geq 1$. *If* $\alpha n \to +\infty$, *then*
(i) $\mathbb{E}(|A|^k) \sim (\alpha n)^k$; *and*
(ii) $|A|^k \sim (\alpha n)^k$ *with probability* $1 - o_k(1)$.

PROOF. Clearly, $|A|$ follows a binomial distribution with n trials and probability of success α. Consequently, (i) is known (see, e.g., [9, Eq. (4.1)]). In turn, (i) implies that

$$\mathbb{V}(|A|^k) = \mathbb{E}(|A|^{2k}) - \mathbb{E}(|A|^k)^2 = o_k(\mathbb{E}(|A|^k)^2).$$

Hence, by Chebyshev's inequality, for every $\varepsilon > 0$ we have

$$\mathbb{P}\left(\left| |A|^k - \mathbb{E}(|A|^k) \right| \geq \varepsilon \, \mathbb{E}(|A|^k) \right) \leq \frac{\mathbb{V}(|A|^k)}{(\varepsilon \, \mathbb{E}(|A|^k))^2} = o_{k,\varepsilon}(1),$$

so that $|A|^k \sim \mathbb{E}(|A|^k) \sim (\alpha n)^k$ with probability $1 - o_k(1)$. $\square$

The next lemma is an easy bound on the size of a product set.

Lemma 4.2. *Let $A_1, \ldots, A_s$ be finite sets of positive integers, and let $k_1, \ldots, k_s \geq 1$ be integers. Then*

$$\left|\prod_{i=1}^{s} A_i^{k_i}\right| \leq \prod_{i=1}^{s} \binom{|A_i| + k_i - 1}{k_i}.$$

Proof. The claim follows easily considering that $\binom{|A|+k-1}{k}$ is the number of unordered k-tuples of elements from a set A. $\square$

For the rest of this section, let $A_1, \ldots, A_s$ be random sets in $\mathcal{B}(n_1, \alpha_1), \ldots, \mathcal{B}(n_s, \alpha_s)$, respectively; and let $k_1, \ldots, k_s$ be fixed positive integers. Also, assume $\alpha_i n_i \to +\infty$ and

$$\alpha_i = o\bigg(\Big((\log n_1)^{k_1 - 1} \prod_{i=2}^{s} (\log n_i)^{k_i}\Big)^{-(k_1 + \cdots + k_s - 1)/2}\bigg), \tag{2}$$

for $i = 1, \ldots, s$. For brevity, we will omit the dependence of implied constants from $k_1, \ldots, k_s$.

Lemma 4.3. *We have $\mathbb{E}(|A_1^{k_1} \cdots A_s^{k_s}|) \sim \frac{(\alpha_1 n_1)^{k_1}}{k_1!} \cdots \frac{(\alpha_s n_s)^{k_s}}{k_s!}$.*

Proof. Hereafter, in operator subscripts, let $\mathbf{a} := (\mathbf{a}_1, \ldots, \mathbf{a}_s)$, where each $\mathbf{a}_i := \{a_{i,1}, \ldots, a_{i,k_i}\}$ runs over the unordered k_i-tuples of elements of $\{1, \ldots, n_i\}$. Also, put $\|\mathbf{a}\| := \prod_{i=1}^{s} \prod_{j=1}^{k_i} a_{i,j}$. With this notation, for each positive integer x, we have

$$\mathbb{P}(x \in A_1^{k_1} \cdots A_s^{k_s}) = \mathbb{P}\bigg(\bigvee_{\|\mathbf{a}\| = x} E_{\mathbf{a}}\bigg),$$

where

$$E_{\mathbf{a}} := \bigwedge_{i=1}^{s} (\mathbf{a}_i \subseteq A_i).$$

Consequently, by Bonferroni inequalities, we have

$$\mathbb{P}(x \in A_1^{k_1} \cdots A_s^{k_s}) = \mathbb{P}\bigg(\sideset{}{^*}\bigvee_{\|\mathbf{a}\| = x} E_{\mathbf{a}}\bigg) + O\bigg(\sideset{}{^{**}}\sum_{\|\mathbf{a}\| = x} \mathbb{P}(E_{\mathbf{a}})\bigg)$$

$$= \sideset{}{^*}\sum_{\|\mathbf{a}\| = x} \mathbb{P}(E_{\mathbf{a}}) + O\bigg(\sideset{}{^*}\sum_{\substack{\mathbf{a} \neq \mathbf{a}' \\ \|\mathbf{a}\| = \|\mathbf{a}'\| = x}} \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'})\bigg) + O\bigg(\sideset{}{^{**}}\sum_{\|\mathbf{a}\| = x} \mathbb{P}(E_{\mathbf{a}})\bigg),$$

where the superscript * denotes the constraint $|\mathbf{a}_i| = k_i$ for every $i \in \{1, \ldots, s\}$, the superscript ** denotes the complementary constrain $|\mathbf{a}_i| < k_i$

for at least one $i \in \{1, \ldots, k\}$, and $\mathbf{a}' := (\mathbf{a}'_1, \ldots, \mathbf{a}'_s)$ follows the same conventions of $\mathbf{a}$. Therefore,

$$\mathbb{E}(|A_1^{k_1} \cdots A_s^{k_s}|) = \sum_{x \leq n_1^{k_1} \cdots n_s^{k_s}} \mathbb{P}(x \in A_1^{k_1} \cdots A_s^{k_s}) \tag{3}$$

$$= \sum_{\mathbf{a}}^{*} \mathbb{P}(E_{\mathbf{a}}) + O\Bigg(\sum_{\substack{\mathbf{a} \neq \mathbf{a}' \\ \|\mathbf{a}\| = \|\mathbf{a}'\|}}^{*} \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) \Bigg) + O\Bigg(\sum_{\mathbf{a}}^{**} \mathbb{P}(E_{\mathbf{a}}) \Bigg).$$

Since $A_1, \ldots, A_s$ are independent and each A_i belongs to $\mathcal{B}(n_i, \alpha_i)$, we have

$$\mathbb{P}(E_{\mathbf{a}}) = \prod_{i=1}^{s} \mathbb{P}(\mathbf{a}_i \subseteq A_i) = \prod_{i=1}^{s} \alpha_i^{|\mathbf{a}_i|}.$$

Hence, for every positive integers $m_1, \ldots, m_s$, with $m_i \leq k_i$, we have

$$\sum_{\mathbf{a}: |\mathbf{a}_i| = m_i} \mathbb{P}(E_{\mathbf{a}}) = \sum_{\mathbf{a}: |\mathbf{a}_i| = m_i} \prod_{i=1}^{s} \alpha_i^{m_i}$$

$$= \prod_{i=1}^{s} \alpha_i^{m_i} \sum_{|\mathbf{a}_i| = m_i} 1 = \prod_{i=1}^{s} \alpha_i^{m_i} \binom{n_i}{m_i} \binom{k_i - 1}{m_i - 1},$$

where we used the fact that the number of unordered k-tuples of elements of $\{1, \ldots, n\}$ having cardinality equal to m is $\binom{n}{m}\binom{k-1}{m-1}$. Therefore,

$$\sum_{\mathbf{a}}^{*} \mathbb{P}(E_{\mathbf{a}}) \sim \prod_{i=1}^{s} \frac{(\alpha_i n_i)^{k_i}}{k_i!} \quad \text{and} \quad \sum_{\mathbf{a}}^{**} \mathbb{P}(E_{\mathbf{a}}) = o\Bigg(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i} \Bigg), \tag{4}$$

as $\alpha_i n_i \to +\infty$, for $i = 1, \ldots, s$. We have

$$\mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) = \prod_{i=1}^{s} \mathbb{P}(\mathbf{a}_i \cup \mathbf{a}'_i \subseteq A_i) = \prod_{i=1}^{s} \alpha_i^{|\mathbf{a}_i \cup \mathbf{a}'_i|}. \tag{5}$$

Suppose that $\mathbf{a}$ and $\mathbf{a}'$, with $\mathbf{a} \neq \mathbf{a}'$ and $\|\mathbf{a}\| = \|\mathbf{a}'\|$, satisfy the condition of *, that is, $|\mathbf{a}_i| = |\mathbf{a}'_i| = k_i$ for $i = 1, \ldots, s$. We shall find an upper bound for (5). Clearly, $|\mathbf{a}_i \cup \mathbf{a}'_i| \geq |\mathbf{a}_i| \geq k_i$ for $i = 1, \ldots, s$. Moreover, since $\mathbf{a} \neq \mathbf{a}'$, there exists $i_1 \in \{1, \ldots, s\}$ such that $\mathbf{a}_{i_1} \neq \mathbf{a}'_{i_1}$. Since $|\mathbf{a}_{i_1}| = |\mathbf{a}'_{i_1}| = k_i$, it follows that $|\mathbf{a}_{i_1} \cup \mathbf{a}'_{i_1}| \geq k_{i_1} + 1$. On the one hand, if there exists $i_2 \in \{1, \ldots, s\} \setminus \{i_1\}$ such that $\mathbf{a}_{i_2} \neq \mathbf{a}'_{i_2}$, then, similarly, we have $|\mathbf{a}_{i_2} \cup \mathbf{a}'_{i_2}| \geq k_{i_2} + 1$. Hence,

$$\mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) \leq \alpha_{i_1} \alpha_{i_2} \prod_{i=1}^{s} \alpha_i^{k_i}.$$

On the other hand, if $\mathbf{a}_i = \mathbf{a}'_i$ for every $i \in \{1, \dots, s\} \setminus \{i_1\}$, then from $\|\mathbf{a}\| = \|\mathbf{a}'\|$ it follows that $\prod_{j=1}^{k_{i_1}} a_{i_1,j} = \prod_{j=1}^{k_{i_1}} a'_{i_1,j}$. In turn, this implies that $|\mathbf{a}_{i_1} \cup \mathbf{a}'_{i_1}| \geq k_{i_1} + 2$. Hence,

$$\mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) \leq \alpha_{i_1}^2 \prod_{i=1}^{s} \alpha_i^{k_i}.$$

Therefore, using Lemma 3.2 and recalling (2), we obtain

$$\sum_{\substack{\mathbf{a} \neq \mathbf{a}' \\ \|\mathbf{a}\| = \|\mathbf{a}'\|}}^{*} \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) \leq \Big(\max_{1 \leq i,j \leq s} \alpha_i \alpha_j\Big) \prod_{i=1}^{s} \alpha_i^{k_i} \sum_{\|\mathbf{a}\| = \|\mathbf{a}'\|} 1 \tag{6}$$

$$\ll \Big(\max_{1 \leq i,j \leq s} \alpha_i \alpha_j\Big) \bigg((\log n_1)^{k_1 - 1} \prod_{i=2}^{s} (\log n_i)^{k_i}\bigg)^{k_1 + \dots + k_s - 1} \prod_{i=1}^{s} (\alpha_i n_i)^{k_i}$$

$$= o\bigg(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\bigg).$$

Finally, putting together (3), (4), and (6), we obtain the desired claim. $\square$

Proof of Theorem 1.2. Define the random variable

$$X := \prod_{i=1}^{s} \binom{|A_i| + k_i - 1}{k_i} - \bigg|\prod_{i=1}^{s} A_i^{k_i}\bigg|.$$

Thanks to Lemma 4.2, we know that X is nonnegative. Moreover, from Lemma 4.1(i) and Lemma 4.3, it follows that

$$\mathbb{E}(X) = o\bigg(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\bigg).$$

Hence, for every $\varepsilon > 0$, by Markov's inequality, we get

$$\mathbb{P}\bigg(X \geq \varepsilon \prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\bigg) \leq \frac{\mathbb{E}(X)}{\varepsilon \prod_{i=1}^{s} (\alpha_i n_i)^{k_i}} = o_\varepsilon(1),$$

which in turn implies $X = o\left(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\right)$ with probability $1 - o(1)$. Therefore, by Lemma 4.1(ii),

$$\bigg|\prod_{i=1}^{s} A_i^{k_i}\bigg| = \prod_{i=1}^{s} \binom{|A_i| + k_i - 1}{k_i} - X = \prod_{i=1}^{s} \frac{|A_i|^{k_i}}{k_i!} + o\bigg(\prod_{i=1}^{s} |A_i|^{k_i}\bigg),$$

with probability $1 - o(1)$, as claimed. $\square$

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