Acta Math. Hungar., **159** (**1**) (2019), 150–163
https://doi.org/10.1007/s10474-019-00973-y
First published online September 5, 2019

SUMS OF FOUR PRIME CUBES IN SHORT INTERVALS

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(Received July 12, 2018; revised April 1, 2019; accepted May 20, 2019)

Abstract. We prove that a suitable asymptotic formula for the average number of representations of integers $n = p_1^3 + p_2^3 + p_3^3 + p_4^3$, where p_1, p_2, p_3, p_4 are prime numbers, holds in intervals shorter than the the ones previously known.

1. Introduction

Let N be a sufficiently large integer and $1 \le H \le N$ an integer. Let

$$(1)\qquad \sum_{n=N+1}^{N+H} r(n), \quad \text{where } r(n) = \sum_{n=p_1^3+p_2^3+p_3^3+p_4^3} \log p_1 \log p_2 \log p_3 \log p_4,$$

be a suitable short interval average of the number of representation of an integer as a sum of four prime cubes. The problem of representing integers as sum of prime cubes is quite an old one; we recall that Hua [4,5] stated that almost all positive integers satisfying some necessary congruence conditions are the sum of five prime cubes and that Davenport [2] proved that almost all positive integers are the sum of four positive cubes. More recent results on the positive proportions of integers that are the sum of four prime cubes were obtained by Roth [13], Ren [11] and Liu [9]. In fact, see Brüdern [1], it is conjectured that all sufficiently large integers satisfying some necessary congruence conditions are the sum of four prime cubes. Here we prove that

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Key words and phrases: Waring–Goldbach problem, Hardy–Littlewood method.
Mathematics Subject Classification: primary 11P32, secondary 11P55, 11P05.

Theorem 1. *Let $N \geq 2$, $1 \leq H \leq N$ be integers. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that*

$$\sum_{n=N+1}^{N+H} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 HN^{1/3} + \mathcal{O}\Big(HN^{1/3}\exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$$

as $N \to \infty$, uniformly for $N^{13/18+\varepsilon} \leq H \leq N^{1-\varepsilon}$, where Γ is Euler's function.

This should be compared with a recent result about the positive proportion of such integers in short intervals by Liu–Zhao [8] which holds for $H = N^{17/18}$. As an immediate consequence of Theorem 1 we can say that, for N sufficiently large, every interval of size larger than $N^{13/18+\varepsilon}$ contains the expected amount of integers which are a sum of four prime cubes. We remark that this level is essentially optimal given the known density estimates for the non trivial zeroes of the Riemann zeta function. Assuming the Riemann Hypothesis (RH) holds we can further improve the size of H.

Theorem 2. *Let $\varepsilon > 0$, $N \geq 2$, $1 \leq H \leq N$ be integers and assume the Riemann Hypothesis holds. Then there exists a constant $B \geq 3/2$ such that*

$$\sum_{n=N+1}^{N+H} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 HN^{1/3}$$

$$+ \mathcal{O}\bigg(\frac{H^2}{N^{2/3}} + H^{3/4}N^{5/12+\varepsilon} + H^{1/2}N^{2/3}(\log N)^B + N(\log N)^3\bigg)$$

as $N \to \infty$, uniformly for $\infty(N^{2/3}L^{2B}) \leq H \leq o(N)$, where $f = \infty(g)$ means $g = o(f)$ and Γ is Euler's function.

As an immediate consequence of Theorem 2 we can say that, for N sufficiently large, every interval of size larger than $N^{2/3+\varepsilon}$ contains the expected amount of integers which are a sum of four prime cubes. We remark that this level is essentially optimal given the spacing of the cubic sequence.

In both the proofs of Theorems 1-2 we will use the original Hardy–Littlewood generating functions to exploit the easier main term treatment they allow (comparing with the one which would follow using Lemmas 2.3 and 2.9 of Vaughan [14]).

Though technically harder to handle, the infinite-series approach yields more precise results than the finite-sum one; furthermore, it is slightly easier to deal with the main term. A key tool is the appropriate version of Hua's Lemma (see Lemma 5 below), which was originally proved only for finite sums. In the next section we define the most important functions that we need, and then we briefly sketch the plan of the proof.

2. Setting

Let $e(\alpha) = e^{2\pi i\alpha}$, $\alpha \in [-1/2, 1/2]$, $L = \log N$, $z = 1/N - 2\pi i\alpha$,

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha) \quad \text{and} \quad \widetilde{V}_\ell(\alpha) = \sum_{p=2}^{\infty} \log p \, e^{-p^\ell/N} e(p^\ell \alpha).$$

We remark that

$$|z|^{-1} \ll \min\bigl(N, |\alpha|^{-1}\bigr). \tag{2}$$

We further set

$$U(\alpha, H) = \sum_{m=1}^{H} e(m\alpha)$$

and, moreover, we also have the usual numerically explicit inequality

$$|U(\alpha, H)| \le \min\bigl(H; |\alpha|^{-1}\bigr), \tag{3}$$

see, e.g., Montgomery [10, p. 39]. An easy computation now shows that we have

$$\sum_{n=N+1}^{N+H} e^{-n/N} r(n) = \int_{-1/2}^{1/2} \widetilde{V}_3(\alpha)^4 U(-\alpha, H) e(-N\alpha)\, \mathrm{d}\alpha.$$

This identity is our starting point. Replacing $\widetilde{V}_3$ by $\widetilde{S}_3$ gives a small error term (see Lemma 1); next, we basically replace $\widetilde{S}_3$ by its expected main term $\Gamma(1/3)/(3z^{1/3})$, as suggested by Lemmas 2 and 4. In the unconditional case we can not use Lemma 4 on the whole integration interval $[-1/2, 1/2]$, and we need a different type of bound on the "periphery." These considerations lead to the decomposition in (5). We give a more detailed explanation at the beginning of Section 3.

We now list the needed preliminary results.

Lemma 1 [6, Lemma 3]. *Let $\ell \ge 1$ be an integer. Then* $|\widetilde{S}_\ell(\alpha) - \widetilde{V}_\ell(\alpha)| \ll_\ell N^{1/(2\ell)}$.

Lemma 2. *Let $\ell \ge 1$ be an integer, $N \ge 2$ and $\alpha \in [-1/2, 1/2]$. Then*

$$\widetilde{S}_\ell(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma(\rho/\ell) + \mathcal{O}_\ell(1),$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

PROOF. It follows the line of [7, Lemma 2]; we just correct an oversight in its proof. In equation (5) on p. 48 of [7] the term $-\sum_{m=1}^{\ell\sqrt{3}/4}\Gamma(-2m/\ell)z^{2m/\ell}$ is missing. Its estimate is trivially $\ll_\ell |z|^{\sqrt{3}/2} \ll_\ell 1$. Hence such an oversight does not affect the final result of [7, Lemma 2]. □

LEMMA 3 [7, Lemma 4]. *Let N be a positive integer and $\mu > 0$. Then*

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha)\,\mathrm{d}\alpha = e^{-n/N}\frac{n^{\mu-1}}{\Gamma(\mu)} + \mathcal{O}_\mu\Big(\frac{1}{n}\Big),$$

uniformly for $n \geq 1$.

LEMMA 4. *Let ε be an arbitrarily small positive constant, $\ell \geq 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\xi}^{\xi} \Big|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}\Big|^2 \mathrm{d}\alpha \ll_\ell N^{2/\ell-1}\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big)$$

uniformly for $0 \leq \xi < N^{-1+5/(6\ell)-\varepsilon}$. Assuming RH we get

$$\int_{-\xi}^{\xi} \Big|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}\Big|^2 \mathrm{d}\alpha \ll_\ell N^{1/\ell}\xi L^2$$

uniformly for $0 \leq \xi \leq 1/2$.

PROOF. It follows the line of [7, Lemma 3] and of [6, Lemma 1]; we just correct an oversight in their proofs which is based on Lemma 2 above. Both of [7, eq. (8), p. 49] and [6, eq. (6), p. 423] should read as

$$\int_{1/N}^{\xi} \Big|\sum_{\rho\colon\, \gamma>0} z^{-\rho/\ell}\Gamma(\rho/\ell)\Big|^2 \mathrm{d}\alpha \leq \sum_{k=1}^{K}\int_{\eta}^{2\eta} \Big|\sum_{\rho\colon\, \gamma>0} z^{-\rho/\ell}\Gamma(\rho/\ell)\Big|^2 \mathrm{d}\alpha,$$

where $\eta = \eta_k = \xi/2^k$, $1/N \leq \eta \leq \xi/2$ and K is a suitable integer satisfying $K = \mathcal{O}(L)$. The remaining part of the proofs are left untouched. Hence such oversights do not affect the final result of [7, Lemma 3] and [6, Lemma 1]. □

In the unconditional case a crucial role is played by

LEMMA 5 (Hua). *Let N be sufficiently large, ℓ, k integers, $\ell \geq 1$, $1 \leq k \leq \ell$. There exists a suitable positive constant $A = A(k,\ell)$ such that*

$$\int_{-1/2}^{1/2} |\widetilde{S}_\ell(\alpha)|^{2^k}\,\mathrm{d}\alpha \ll_{k,\ell} N^{(2^k-k)/\ell}L^A$$

and

$$\int_{-1/2}^{1/2} |\widetilde{V}_\ell(\alpha)|^{2^k} \, \mathrm{d}\alpha \ll_{k,\ell} N^{(2^k-k)/\ell} L^A.$$

PROOF. We just prove the first part since the second one follows immediately by remarking that the primes are supported on a thinner set than the prime powers. Let $P = (10NL/\ell)^{1/\ell}$. A direct estimate gives $\widetilde{S}_\ell(\alpha) = \sum_{n \le P} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha) + \mathcal{O}_\ell(1)$. Recalling that the Prime Number Theorem implies $S_\ell(\alpha; t) := \sum_{n \le t} \Lambda(n) e(n^\ell \alpha) \ll t$, a partial integration argument gives

$$\sum_{n \le P} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha) = -\frac{\ell}{N} \int_1^P t^{\ell-1} e^{-t^\ell/N} S_\ell(\alpha; t) \, \mathrm{d}t + \mathcal{O}_\ell(1).$$

Using the inequality $(|a| + |b|)^{2^k} \ll_k |a|^{2^k} + |b|^{2^k}$, Hölder's inequality and interchanging the integrals, we get that

$$\int_{-1/2}^{1/2} |\widetilde{S}_\ell(\alpha)|^{2^k} \, \mathrm{d}\alpha \ll_{k,\ell} \int_{-1/2}^{1/2} \left| \frac{1}{N} \int_1^P t^{\ell-1} e^{-t^\ell/N} S_\ell(\alpha; t) \, \mathrm{d}t \right|^{2^k} \mathrm{d}\alpha + 1$$

$$\ll_{k,\ell} \frac{1}{N^{2^k}} \Big(\int_1^P t^{\ell-1} e^{-t^\ell/N} \, \mathrm{d}t \Big)^{2^k-1} \Big(\int_1^P t^{\ell-1} e^{-t^\ell/N} \int_{-1/2}^{1/2} |S_\ell(\alpha; t)|^{2^k} \, \mathrm{d}\alpha \, \mathrm{d}t \Big) + 1.$$

Theorem 4 of Hua [5] implies, remarking that the von Mangoldt function is supported on a thinner set than the integers and inserts a logarithmic weight whose total contribution can be inserted in the power of L, that there exists a positive constant $A = A(k, \ell)$ such that

$$\int_{-1/2}^{1/2} |S_\ell(\alpha; t)|^{2^k} \, \mathrm{d}\alpha \ll_{k,\ell} t^{2^k-k} (\log t)^A.$$

Using such an estimate and remarking that $\int_1^P t^{\ell-1} e^{-t^\ell/N} \, \mathrm{d}t \ll_\ell N$, we obtain that

$$\int_{-1/2}^{1/2} |\widetilde{S}_\ell(\alpha)|^{2^k} \, \mathrm{d}\alpha$$

$$\ll_{k,\ell} \frac{1}{N} \int_1^P t^{\ell-1+2^k-k} e^{-t^\ell/N} (\log t)^A \, \mathrm{d}t + L^{2^k/\ell} \ll_{k,\ell} N^{(2^k-k)/\ell} L^A$$

by a direct computation. This proves the first part of the lemma. □

In fact the argument used in the proof of Lemma 5 can be used to derive other estimates on $\widetilde{S}_\ell(\alpha)$ from the ones on $S_\ell(\alpha; t)$. Another instance of this fact is the following lemma about the truncated fourth-mean average of $\widetilde{S}_\ell(\alpha)$ which is based on a result by Robert-Sargos [12].

Lemma 6. *Let $N \in \mathbb{N}$, $\varepsilon > 0$, $\ell > 1$ and $\tau > 0$. Then we have*

$$\int_{-\tau}^{\tau} |\widetilde{S}_\ell(\alpha)|^4 \,\mathrm{d}\alpha \ll_\ell \left(\tau N^{2/\ell} + N^{4/\ell-1}\right) N^\varepsilon$$

and

$$\int_{-\tau}^{\tau} |\widetilde{V}_\ell(\alpha)|^4 \,\mathrm{d}\alpha \ll_\ell \left(\tau N^{2/\ell} + N^{4/\ell-1}\right) N^\varepsilon.$$

Proof. We can argue as in the proof of Lemma 5 using [3, Lemma 4] on $S_\ell(\alpha;t) = \sum_{n\le t} \Lambda(n) e(n^\ell \alpha)$ instead of [5, Theorem 4]. □

The last lemma is a consequence of Lemma 6.

Lemma 7. *Let $N \in \mathbb{N}$, $\varepsilon > 0$, $\ell > 2$, $c \ge 1$ and $N^{-c} \le \omega \le N^{2/\ell-1}$. Let further $I(\omega) := [-1/2, -\omega] \cup [\omega, 1/2]$. Then we have*

$$\int_{I(\omega)} |\widetilde{S}_\ell(\alpha)|^4 \frac{\mathrm{d}\alpha}{|\alpha|} \ll_\ell \frac{N^{4/\ell-1+\varepsilon}}{\omega} \quad \textit{and} \quad \int_{I(\omega)} |\widetilde{V}_\ell(\alpha)|^4 \frac{\mathrm{d}\alpha}{|\alpha|} \ll_\ell \frac{N^{4/\ell-1+\varepsilon}}{\omega}.$$

Proof. By partial integration and Lemma 6 we get that

$$\int_{\omega}^{1/2} |\widetilde{S}_\ell(\alpha)|^4 \frac{\mathrm{d}\alpha}{\alpha}$$

$$\ll \frac{1}{\omega} \int_{-\omega}^{\omega} |\widetilde{S}_\ell(\alpha)|^4 \;\mathrm{d}\alpha + \int_{-1/2}^{1/2} |\widetilde{S}_\ell(\alpha)|^4 \;\mathrm{d}\alpha + \int_{\omega}^{1/2} \left(\int_{-\xi}^{\xi} |\widetilde{S}_\ell(\alpha)|^4 \;\mathrm{d}\alpha \right) \frac{\mathrm{d}\xi}{\xi^2}$$

$$\ll_\ell N^{2/\ell+\varepsilon} + N^\varepsilon \int_{\omega}^{1/2} \frac{\xi N^{2/\ell} + N^{4/\ell-1}}{\xi^2} \,\mathrm{d}\xi \ll_\ell \frac{N^{4/\ell-1+\varepsilon}}{\omega}$$

since $N^{-c} \le \omega \le N^{2/\ell-1}$. A similar computation proves the result in $[-1/2, -\omega]$ too. The estimate on $\widetilde{V}_\ell(\alpha)$ can be obtained analogously. □

3. The unconditional case

Let $H > 2B$, where

$$B = N^{2\varepsilon}. \tag{4}$$

Letting $I(B/H) := [-1/2, -B/H] \cup [B/H, 1/2]$, and recalling (1), we have

$$\sum_{n=N+1}^{N+H} e^{-n/N} r(n) = \int_{-1/2}^{1/2} \widetilde{V}_3(\alpha)^4 U(-\alpha, H) e(-N\alpha) \,\mathrm{d}\alpha \tag{5}$$

$$= \int_{-B/H}^{B/H} \widetilde{S}_3(\alpha)^4 U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \int_{I(B/H)} \widetilde{S}_3(\alpha)^4 U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha$$
$$+ \int_{-1/2}^{1/2} \bigl(\widetilde{V}_3(\alpha)^4 - \widetilde{S}_3(\alpha)^4\bigr) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha = I_1 + I_2 + I_3,$$

say. Now we evaluate these terms. The main term will be provided by I_1, which is evaluated below by means of Lemma 3, followed by standard computations needed to bound the difference between $\widetilde{S}_3$ and its "dominant term" $\Gamma(1/3)/(3z^{1/3})$. The other summands are error terms, which are treated by means of a combination of techniques, essentially the Hölder inequality in order to use Lemmas 5, 6 and 7. The next several subsections are devoted to the fairly tedious verification that these error terms do not contribute too much. Finally, we remove the exponential weight on the left-hand side of (5) and we achieve the proof of Theorem 1.

3.1. Estimation of I_2. Using (3) and Lemma 7 with $\omega = B/H$ and $\ell = 3$, we obtain

$$I_2 \ll \int_{I(B/H)} |\widetilde{S}_3(\alpha)|^4 \frac{\mathrm{d}\alpha}{|\alpha|} \ll \frac{H}{B} N^{1/3+\varepsilon}, \tag{6}$$

provided that $H \gg N^{1/3}B$.

3.2. Estimation of I_3. Clearly

$$|\widetilde{V}_3(\alpha)^4 - \widetilde{S}_3(\alpha)^4|$$
$$= |\widetilde{V}_3(\alpha) - \widetilde{S}_3(\alpha)|\,|\widetilde{V}_3(\alpha)^3 + \widetilde{V}_3(\alpha)^2\widetilde{S}_3(\alpha) + \widetilde{V}_3(\alpha)\widetilde{S}_3(\alpha)^2 + \widetilde{S}_3(\alpha)^3|$$
$$\ll |\widetilde{V}_3(\alpha) - \widetilde{S}_3(\alpha)|(|\widetilde{V}_3(\alpha)| + |\widetilde{S}_3(\alpha)|)^3$$
$$\ll |\widetilde{V}_3(\alpha) - \widetilde{S}_3(\alpha)| \max(|\widetilde{V}_3(\alpha)|^3; |\widetilde{S}_3(\alpha)|^3).$$

Hence by Lemma 1 we have

$$I_3 \ll N^{1/6} \int_{-1/2}^{1/2} \bigl(|\widetilde{V}_3(\alpha)|^3 + |\widetilde{S}_3(\alpha)|^3\bigr) |U(-\alpha,H)|\,\mathrm{d}\alpha = N^{1/6}(K_1 + K_2), \tag{7}$$

say. By (3) we get

$$K_2 \ll H \int_{-1/H}^{1/H} |\widetilde{S}_3(\alpha)|^3\,\mathrm{d}\alpha + \int_{I(1/H)} |\widetilde{S}_3(\alpha)|^3 \frac{\mathrm{d}\alpha}{|\alpha|} = K_{2,1} + K_{2,2}, \tag{8}$$

say. Using the Hölder inequality and Lemma 6 with $\tau = 1/H$ and $\ell = 3$ we get

$$K_{2,1} \ll H^{3/4}\Big(\int_{-1/H}^{1/H} |\widetilde{S}_3(\alpha)|^4 \,\mathrm{d}\alpha\Big)^{3/4} \ll H^{3/4}N^{1/4+\varepsilon}, \tag{9}$$

provided that $H \gg N^{1/3}$. Using the Hölder inequality and Lemma 7 with $\omega = 1/H$ and $\ell = 3$ we get

$$\begin{aligned} K_{2,2} &\ll \Big(\int_{I(1/H)} |\widetilde{S}_3(\alpha)|^4 \frac{\mathrm{d}\alpha}{|\alpha|}\Big)^{3/4}\Big(\int_{I(1/H)} \frac{\mathrm{d}\alpha}{|\alpha|}\Big)^{1/4} \\ &\ll \big(HN^{1/3+\varepsilon}\big)^{3/4}L^{1/4} \ll H^{3/4}N^{1/4+\varepsilon}, \end{aligned} \tag{10}$$

provided that $H \gg N^{1/3}$. Combining (8)–(10) we obtain

$$K_2 \ll H^{3/4}N^{1/4+\varepsilon}, \tag{11}$$

provided that $H \gg N^{1/3}$. An analogous computation gives

$$K_1 \ll H^{3/4}N^{1/4+\varepsilon}, \tag{12}$$

and, by (7) and (11)–(12), we can finally write

$$I_3 \ll H^{3/4}N^{5/12+\varepsilon}, \tag{13}$$

provided that $H \gg N^{1/3}$.

3.3. Evaluation of I_1. Since $\Gamma(4/3) = (1/3)\Gamma(1/3)$, we have that

$$\begin{aligned} I_1 = &\int_{-B/H}^{B/H} \frac{\Gamma(4/3)^4}{z^{4/3}} U(-\alpha, H)e(-N\alpha)\,\mathrm{d}\alpha \\ &+ \int_{-B/H}^{B/H} \Big(\widetilde{S}_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}}\Big)U(-\alpha, H)e(-N\alpha)\,\mathrm{d}\alpha = J_1 + J_2, \end{aligned}$$

say. By (2)–(3) and Lemma 3, a direct calculation gives

$$\begin{aligned} J_1 &= \Gamma\Big(\frac{4}{3}\Big)^3 \sum_{n=N+1}^{N+H} e^{-n/N}n^{1/3} + \mathcal{O}\Big(\frac{H}{N}\Big) + \mathcal{O}\Big(\int_{B/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{7/3}}\Big) \\ &= \frac{\Gamma(4/3)^3}{e} \sum_{n=N+1}^{N+H} n^{1/3} + \mathcal{O}\Big(\frac{H}{N} + \frac{H^2}{N^{2/3}} + \frac{H^{4/3}}{B^{4/3}}\Big) \end{aligned} \tag{14}$$

$$= \Gamma\Big(\frac{4}{3}\Big)^3 \frac{HN^{1/3}}{e} + \mathcal{O}\Big(\frac{H^{4/3}}{B^{4/3}} + N^{1/3}\Big).$$

From now on, we denote $\widetilde{E}_3(\alpha) := \widetilde{S}_3(\alpha) - \frac{\Gamma(4/3)}{z^{1/3}}$. By $f^2 - g^2 = 2g(f-g) + (f-g)^2$, (2) and $\widetilde{S}_3(\alpha) \ll N^{1/3}$ we get

$$\widetilde{S}_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}} = \Big(\widetilde{S}_3(\alpha)^2 + \frac{\Gamma(4/3)^2}{z^{2/3}}\Big)\Big(\widetilde{S}_3(\alpha)^2 - \frac{\Gamma(4/3)^2}{z^{2/3}}\Big) \tag{15}$$

$$= \Big(\widetilde{S}_3(\alpha)^2 + \frac{\Gamma(4/3)^2}{z^{2/3}}\Big)\Big(2\frac{\Gamma(4/3)}{z^{1/3}}\widetilde{E}_3(\alpha) + \widetilde{E}_3(\alpha)^2\Big)$$

$$\ll |\widetilde{S}_3(\alpha)|^2 \frac{|\widetilde{E}_3(\alpha)|}{|z|^{1/3}} + \frac{|\widetilde{E}_3(\alpha)|}{|z|} + N^{2/3}|\widetilde{E}_3(\alpha)|^2.$$

Using (15) and (2) we get

$$J_2 \ll H\int_{-B/H}^{B/H} |\widetilde{S}_3(\alpha)|^2 \frac{|\widetilde{E}_3(\alpha)|}{|z|^{1/3}}\,\mathrm{d}\alpha + H\int_{-B/H}^{B/H} \frac{|\widetilde{E}_3(\alpha)|}{|z|}\,\mathrm{d}\alpha \tag{16}$$

$$+ HN^{2/3}\int_{-B/H}^{B/H} |\widetilde{E}_3(\alpha)|^2\,\mathrm{d}\alpha = H(E_1 + E_2 + N^{2/3}E_3),$$

say. By (3) and Lemma 4 we obtain that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$E_3 \ll N^{-1/3}\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \tag{17}$$

provided that $B/H \le N^{-13/18-\varepsilon}$, i.e., $H \ge BN^{13/18+\varepsilon}$. By the Cauchy–Schwarz inequality, (2) and (17) we obtain that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$E_2 \ll E_3^{1/2}\Big(\int_{-B/H}^{B/H}\frac{\mathrm{d}\alpha}{|z|^2}\Big)^{1/2} \ll E_3^{1/2}N^{1/2} \ll N^{1/3}\exp\Big(-\frac{c_1}{2}\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{18}$$

provided that $H \ge BN^{13/18+\varepsilon}$. By using twice the Cauchy–Schwarz inequality, Lemma 5, (2) and (17) we obtain that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$E_1 \ll E_3^{1/2}\Big(\int_{-B/H}^{B/H}\frac{|\widetilde{S}_3(\alpha)|^4}{|z|^{2/3}}\ \mathrm{d}\alpha\Big)^{1/2} \tag{19}$$

$$\ll E_3^{1/2}\Big(\int_{-1/2}^{1/2} |\widetilde{S}_3(\alpha)|^8 \ \mathrm{d}\alpha\Big)^{1/4}\Big(\int_{-B/H}^{B/H} \frac{\mathrm{d}\alpha}{|z|^{4/3}}\Big)^{1/4}$$
$$\ll E_3^{1/2} N^{1/2} L^{A/4} \ll N^{1/3} \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big),$$

provided that $H \ge BN^{13/18+\varepsilon}$. Hence by (16)–(19) we finally can write that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$J_2 \ll HN^{1/3} \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{20}$$

provided that $H \ge BN^{13/18+\varepsilon}$. Summing up, by (3.3)-(14) and (20) we have that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$I_1 = \Gamma\Big(\frac{4}{3}\Big)^3 \frac{HN^{1/3}}{e} + \mathcal{O}\Big(HN^{1/3} \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big)\Big) \tag{21}$$

provided that $H \ge BN^{13/18+\varepsilon}$.

3.4. Final words. Summing up, by (5)-(6), (13) and (21) we have that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 \frac{HN^{1/3}}{e}$$
$$+ \mathcal{O}\Big(HN^{1/3} \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big) + \frac{H}{B} N^{1/3+\varepsilon}\Big)$$

provided that $H \ge BN^{13/18+\varepsilon}$. The second error term is dominated by the first one since $B = N^{2\varepsilon}$ by (4). Hence we can write that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 \frac{HN^{1/3}}{e} + \mathcal{O}\Big(HN^{1/3} \exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big)\Big) \tag{22}$$

provided that $H \ge N^{13/18+3\varepsilon}$. From $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$, $1 \le H \le N$, we get that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 HN^{1/3}$$

$$+\,\mathcal{O}\Big(HN^{1/3}\exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big)\Big)+\mathcal{O}\Big(\frac{H}{N}\sum_{n=N+1}^{N+H} r(n)\Big)$$

provided that $H \ge N^{13/18+3\varepsilon}$ and $H \le N$. Using $e^{n/N} \le e^2$ and (22), the last error term is $\ll H^2N^{-2/3}$. Hence we get that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 HN^{1/3} + \mathcal{O}\Big(HN^{1/3}\exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big)\Big),$$

provided that $N^{13/18+3\varepsilon} \le H \le N^{1-\varepsilon}$. Theorem 1 follows by rescaling ε.

4. The conditional case

From now on we assume the Riemann Hypothesis holds. Comparing with Section 3 we can simplify the setting, since Lemma 4 now applies to the whole integration interval, though the basic strategy does not change. Recalling (1) and $\Gamma(4/3) = (1/3)\Gamma(1/3)$, we have

$$\begin{aligned}(23)\qquad \sum_{n=N+1}^{N+H} e^{-n/N} r(n) &= \int_{-1/2}^{1/2} \widetilde{V}_3(\alpha)^4 U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\ &= \Gamma\Big(\frac{4}{3}\Big)^4 \int_{-1/2}^{1/2} \frac{U(-\alpha,H)}{z^{4/3}} e(-N\alpha)\,\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \Big(\widetilde{S}_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}}\Big) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \big(\widetilde{V}_3(\alpha)^4 - \widetilde{S}_3(\alpha)^4\big) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,\end{aligned}$$

say. Now we evaluate these terms.

4.1. Evaluation of $\mathcal{J}_1$. By Lemma 3, a direct calculation gives

$$\begin{aligned}(24)\qquad \mathcal{J}_1 &= \Gamma\Big(\frac{4}{3}\Big)^3 \sum_{n=N+1}^{N+H} e^{-n/N} n^{1/3} + \mathcal{O}\Big(\frac{H}{N}\Big) = \frac{\Gamma(4/3)^3}{e} \sum_{n=N+1}^{N+H} n^{1/3} \\ &+ \mathcal{O}\Big(\frac{H}{N} + \frac{H^2}{N^{2/3}}\Big) = \Gamma\Big(\frac{4}{3}\Big)^3 \frac{HN^{1/3}}{e} + \mathcal{O}\Big(\frac{H^2}{N^{2/3}} + N^{1/3}\Big).\end{aligned}$$

4.2. Estimate of $\mathcal{J}_2$. Recall that $\widetilde{E}_3(\alpha) := \widetilde{S}_3(\alpha) - \frac{\Gamma(4/3)}{z^{1/3}}$. Using $f^2 - g^2 = 2g(f-g) + (f-g)^2$ we can write that

$$\widetilde{S}_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}} = \Big(\widetilde{S}_3(\alpha)^2 + \frac{\Gamma(4/3)^2}{z^{2/3}}\Big)\Big(\widetilde{S}_3(\alpha)^2 - \frac{\Gamma(4/3)^2}{z^{2/3}}\Big)$$
$$\ll (|\widetilde{S}_3(\alpha)|^2 + |z|^{-2/3})(|z|^{-1/3}|\widetilde{E}_3(\alpha)| + |\widetilde{E}_3(\alpha)|^2).$$

Hence

(25)
$$\mathcal{J}_2 \ll \int_{-1/2}^{1/2} \frac{|\widetilde{S}_3(\alpha)|^2}{|z|^{1/3}} |\widetilde{E}_3(\alpha)||U(-\alpha,H)| \,\mathrm{d}\alpha + \int_{-1/2}^{1/2} \frac{|\widetilde{E}_3(\alpha)|}{|z|} |U(-\alpha,H)| \,\mathrm{d}\alpha$$
$$+ \int_{-1/2}^{1/2} (|\widetilde{S}_3(\alpha)|^2 + |z|^{-2/3})|\widetilde{E}_3(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

say. Let

$$\mathcal{E} := \int_{-1/2}^{1/2} |\widetilde{E}_3(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha.$$

By (3), Lemma 4 and an integration by parts we obtain, recalling that $I(1/H) = [-1/2,-1/H] \cup [1/H,1/2]$, that

$$\mathcal{E} \ll H \int_{-1/H}^{1/H} |\widetilde{E}_3(\alpha)|^2 \,\mathrm{d}\alpha + \int_{I(1/H)} \frac{|\widetilde{E}_3(\alpha)|^2}{|\alpha|} \,\mathrm{d}\alpha \tag{26}$$
$$\ll H \frac{N^{1/3}L^2}{H} + N^{1/3}L^2 + \int_{1/H}^{1/2} \frac{N^{1/3}L^2}{\xi} \,\mathrm{d}\xi \ll N^{1/3}L^3.$$

By (2), $\widetilde{S}_3(\alpha) \ll N^{1/3}$ and (26) we obtain

$$\mathcal{I}_3 \ll N^{2/3}\mathcal{E} \ll NL^3. \tag{27}$$

By the Cauchy–Schwarz inequality, (2)-(3) and (26), we obtain

$$\mathcal{I}_2 \ll \mathcal{E}^{1/2} \Big(\int_{-1/2}^{1/2} \frac{|U(-\alpha,H)|}{|z|^2} \,\mathrm{d}\alpha \Big)^{1/2} \tag{28}$$
$$\ll \mathcal{E}^{1/2} \Big(HN^2 \int_{-1/N}^{1/N} \mathrm{d}\alpha + H \int_{1/N}^{1/H} \frac{\mathrm{d}\alpha}{\alpha^2} + \int_{1/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^3} \Big)^{1/2} \ll H^{1/2}N^{2/3}L^{3/2}.$$

By the Cauchy–Schwarz inequality we obtain

$$\mathcal{I}_1 \ll \mathcal{E}^{1/2}\Big(\int_{-1/2}^{1/2} |\widetilde{S}_3(\alpha)|^4 \frac{|U(-\alpha,H)|}{|z|^{2/3}}\,\mathrm{d}\alpha\Big)^{1/2}.$$

Again by the Cauchy–Schwarz inequality, (2)–(3) and (26), we obtain

$$\mathcal{I}_1 \ll \mathcal{E}^{1/2}\Big(\int_{-1/2}^{1/2} |\widetilde{S}_3(\alpha)|^8\,\mathrm{d}\alpha\Big)^{1/4}\Big(\int_{-1/2}^{1/2} \frac{|U(-\alpha,H)|^2}{|z|^{4/3}}\,\mathrm{d}\alpha\Big)^{1/4} \tag{29}$$

$$\ll N^{1/6}L^{3/2}N^{5/12}L^{A/4}\Big(H^2N^{4/3}\int_{-1/N}^{1/N}\mathrm{d}\alpha + H^2\int_{1/N}^{1/H}\frac{\mathrm{d}\alpha}{\alpha^{4/3}} + \int_{1/H}^{1/2}\frac{\mathrm{d}\alpha}{\alpha^{10/3}}\Big)^{1/4}$$
$$\ll H^{1/2}N^{2/3}L^{3/2+A/4}.$$

Summing up by (25) and (27)-(29), we can finally write that

$$\mathcal{J}_2 \ll H^{1/2}N^{2/3}L^{3/2+A/4} + NL^3. \tag{30}$$

4.3. Estimate of $\mathcal{J}_3$. It is clear that $\mathcal{J}_3 = I_3$ of section 3.2. Hence by (13) we obtain

$$\mathcal{J}_3 \ll H^{3/4}N^{5/12+\varepsilon}. \tag{31}$$

4.4. Final words. Summing up, by (23)-(24), (30) and (31), there exists $B = B(A) > 0$ such that we have

$$\sum_{n=N+1}^{N+H} e^{-n/N} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 \frac{HN^{1/3}}{e} \tag{32}$$
$$+ \mathcal{O}\Big(\frac{H^2}{N^{2/3}} + H^{3/4}N^{5/12+\varepsilon} + H^{1/2}N^{2/3}L^B + NL^3\Big)$$

which is an asymptotic formula $\infty(N^{2/3}L^{2B}) \le H \le o\,(N)$. From $e^{-n/N} = e^{-1} + \mathcal{O}\,(H/N)$ for $n \in [N+1, N+H]$, $1 \le H \le N$, we get

$$\sum_{n=N+1}^{N+H} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 HN^{1/3}$$
$$+ \mathcal{O}\Big(\frac{H^2}{N^{2/3}} + H^{3/4}N^{5/12+\varepsilon} + H^{1/2}N^{2/3}L^B + NL^3\Big) + \mathcal{O}\Big(\frac{H}{N}\sum_{n=N+1}^{N+H} r(n)\Big).$$

Using $e^{n/N} \le e^2$ and (32), the last error term is $\ll H^2 N^{-2/3} + H^{7/4} N^{-7/12+\varepsilon} + H^{3/2} N^{-2/3} L^B + HL^3$. Hence we get

$$\sum_{n=N+1}^{N+H} r(n) = \Gamma\Big(\frac{4}{3}\Big)^3 H N^{1/3}$$

$$+\mathcal{O}\bigg(\frac{H^2}{N^{2/3}} + H^{3/4} N^{5/12+\varepsilon} + H^{1/2} N^{2/3} L^B + NL^3\bigg),$$

uniformly for $\infty(N^{2/3} L^{2B}) \le H \le o(N)$, $B \ge 3/2$. Theorem 2 follows.

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