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DILWORTH'S DECOMPOSITION THEOREM FOR POSETS IN ZF

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Abstract. In set theory without the Axiom of Choice (AC), we investigate the set-theoretic strength of Dilworth's theorem for infinite posets with finite width, and its possible placement in the hierarchy of weak choice principles.

1. Introduction

Dilworth [4] established the following deep result about partial orders with bounded finite antichains, which is of great combinatorial and order-theoretical interest: *Let P be an arbitrary poset and let k be a natural number. If P has no antichains of size $k+1$ while at least one k-element subset of P is an antichain, then P can be partitioned into k chains.*

In other words, the above result, which is uniformly known as *Dilworth's Theorem*—abbreviated here by DT—states that if the maximum number of elements in an antichain of a poset P is finite, then it is equal to the minimum number of pairwise disjoint chains into which P can be decomposed. Dilworth's original proof was somewhat complicated and consisted of two parts: in the first part, the theorem was proved for the case where P is finite (with a fairly involved argument), and in the second part, the proof of the general case was based on the finite case and the Teichmüller–Tukey Lemma ("If a non-empty subset $\mathcal{U} \subseteq \wp(X)$ is of finite character, i.e., $A \in \mathcal{U}$ if and only if every finite subset of A belongs to $\mathcal{U}$, has a $\subseteq$-maximal element"). The latter Maximal Principle is well-known to be equivalent to the full AC in ZFA, i.e., Zermelo–Fraenkel set theory with atoms (see for example Herrlich [7, Theorem 2.2]).

Since Dilworth's original proof, several other proofs of DT in the finite case (i.e., DT for finite posets) were accomplished by many researchers,

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which were considerable simplifications of the original corresponding one. We would like to single out the notably elegant and short proofs by Galvin [6], Perles [15], and Tverberg [19]. Furthermore, Perles [16] showed that DT ceases to be true if the given poset contains arbitrarily large finite antichains (and no infinite antichains). Usually, the supremum of the cardinalities of antichains in a poset is called the *width of the poset*. So in the finite case, the width of a (finite) poset P is equal to the minimum cardinality of a partition of P into chains, and by DT, this is also true for infinite posets with finite width.

DT in the finite case is certainly a theorem of ZF (i.e., Zermelo–Fraenkel set theory without AC). However, as Dilworth's original proof explicitly indicated, AC comes into the picture for the infinite case (that is, for infinite posets with finite width). Now, it is part of the folklore that DT does not imply back AC in ZF. In particular, DT is derivable from the Boolean Prime Ideal Theorem (BPI), which is well-known to be strictly weaker than AC in ZF (see [8]). Furthermore, a proof of DT can also be achieved by using the n-Coloring Theorem ("If (V, E) is a graph whose finite subgraphs are n-colorable then so is the whole graph (V, E)", where an n-coloring of (V, E) is a map $C\colon V \to n$ such that adjacent vertices have different colors, i.e., $\{u, v\} \in E \Rightarrow C(u) \neq C(v)$) by De Bruijn and Erdős [3], who used Rado's Selection Lemma in conjunction with the Axiom of Choice for Finite Sets ($\mathsf{AC_{fin}}$) for its proof (complete definitions will be given in Section 2). We note that for any integer $n \geq 3$, the n-Coloring Theorem is equivalent to BPI, as shown by Läuchli, and so is 'Rado's Selection Lemma + $\mathsf{AC_{fin}}$', as shown independently by Rav, Wolk, and Blass (see Howard–Rubin [8] for complete references to the above results; in particular, we point out that Blass' proof appears in Note 33 of [8]). A quite simple and short proof of DT, which employs an equivalent form of BPI (called Intersection Lemma), has been given by Erné [5] (both the Intersection Lemma as well as the so-called Finite Cutset Lemma are due to Erné, who proved in [5] their equivalence to BPI).

We will also provide a very simple proof of DT using the Propositional Compactness Theorem, which is equivalent to BPI (see [8]). Furthermore, we will prove that the implication '$\mathsf{BPI} \to \mathsf{DT}$' is *not* reversible in ZFA, which is a *new* result. In particular, we will establish that DT does not imply $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ (Axiom of Choice for countably infinite families of non-empty finite sets) in ZFA (using a suitable Fraenkel–Mostowski permutation model), which is known to be strictly weaker than BPI in ZF (see [8]). Whether or not DT in conjunction with $\mathsf{AC_{fin}}$ implies BPI, is *unknown* to us.

Although the implication '$\mathsf{BPI} \to \mathsf{DT}$' is undoubtedly a strong and informative result, beyond this there seems to be a *considerable gap in information* (at least to the best of our knowledge) about the set-theoretic strength of DT and its more precise placement in the hierarchy of weak choice principles. It is also a surprising fact that DT *does not* appear in the important encyclopedic book [8] by Howard and Rubin on the consequences of AC. The *motivation* for the research in this paper stems exactly from this lack of source of information.

Among other results, we will show that DT is *not provable in* ZF, by arguing that it implies AC for linearly ordered sets of n-element sets ($\mathsf{AC}_n^{\mathsf{LO}}$), for any integer $n \geq 2$. This answers the first plausible question that arises about the strength of DT, and completely settles that it is in vain to expect that DT might be proved without invoking any form of choice. The above result also gives that the Axiom of Multiple Choice (MC)—which is equivalent to AC in ZF, but *not* equivalent to AC in ZFA— does not imply DT in ZFA (while it implies DT in ZF).

We will also prove that the Axiom of Choice for well-ordered families of non-empty sets ($\mathsf{AC}^{\mathsf{WO}}$) does not imply DT in ZFA. In particular, we will establish that for any integer $n \geq 2$, $\mathsf{AC}^{\mathsf{WO}}$ does not imply $\mathsf{AC}_n^{\mathsf{LO}}$ in ZFA by introducing a *new* Fraenkel–Mostowski model with the required properties. The latter independence result also *settles the corresponding open problem* (for ZFA) in Howard and Rubin [8]. Furthermore, we will show that for any regular cardinal $\aleph_\alpha$, the statement "for every infinite cardinal $\lambda < \aleph_\alpha$, DC_λ" does not imply DT in ZF; in particular, neither the (weaker than $\mathsf{AC}^{\mathsf{WO}}$) Principle of Dependent Choice (DC) nor the (weaker than DC) Axiom of Countable Choice (AC^ω) implies DT in ZF. The non-provability of DT from MC, $\mathsf{AC}^{\mathsf{WO}}$, and $\forall\lambda < \aleph_\alpha(\mathsf{DC}_\lambda)$ (for any regular cardinal $\aleph_\alpha$) indicates that DT is actually a *strong axiom*.

Yet, we will prove that DT *does not* imply Marshall Hall's Theorem in ZFA (which is deducible from BPI, and it is unknown whether it is equivalent to BPI), and is *not* derivable from Rado's Selection Lemma in ZFA. (We recall here that DT in the finite case is equivalent to Philip Hall's theorem, see for example Cameron [2].) Now, Perles [15] mentions that DT can be easily deduced from the finite case (which requires *no* choice) and Rado's Selection Lemma. However, our aforementioned result shows that **Perles' assertion is incorrect** in the setting of ZFA. (As mentioned above, what is certainly true is that Rado's Lemma *in conjunction with* $\mathsf{AC}_{\mathsf{fin}}$ implies DT.) A proof of DT using Rado's Selection Lemma in conjunction with $\mathsf{AC}_{\mathsf{fin}}$ can be found in Mirsky [12, Theorem 4.4.1], where minor adjustments to the proof are required (so that the use of $\mathsf{AC}_{\mathsf{fin}}$ to be clarified).

Last but not least, we will give a *new* ZF-*proof* of DT for well-orderable (infinite) posets with finite width. This will be useful in establishing some of our forthcoming main results.

2. Notation, terminology, and known results

DEFINITION 2.1. Let $(P, \leq)$ be a partially ordered set (poset). A subset $C \subseteq P$ is called a *chain* in P, if $(C, \leq\restriction C)$ is linearly ordered.

A subset $A \subseteq P$ is called an *antichain* in P, if no two elements of A are comparable under $\leq$.

A $\subseteq$-maximal subset $M \subseteq P$ of pairwise incomparable (under $\leq$) elements is called a *maximal antichain* in P.

A set X is called *Dedekind-finite* if there is no injection $f\colon \omega \to X$ (as usual, ω denotes the set of natural numbers). Otherwise, X is called *Dedekind-infinite*.

An infinite set X is called *amorphous* if X cannot be written as a disjoint union of two infinite subsets.

A topological space (X, τ) is called *compact* if for every $\mathcal{U} \subseteq \tau$ such that $\bigcup \mathcal{U} = X$ there is a finite subset $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{V} = X$.

DEFINITION 2.2.

(1) The *Axiom of Choice*, AC (Form 1 in [8]): Every family of non-empty sets has a choice function.

(2) The *Axiom of Choice for Finite Sets*, $\mathsf{AC}_{\mathsf{fin}}$ (Form 62 in [8]): Every family of non-empty finite sets has a choice function.

(3) $\mathsf{AC}^{\mathsf{WO}}$ (Form 40 in [8]): Every well-ordered family of non-empty sets has a choice function.

(4) The *Axiom of Countable Choice*, AC^{ω} (Form 8 in [8]): Every countably infinite family of non-empty sets has a choice function.

(5) $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ (Form 10 in [8]): Every countably infinite family of non-empty finite sets has a choice function.

(6) The *Principle of Dependent Choice*, DC (Form 43 in [8]): Let X be a non-empty and let R be a binary relation on X such that $(\forall x \in X)$ $(\exists y \in X)(x \mathrel{R} y)$. Then there exists a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_n \mathrel{R} x_{n+1}$ for all $n \in \omega$.

(7) Let κ be an infinite well-ordered cardinal number. DC_{κ} (Form 87(κ) in [8]): Let S be a non-empty set and let R be a binary relation such that for every $\alpha < \kappa$ and every α-sequence $s = (s_{\xi})_{\xi<\alpha}$ of elements of S there exists $y \in S$ such that $s \mathrel{R} y$. Then there is a function $f\colon \kappa \to S$ such that for every $\alpha < \kappa$, $(f \restriction \alpha) \mathrel{R} f(\alpha)$.
(Note that DC_{ω} is a reformulation of DC.)

(8) $\mathsf{DF} = \mathsf{F}$ (Form 9 in [8]): Every Dedekind-finite set is finite.

(9) $\mathsf{AC}^{\mathsf{LO}}$ (Form 202 in [8]): Every linearly ordered family of non-empty sets has a choice function.

(10) Let $n \in \omega \setminus \{0, 1\}$. $\mathsf{AC}^{\mathsf{LO}}_{n}$ (Form 33(n) in [8]): Every linearly ordered family of n-element sets has a choice function.

(11) The *Axiom of Multiple Choice*, MC (Form 67 in [8]): Every family $\mathcal{A}$ of non-empty sets has a multiple choice function, i.e., there is a function F with domain $\mathcal{A}$ such that for every $A \in \mathcal{A}$, $f(A)$ is a non-empty finite subset of A.

(12) The *Boolean Prime Ideal Theorem*, BPI (Form 14 in [8]): Every Boolean algebra has a prime ideal.

(13) *Marshall Hall's Theorem*, MHT (Form 107 in [8]): Let $\{S(a) : a \in A\}$ be a family of finite subsets of a set X. Then if

($*$) for each finite $F \subseteq A$ there is an injective choice function for $\{S(a) : a \in F\}$

then there is an injective choice function for $\{S(a) : a \in A\}$.

Philip Hall's Theorem states that ($*$) is equivalent to "for any finite $F \subseteq A$, $|\bigcup\{S(a) : a \in F\}| \geq |F|$"—the latter property is known as *Hall's Condition*. The proof of P. Hall's theorem does not require any choice principle.

(14) *Rado's Selection Lemma*, RSL (Form 99 in [8]): Let $\mathfrak{F}$ be a family of finite sets and suppose that to every finite subset F of $\mathfrak{F}$ there corresponds a choice function ϕ_F whose domain is F such that $\phi_F(T) \in T$ for each $T \in F$. Then there is a choice function f whose domain is $\mathfrak{F}$ with the property that for every finite subset F of $\mathfrak{F}$, there is a finite subset F' of $\mathfrak{F}$ such that $F \subseteq F'$ and $f(T) = \phi_{F'}(T)$ for all $T \in F$.

(15) *Ramsey's Theorem*, RT (Form 17 in [8]): If A is an infinite set and $[A]^2$ (the family of all 2-element subsets of A) is partitioned into two sets X and Y, then there is an infinite subset $B \subseteq A$ such that either $[B]^2 \subseteq X$ or $[B]^2 \subseteq Y$.

(16) The *Chain-Antichain Principle*, CAC (Form 217 in [8]): Every infinite poset has either an infinite chain or an infinite antichain.

THEOREM 2.3 ([8], [10]). *The following hold*:

(i) *In* ZFA, AC $\to$ MC $\to$ *"Antichain Principle"* (*"every poset has a maximal antichain"*). *None of the above implications is reversible in* ZFA.

(ii) *In* ZF, AC $\leftrightarrow$ MC $\leftrightarrow$ *"Antichain Principle"*.

(iii) BPI *is equivalent to each of "Propositional Compactness Theorem"* (*"a set X of propositional formulas is satisfiable if each finite subset of X is satisfiable"*), *and "*RSL + $\mathsf{AC_{fin}}$*". Furthermore*, RSL *is strictly weaker than* BPI *in* ZFA, *and* $\mathsf{AC_{fin}}$ *is strictly weaker than* BPI *in* ZF.

(iv) ([17]) CAC *is strictly weaker than* RT *in* ZF.

(v) DT *in the finite case is equivalent to P. Hall's theorem.*

(vi) BPI $\to$ MHT $\to$ $\mathsf{AC_{fin}}$.

(vii) ([13]) *In* ZFA, MC *implies* RSL.

(ix) DF = F *implies* RT.

We note that it is an open problem whether RSL implies BPI in ZF. For recent research on RSL and a topological variant of RSL, which is equivalent to BPI, the reader is referred to Howard and Tachtsis [9].

It is also an open problem whether or not any of the implications in Theorem 2.3(vi) is reversible.

THEOREM 2.4. BPI *implies* DT. *Hence* DT *is strictly weaker than* AC *in* ZF. *Furthermore*, DT *does not imply* RT *in* ZF, *and hence* (*by Theorem* 2.3(ix)) *neither does it imply* DF = F *in* ZF.

Proof. Let $(P, \leq)$ be an arbitrary poset with finite width k (where k is some natural number). Let L be a propositional language with propositional variables p_{xi}, where $x \in P$ and $1 \leq i \leq k$. The variable p_{xi} has the intended meaning that the element x of P belongs to the i-th chain. Let $\mathcal{F}$ be the set of all formulas of L, and also let Σ be the subset of $\mathcal{F}$ which comprises the following formulas:

(1) $p_{x1} \vee p_{x2} \vee \cdots \vee p_{xk}$ for $x \in P$;

(2) $\neg(p_{xi} \wedge p_{yi})$ for incomparable elements x, y of P and $1 \leq i \leq k$;

(3) $\neg(p_{xi} \wedge p_{xj})$ for $x \in P$ and $1 \leq i, j \leq k$ with $i \neq j$.

The formulas in (1) suggest that every element of P belongs to at least one of the k chains; the formulas in (2) suggest that each of the k many sets (which cover P) is a chain; and the formulas in (3) suggest the k many chains which cover P are pairwise disjoint.

Now by DT in the finite case (recall that this is provable without choice), it is fairly easy to see that for every finite subset $\Sigma_0 \subset \Sigma$ there is a valuation mapping $f \in 2^{\mathcal{F}}$ which satisfies Σ_0, that is, $f(\phi) = 1$ for all $\phi \in \Sigma_0$. Hence, by Theorem 2.3(iii), there is a valuation mapping $f \in 2^{\mathcal{F}}$ which satisfies Σ. For each $i \in \{1, \ldots, k\}$, we let

$$C_i = \{x \in P : f(p_{xi}) = 1\}.$$

Clearly $\mathcal{C} = \{C_i : 1 \leq i \leq k\}$ is a partition of P into k chains.

For the second and third assertions of the theorem, consider the basic Cohen model of ZF, which is labeled as Model $\mathcal{M}1$ in [8]. It is well-known that BPI (and hence DT) is true in $\mathcal{M}1$, whereas RT (and hence, by Theorem 2.3(ix), DF = F) is false in $\mathcal{M}1$ (see [1], [8]). □

Theorem 2.5 [11]. *Let $\{X_i : i \in I\}$ be a family of compact spaces which is indexed by a set I on which there is a well-ordering $\leq$. If I is an infinite set, let there also be a choice function F on the collection $\{C : C$ is closed, $C \neq \emptyset$, $C \subseteq X_i$ for some $i \in I\}$. Then the product space $\prod_{i \in I} X_i$ is compact in the product topology.*

3. Main results

We start this section by proving that, in ZF, Dilworth's Theorem is true for well-ordered infinite posets with finite width. The argument is a slight variant of the one for the proof of Theorem 2.4 and could also be used to establish the latter theorem. The reason is that BPI is equivalent to Tychonoff's product theorem for compact Hausdorff spaces (i.e., "products of compact Hausdorff spaces are compact"), see [8, Form 14] for complete references to this result. Furthermore, we note that we could argue for the next result using the proof of Theorem 2.4 and transfinite induction; however, we prefer to give the following proof so that the interested reader attains further insight and information.

We also prove that for any integer $n \geq 2$, DT implies $\mathsf{AC}_n^{\mathsf{LO}}$, and thus DT is *not* provable in ZF.

THEOREM 3.1. *The following hold*:

(i) DT *for well-ordered infinite posets with finite width is provable in* ZF.

(ii) *For any integer* $n \geq 2$, DT *implies* $\mathsf{AC}_n^{\mathsf{LO}}$. *Hence* DT *is not provable in* ZF. *Furthermore*, MC *does not imply* DT *in* ZFA, *and thus by* (i) *and* (vii) *of Theorem* 2.3, *neither the Antichain Principle nor* RSL *implies* DT *in* ZFA.

(iii) DT *does not imply the Antichain Principle in* ZFA.

PROOF. (i) Let $(P, \leq)$ be an infinite poset with finite width k, and such that P is well-ordered. Let L, p_{xi} (where $x \in P$ and $1 \leq i \leq k$), $\mathcal{F}$, and $\Sigma \subseteq \mathcal{F}$ be defined as in the proof of Theorem 2.4.

Let $\mathrm{Var} = \{p_{xi} : x \in P, i \in \{1, \dots, k\}\}$. Since $P \times \{1, \dots, k\}$ is well-orderable (for P is well-ordered), so is Var. For every $W \in [P]^{<\omega} \setminus \{\emptyset\}$ (the set of non-empty finite subsets of P), we let Σ_W be the subset of $\mathcal{F}$, which is defined as Σ except that the subscripts in the formulas run through the set $W \cup \{1, \dots, k\}$. We let

$$F_W = \{f \in 2^{\mathrm{Var}} : \forall \phi \in \Sigma_W (f'(\phi) = 1)\},$$

where $W \in [P]^{<\omega} \setminus \{\emptyset\}$ and for $f \in 2^{\mathrm{Var}}$, the element f' of $2^{\mathcal{F}}$ denotes the valuation mapping determined by f.

Now using DT in the finite case, one easily verifies that the family

$$\mathcal{Z} = \{F_W : W \in [P]^{<\omega} \setminus \{\emptyset\}\}$$

has the finite intersection property (i.e. any finite subfamily of $\mathcal{Z}$ has a non-empty intersection). It is also easy to see that for each $W \in [P]^{<\omega} \setminus \{\emptyset\}$, F_W is closed in the product space 2^{Var} (where 2 is the discrete 2-element space $\{0, 1\}$). By Theorem 2.5, we derive that 2^{Var} is compact, and consequently $\bigcap \mathcal{Z} \neq \emptyset$.

Let $f \in \bigcap \mathcal{Z}$ and also let $f' \in 2^{\mathcal{F}}$ be the unique valuation mapping which extends f. Then $f'(\phi) = 1$ for all $\phi \in \Sigma$. We may finish now the proof as in Theorem 2.4 in order to write P as a disjoint union of k chains.

(ii) Fix $n \in \omega \setminus \{0, 1\}$. Let $\mathcal{A} = \{A_i : i \in I\}$ be an infinite family of n-element sets (the mapping $i \mapsto A_i$ $(i \in I)$ is a bijection), where the index set I is equipped with a linear order, say $\leq$. Without loss of generality we assume that $\mathcal{A}$ is disjoint (otherwise, we may work with the disjoint family $\mathcal{B} = \{A_i \times \{i\} : i \in I\}$). We define a binary relation $\preceq$ on $A = \bigcup \mathcal{A}$ as follows: for all $x \in A$, $x \preceq x$, and for distinct $x, y \in A$,

$$x \prec y \iff x \in A_i,\ y \in A_j,\ \text{and } i < j.$$

Clearly, $\preceq$ is a partial order on A such that the only antichains in A are the subsets of the sets A_i $(i \in I)$. Thus the width of $(A, \preceq)$ is n, and consequently by DT, A has an n-sized partition $\mathcal{C} = \{C_1, \ldots, C_n\}$, where C_i is a chain in A for every $i \in \{1, \ldots, n\}$.

As $|A_i \cap C_j| \leq 1$ for all $i \in I$ and all $j \in \{1, \ldots, n\}$, the only way that the union of the C_j's covers each A_i is that $|A_i \cap C_j| = 1$ $(i \in I$ and $j \in \{1, \ldots, n\})$. Then, each C_j determines a choice function for the system $\mathcal{A}$.

For the second assertion of (ii), fix any $n \in \omega \setminus \{0, 1\}$. Then in Pincus' model $\mathcal{M}47(n, M)$ in [8], $\mathsf{AC}_n^{\mathsf{LO}}$ is false (see [8] for complete references to this result), and hence by the above argument, so is DT.

For the third assertion of (ii), we note that in the second Fraenkel model (Model $\mathcal{N}2$ in [8]), MC (and hence the Antichain Principle and RSL) is true, whereas the axiom of countable choice for pairs is false (see [8], [10]), and thus so is DT.

(iii) In Mostowski's linearly ordered model—Model $\mathcal{N}3$ in [8]—BPI is true (see [8]). Thus, by Theorem 2.4, DT is true in $\mathcal{N}3$. On the other hand, in $\mathcal{N}3$, the set A of atoms is linearly ordered, but it is not well-orderable. Since the Antichain Principle implies "every linearly ordered set can be well-ordered" (see Jech [10, Theorem 9.1(a)]), it follows that the Antichain Principle is false in $\mathcal{N}3$. $\square$

It is easy to see that the statement "for every integer $n \geq 2$, $\mathsf{AC}_n^{\mathsf{LO}}$" is equivalent to "for every integer $n \geq 2$, the union of a linearly orderable family of n-element sets is linearly orderable". (Assuming that $\mathsf{AC}_n^{\mathsf{LO}}$ is true for all integers $n \geq 2$, fix an integer $n \geq 2$, and let $\mathcal{A} = \{A_i : i \in I\}$ be an infinite family of n-element sets (the mapping $i \mapsto A_i$ $(i \in I)$ is a bijection), where the index set I is equipped with some linear order $\leq$. For each $i \in I$, let $B_i = \{f : f$ is a one-to-one function from n onto $A_i\}$. Then the family $\mathcal{B} = \{B_i : i \in I\}$ is linearly orderable, and $|B_i| = n!$ for all $i \in I$. By $\mathsf{AC}_{n!}^{\mathsf{LO}}$, let F be a choice function for $\mathcal{B}$. On the basis of the functions $F(i)$ $(i \in I)$ and the binary relation $\preceq$ on $\bigcup \mathcal{A}$, as this was defined in the proof of (ii) of Theorem 3.1, it is straightforward to define a linear order on $\bigcup \mathcal{A}$.)

From the above observation and Theorem 3.1(ii), we immediately obtain the following corollary.

Corollary 3.2. *For any integer $n \geq 2$, DT implies the statement "the union of a linearly orderable family of n-element sets is linearly orderable".*

Remark 3.3. We recall that the Antichain Principle is equivalent to AC in ZF (see Theorem 2.3(ii)), and thus it implies DT in ZF.

Also, as mentioned in Section 1, for every integer $n \geq 3$, the n-Coloring Theorem (if (V, E) is a graph whose finite subgraphs are n-colorable, then so is the whole graph (V, E)) is equivalent to BPI, and thus implies DT (see Theorem 2.4). On the other hand, Mycielski [14] showed that *the 2-Coloring*

Theorem is equivalent to AC_2 (the axiom of choice for families of pairs, and Form 88 in [8]), which is strictly weaker than BPI in ZF (see [8]).

Now, in the permutation model $\mathcal{N}2^*(3)$ of [8] (due to P. Howard), AC_2, and thus the 2-Coloring Theorem, is true. However, in $\mathcal{N}2^*(3)$, there is a countably infinite family of 3-element sets of atoms which has no choice function in the model. By Theorem 3.1(ii), it follows that DT is false in $\mathcal{N}2^*(3)$. Hence, *the 2-Coloring Theorem does not imply* DT *in* ZFA.

THEOREM 3.4. DT *does not imply* $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ *in* ZFA. *Hence, by Theorem* 2.3(*vi*), DT *does not imply* MHT (*and hence* BPI) *in* ZFA.

PROOF. For our independence result we will use Lévy's permutation model, which is labeled as Model $\mathcal{N}6$ in [8]. The description of $\mathcal{N}6$ is as follows: We start with a ground model M of $\mathsf{ZFA} + \mathsf{AC}$ with a countably infinite set A of atoms which is written as a disjoint union $\bigcup\{P_n : n \in \omega\}$, where $P_n = \{a^n_1, a^n_2, \dots, a^n_{p_n}\}$, p_n being the n-th prime number ($p_0 = 2$, $p_1 = 3$, etc.). Let G be the group generated by the following permutations π_n of A:

$$\pi_n : a^n_1 \mapsto a^n_2 \mapsto \cdots \mapsto a^n_{p_n} \mapsto a^n_1,$$
$$\pi_n(x) = x \quad \text{for all } x \in A \setminus P_n.$$

(G is the weak direct product of cyclic groups of order p_n.) For any element x of M, $\mathrm{fix}_G(x)$ denotes the subgroup $\{\phi \in G : \forall y \in x(\phi(y) = y)\}$ of G and $\mathrm{Sym}_G(x)$ denotes the subgroup $\{\phi \in G : \phi(x) = x\}$ of G. Let $\mathcal{F}$ be the filter of subgroups of G generated by $\{\mathrm{fix}_G(E) : E \in [A]^{<\omega}\}$. An element x of M is called *symmetric* if $\mathrm{Sym}_G(x) \in \mathcal{F}$, and thus x is symmetric if there exists a finite subset $E \subset A$ such that $\mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(x)$. Under these circumstances, E is called a *support* of x. An element x of M is called *hereditarily symmetric* if x and every element in the transitive closure of x are symmetric. Lévy's model $\mathcal{N}6$ is the permutation model which is determined by M, G, and $\mathcal{F}$, that is $\mathcal{N}6$ consists exactly of the hereditarily symmetric elements of M.

In $\mathcal{N}6$, AC_n (the axiom of choice for families of n-element sets) is true for all integers $n \geq 2$ (see [10, Theorem 7.11]). On the other hand, $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ is false in $\mathcal{N}6$ (the countably infinite family $\{P_n : n \in \omega\}$ has no infinite subfamily with a choice function in $\mathcal{N}6$, see [10]). Thus, by Theorem 2.3(vi), MHT (and hence BPI) is also false in $\mathcal{N}6$.

We show now that DT is true in $\mathcal{N}6$. We will first prove a couple of claims.

CLAIM 3.5. *Let* $(P, \leq)$ *be a partially ordered set in* $\mathcal{N}6$ *with support* E. *Then for each* $p \in P$, *the* $\mathrm{fix}_G(E)$-*orbit* $\mathrm{Orb}_E(p)$ *of* p, *i.e., the set* $\mathrm{Orb}_E(p) = \{\phi(p) : \phi \in \mathrm{fix}_G(E)\}$, *is an antichain in* P.

PROOF. Fix $p \in P$. By way of contradiction, we assume that $\mathrm{Orb}_E(p)$ is not an antichain. It follows that for some $\phi, \psi \in \mathrm{fix}_G(E)$, $\phi(p)$ and $\psi(p)$

are comparable, say $\phi(p) < \psi(p)$. Since every permutation of A in G moves only finitely many atoms, we must have that for all $\pi \in G$ there exists $k \in \omega$ such that $\pi^k = 1_A$ (the identity function on A). Letting $\pi = \psi^{-1}\phi$, we have $\pi(p) < p$ and $\pi^k = 1_A$ for some $k \in \omega$. It follows that $p = \pi^k(p) < \pi^{k-1}(p) < \cdots < \pi^2(p) < \pi(p) < p$, and thus $p < p$, a contradiction. □

CLAIM 3.6. *In* $\mathcal{N}6$, *every poset with finite width can be well-ordered.*

PROOF. Let $(P, \leq)$ be a poset in $\mathcal{N}6$ with finite width k. Let $E \subset A$ be a (finite) support of $(P, \leq)$. Then P is written as a disjoint union of $\mathrm{fix}_G(E)$-orbits, i.e.,

$$P = \bigcup\{\mathrm{Orb}_E(p) : p \in P\},$$

where $\mathrm{Orb}_E(p) = \{\phi(p) : \phi \in \mathrm{fix}_G(E)\}$ for all $p \in P$. The family $\{\mathrm{Orb}_E(p) : p \in P\}$ is well-orderable in $\mathcal{N}6$ since $\mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(\mathrm{Orb}_E(p))$ for all $p \in P$.

Now, by Claim 3.5, we have that $\mathrm{Orb}_E(p)$ is an antichain in P for all $p \in P$. Since the width of P is k, it follows that $|\mathrm{Orb}_E(p)| \leq k$ for all $p \in P$. As AC_n is true in $\mathcal{N}6$ for all integers $n \geq 2$, it is easy to verify that $\bigcup\{\mathrm{Orb}_E(p) : p \in P\}$, and hence P, is well-orderable in $\mathcal{N}6$. □

By Claim 3.6 and Theorem 3.1(i), it follows that DT is true in $\mathcal{N}6$. □

THEOREM 3.7. DT *does not imply "there are no amorphous sets"* (*Form* 64 *in* [8]) *in* ZFA. *In particular*, DT *is true in the basic Fraenkel model in which there are amorphous sets.*

PROOF. We first recall the description of the basic Fraenkel model, which is labeled as Model $\mathcal{N}1$ in [8]: We start with a ground model M of $\mathsf{ZFA} + \mathsf{AC}$ with a countably infinite set A of atoms. Let G be the group of all permutations of A, and let $\mathcal{F}$ be the filter of subgroups of G which is generated by the filter base $\{\mathrm{fix}_G(E) : E \in [A]^{<\omega}\}$. The basic Fraenkel model, $\mathcal{N}1$, is the permutation model determined by M, G, and $\mathcal{F}$.

In $\mathcal{N}1$, the set A of atoms is amorphous (see [8]). Thus we only need to show that DT is true in $\mathcal{N}1$. To this end, we prove that in $\mathcal{N}1$ every poset with finite width can be well-ordered; then the conclusion will follow from Theorem 3.1(i).

So, let $(P, \leq)$ be a poset in $\mathcal{N}1$ with finite width. Let $E \subset A$ be a finite support of $(P, \leq)$. Then $P = \bigcup\{\mathrm{Orb}_E(p) : p \in P\}$, where $\mathrm{Orb}_E(p) = \{\phi(p) : \phi \in \mathrm{fix}_G(E)\}$ for all $p \in P$. Now, from the proof of Jech's Theorem 9.2(ii) in [10] (in particular, see the proof of Lemma 9.3 in [10]), one immediately obtains that for all $p \in P$, $\mathrm{Orb}_E(p)$ is an antichain in P. Thus $\mathrm{Orb}_E(p)$ is finite, for all $p \in P$. Furthermore, since $\{\mathrm{Orb}_E(p) : p \in P\}$ is well-orderable in $\mathcal{N}1$ (every element of this family is supported by E) and "the union of a well-orderable family of well-orderable sets is well-orderable" is true in $\mathcal{N}1$ (see [8]), it follows that P is well-orderable as desired. □

Remark 3.8. In order to provide further information to the reader, we mention here another permutation model in which DT is true, whereas "there are no amorphous sets" is false. This model was constructed by Tachtsis in [17], and its description is as follows: We start with a model M of ZFA + AC with a set of atoms $A = \bigcup\{A_i : i \in \omega\}$ which is a countable disjoint union of pairs $A_i = \{a_i, b_i\}$, $i \in \omega$. Let G be the group of all permutations ϕ of A such that ϕ moves only finitely many atoms and for all $i \in \omega$, $\phi(A_i) = A_k$ for some $k \in \omega$. Let Γ be the filter of subgroups of G generated by $\{\mathrm{fix}_G(E) : E \in [A]^{<\omega}\}$. Let $\mathcal{N}$ be the Fraenkel–Mostowski model determined by M, G, and Γ.

In [17], it is shown that, in $\mathcal{N}$, every poset can be expressed as a well-orderable union of antichains (in particular, for any poset $(P, \leq)$ in $\mathcal{N}$, $\mathrm{Orb}_E(p)$ is an antichain for all $p \in P$), and also that "the union of a well-orderable family of well-orderable sets is well-orderable" is true. Hence, as in the proof of Theorem 3.7, we may conclude that DT is true in $\mathcal{N}$.

On the other hand, in $\mathcal{N}$, there are amorphous sets; both A and $\mathcal{A} = \{A_i : i \in \omega\}$ are amorphous, as shown in [17]. Furthermore, in [17], it has been established that CAC is true in $\mathcal{N}$, whereas RT is false in $\mathcal{N}$.

We would also like to note here that the Ordering Principle OP (i.e., every set can be linearly ordered, and Form 30 in [8]) lies in strength between BPI and $\mathsf{AC}_{\mathsf{fin}}$ (the latter principle implies "there are no amorphous sets", see [8]). Thus OP is false in the models $\mathcal{N}1$, $\mathcal{N}6$, and the above model of [17], and hence DT *does not imply* OP *in* ZFA. We *do not know* whether or not OP implies DT. We also point out that the relationship between OP and MHT is *unknown* (this is also stated in [8]).

Next, we prove that $\mathsf{AC}^{\mathsf{WO}}$ does not imply DT in ZFA. In fact, we show something stronger, namely that $\mathsf{AC}^{\mathsf{WO}}$ does not imply $\mathsf{AC}_n^{\mathsf{LO}}$ in ZFA, for any integer $n \geq 2$, and hence by Theorem 3.1(ii), it does not imply DT in ZFA either. We will establish the above result by *introducing a new Fraenkel–Mostowski model*. Furthermore, our result on the independence of $\mathsf{AC}_n^{\mathsf{LO}}$ from $\mathsf{AC}^{\mathsf{WO}}$ *settles the corresponding open problem* (for ZFA) in [8].

Theorem 3.9. *For any integer $n \geq 2$,* $\mathsf{AC}^{\mathsf{WO}}$ *does not imply* $\mathsf{AC}_n^{\mathsf{LO}}$ *in* ZFA. *Hence, by Theorem* 3.1(ii), $\mathsf{AC}^{\mathsf{WO}}$ *does not imply* DT *in* ZFA *either.*

Proof. Fix $n \in \omega \setminus \{0, 1\}$. We start with a ground model M of ZFA + AC with a set of atoms, $A = \bigcup\{A_q : q \in \mathbb{Q}\}$ (where $\mathbb{Q}$ is the set of rational numbers), which is a disjoint union of the n-element sets $A_q = \{a_{q1}, a_{q2}, \ldots, a_{qn}\}$ ($q \in \mathbb{Q}$). Let G be the group of all permutations π of A with the following two properties:

(1) for all $q \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ such that $\pi(A_q) = A_r$;

(2) for all $q, q' \in \mathbb{Q}$, $q < q'$, if and only if, $A_r = \pi(A_q)$, $A_{r'} = \pi(A_{q'})$ and $r < r'$ (where $<$ is the usual dense linear order on $\mathbb{Q}$).

Let $\mathcal{F}$ be the filter of subgroups of G which is generated by the subgroups $\mathrm{fix}_G(E)$ of G, where $E = \bigcup\{A_q : q \in S\}$ for some bounded subset $S \subset \mathbb{Q}$. Let $\mathcal{V}$ be the Fraenkel–Mostowski model determined by M, G, and $\mathcal{F}$. (We also note the following about the elements of G: For every order automorphism ψ of $(\mathbb{Q}, \leq)$, let ϕ_ψ be the element of G defined by $\phi_\psi(a_{qi}) = a_{\psi(q)i}$ for all $q \in \mathbb{Q}$ and $i \in \{1, \ldots, n\}$. Then every element π of G can be given by the formula $\pi = \rho\phi_\psi$, where ψ is an order automorphism of $(\mathbb{Q}, \leq)$, ϕ_ψ is the corresponding element of G defined above, and ρ is an element of G that fixes $\{A_q : q \in \mathbb{Q}\}$ pointwise.)

Claim 3.10. $\mathsf{AC}_n^{\mathsf{LO}}$ *is false in* $\mathcal{V}$, *and hence* (*by Theorem* 3.1(ii)) DT *is also false in* $\mathcal{V}$.

Proof. Let $\mathcal{A} = \{A_q : q \in \mathbb{Q}\}$. Then $\mathcal{A}$ is an element of $\mathcal{V}$, and is also linearly orderable in $\mathcal{V}$. Indeed, the binary relation $\preceq$ on $\mathcal{A}$ defined by:

$$A_q \preceq A_r \iff q \leq r$$

is clearly a (dense) linear order on $\mathcal{A}$. Furthermore, $\emptyset$ is a support for $(\mathcal{A}, \preceq)$, i.e., every permutation of A in G fixes both $\mathcal{A}$ and $\preceq$.

We assert that $\mathcal{A}$ has no choice function in $\mathcal{V}$. Assume the contrary; thus we may let f be a choice function for $\mathcal{A}$ in $\mathcal{V}$. Let $E = \bigcup\{A_q : q \in S\}$ for some bounded $S \subset \mathbb{Q}$, be a support of f. Let $r \in \mathbb{Q}$ be such that $q < r$ for all $q \in S$. Choose an element x of A_r such that $x \neq f(A_r)$ and consider the transposition $\pi = (x, f(A_r))$, i.e., π interchanges x and $f(A_r)$, but fixes all the other atoms. Clearly $\pi(A_r) = A_r$ and $\pi \in \mathrm{fix}_G(E)$; hence $\pi(f) = f$. However, we have

$$(A_r, f(A_r)) \in f \Rightarrow (\pi(A_r), \pi(f(A_r))) \in \pi(f) \Rightarrow (A_r, x) \in f,$$

which contradicts the fact that f is a function, since $x \neq f(A_r)$. Thus $\mathcal{A}$ has no choice function in $\mathcal{V}$. □

Claim 3.11. $\mathsf{AC}^{\mathsf{WO}}$ *is true in* $\mathcal{V}$.

Proof. Let $\mathcal{X}$ be a well-ordered set in $\mathcal{V}$ comprising non-empty sets. Let $E = \bigcup\{A_q : q \in S\}$ for some bounded $S \subset \mathbb{Q}$, be a support of every element of $\mathcal{X}$.[1] Let $K = [r, r']$ be an interval in the ordering of $\mathbb{Q}$ such that

[1] The permutation model $\mathcal{V}$ contains all elements of the kernel V ($V = \bigcup\{V_\alpha : \alpha \in \mathrm{On}\}$, where $V_0 = \emptyset$, $V_{\alpha+1} = \wp(V_\alpha)$, and $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$ if α is a limit ordinal)—this is true for any permutation model—and so AC is true in the kernel; thus every element of V can be well-ordered. Hence an element x of $\mathcal{V}$ can be well-ordered (in $\mathcal{V}$) if and only if there is (in $\mathcal{V}$) an injection f of x into V. For any such f, $\pi f = f$ if and only if $\pi \in \mathrm{fix}_G(x)$, and thus an element x of $\mathcal{V}$ can be well-ordered in $\mathcal{V}$ if and only if $\mathrm{fix}_G(x) \in \mathcal{F}$. By definition of $\mathcal{F}$, '$\mathrm{fix}_G(x) \in \mathcal{F}$' means that there is a subset $F = \bigcup\{A_q : q \in R\} \subset A$ for some bounded $R \subset \mathbb{Q}$ such that $\mathrm{fix}_G(F) \subseteq \mathrm{fix}_G(x)$, and so any $x \in \mathcal{V}$ can be well-ordered if and only if there is a subset $F = \bigcup\{A_q : q \in R\} \subset A$ for some bounded $R \subset \mathbb{Q}$, which is a support of every element of x.

(1) $E \subseteq L$, where $L = \bigcup\{A_q : q \in K\}$, and
(2) there are $t, t' \in A$ such that $r < t < t' < r'$ and

$$\bigcup\{A_q : q \in [r,t]\} \cap E = \emptyset = \bigcup\{A_q : q \in [t',r']\} \cap E.$$

In the ground model M, in which AC is true, we let F be a choice function for $\mathcal{X}$ (so that F may not be in $\mathcal{V}$). For every $x \in \mathcal{X}$, we let $D_{F(x)}$ be a support of $F(x)$. Since $\preceq$ is a dense linear order on $\mathcal{A} = \{A_q : q \in \mathbb{Q}\}$ (see the proof of Claim 3.10) and $D_{F(x)} = \bigcup\{A_q : q \in T\}$ for some bounded $T \subset \mathbb{Q}$, there exists a permutation $\pi_{F(x)} \in \mathrm{fix}_G(E)$ such that

$$\pi_{F(x)}(D_{F(x)}) \subseteq L$$

(see also the parenthetic note following the construction of $\mathcal{V}$). Now we define

$$f = \{(x, \pi_{F(x)}(F(x))) : x \in \mathcal{X}\}.$$

It is clear that f is a function on $\mathcal{X}$, and we assert that $f \in \mathcal{V}$. In order to prove our assertion, we argue that $\mathrm{fix}_G(L) \subseteq \mathrm{fix}_G(f)$. Let $\phi \in \mathrm{fix}_G(L)$ and also let $x \in \mathcal{X}$. Since $E \subseteq L$, we have $\mathrm{fix}_G(L) \subseteq \mathrm{fix}_G(E)$, and hence $\phi(x) = x$. Furthermore, since $\pi_{F(x)}(a) \in \pi_{F(x)}(D_{F(x)}) \subseteq L$ for all $a \in D_{F(x)}$, and $\phi \in \mathrm{fix}_G(L)$, we have

$$\begin{aligned} \phi(\pi_{F(x)}(a)) = \pi_{F(x)}(a), &\quad \text{for all } a \in D_{F(x)} \\ \Rightarrow\ \pi_{F(x)}^{-1}\phi\pi_{F(x)}(a) = a, &\quad \text{for all } a \in D_{F(x)} \\ \Rightarrow\ \phi\pi_{F(x)}(F(x)) = \pi_{F(x)}(F(x)), & \end{aligned}$$

since $D_{F(x)}$ is a support of $F(x)$. Thus $\phi \in \mathrm{fix}_G(f)$.

Furthermore, f is a choice function for $\mathcal{X}$. Indeed, since for every $x \in \mathcal{X}$, $F(x) \in x$ and $\pi_{F(x)} \in \mathrm{fix}_G(E)$, we conclude that $\pi_{F(x)}(F(x)) \in \pi_{F(x)}(x) = x$. □

The above arguments complete the proof of the theorem. □

THEOREM 3.12. *Let $\aleph_\alpha$ be a regular aleph. Then "for every infinite well-ordered cardinal $\lambda < \aleph_\alpha$,* DC_λ*" does not imply* DT *in* ZF. *Hence, neither does the statement "for every infinite well-ordered cardinal $\lambda < \aleph_\alpha$,* AC^λ*" (where* AC^λ *is the axiom of choice for λ-sized families of non-empty sets) imply* DT *in* ZF.

In particular, DC *does not imply* DT *in* ZF, *and hence neither does* AC^ω. *Furthermore,* RT *does not imply* DT *in* ZF.

PROOF. In the proof of Jech's Theorem 8.3 in [10], a permutation model $\mathscr{V}$ is constructed so that DC_λ is true in $\mathscr{V}$ for all infinite well-ordered cardinals $\lambda < \aleph_\alpha$, whereas there exists an $\aleph_\alpha$-sized family of unordered pairs which has no choice function in $\mathscr{V}$. Then Jech embeds $\mathscr{V}$ in a symmetric model $\mathcal{N}$ of ZF so that $\mathcal{N} \models \forall\lambda(\lambda < \aleph_\alpha \rightarrow \mathsf{DC}_\lambda) \wedge \neg\mathsf{AC}_2^{\aleph_\alpha}$ (where $\mathsf{AC}_2^{\aleph_\alpha}$ is the axiom of choice for $\aleph_\alpha$-sized families of pairs). By Theorem 3.1(ii), it follows that DT is false in $\mathcal{N}$.

The second assertion of the theorem follows from the fact that for any infinite well-ordered cardinal κ, $\mathsf{DC}_\kappa \rightarrow \mathsf{AC}^\kappa$ (see [10, Theorem 8.1]).

The last assertion of the theorem follows from the first one and the fact that $\mathsf{DF} = \mathsf{F}$ (and hence AC^ω) implies RT (see Theorem 2.3(ix)). □

THEOREM 3.13. CAC *does not imply* DT *in* ZF, *and* DT *does not imply* CAC *in* ZFA.

PROOF. It is known that $\mathsf{DF} = \mathsf{F}$ implies CAC (see [8]), and since AC^ω implies $\mathsf{DF} = \mathsf{F}$ (see [8], [10]), it follows (by Theorem 3.12) that CAC does not imply DT in ZF.

On the other hand, CAC implies $\mathsf{AC}_{\mathsf{fin}}^\omega$ (see Tachtsis [18]), and hence (by Theorem 3.4) DT does not imply CAC in ZFA. □

4. Open questions

(1) Is there a model of ZFA in which $\mathsf{AC}^{\mathsf{LO}}$ is true, but DT is false? (We recall here that $\mathsf{AC}^{\mathsf{LO}}$ is equivalent to AC in ZF, but *not* equivalent to AC in ZFA (see [8]). It follows that $\mathsf{AC}^{\mathsf{LO}} \rightarrow \mathsf{DT}$ in ZF.)

(2) Does OP imply DT?

(3) Does MHT imply DT?

(4) Does RSL imply DT in ZF? (Recall that, by Theorem 3.1(ii), RSL *does not imply* DT *in* ZFA.)

(5) Does $\mathsf{DT} + \mathsf{AC}_{\mathsf{fin}}$ imply BPI?

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