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AREA MINIMIZATION OF SPECIAL POLYGONS

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Abstract. It is quite rare that a simple area optimization result bears somebody's name. One of these statements, called Hajós' Lemma, became particularly known, mainly because of its esthetic appearance and due to its application at solving the densest circle packing problem. Hajós considered a pair of concentric circles and wanted to find the minimum area polygon among those polygons which contain the smaller circle and whose vertices are outside of the larger circle. In this paper we state and prove two generalizations of Hajós' Lemma. In the first version we allow the circles to be non concentric, in the second version we consider disc polygons instead of usual polygons.

1. Introduction

Area, diameter, perimeter and width are standard characteristics of planar polygons. Typically, complex geometric proofs are reduced to the optimization of one of these characteristics under certain geometric constraints.

György Hajós proved the following theorem (see the paper of Molnár [3]):

Theorem 1 (Hajós Lemma). *Let* $0 < r < 1$. *Among all convex polygons, which contain a circle of radius* r, *and have no vertices inside of the concentric unit circle*, *the one which is inscribed in the unit circle so that all sides*, *with the exception of at most one*, *are tangent to the smaller circle is of minimum area.*

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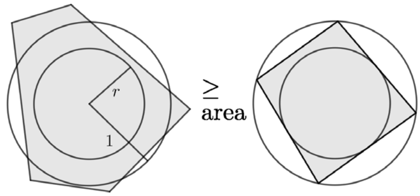

Fig. 1: An illustration for Hajós Lemma

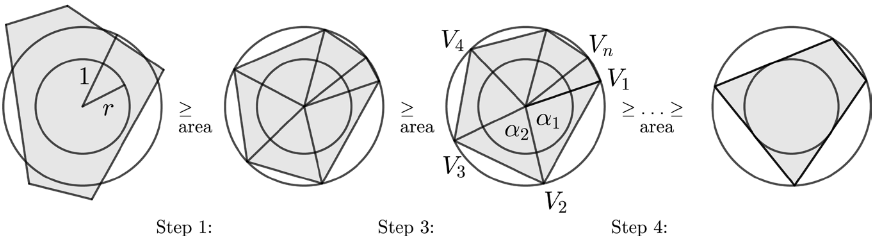

Fig. 2: Proof of Hajós Lemma

For visual illustration of Hajós Lemma see Fig. 1. What makes Hajós Lemma important is its application to the densest circle packing problem.

Corollary 1. *That arrangement of circles of radius r, where each disc is touched by exactly six other ones, has the largest density (largest percent of the plane occupied by discs) among packings of circles of radius r in the plane.*

We include the proofs of Theorem 1 and Corollary 1 in Section 2 partly for completeness, but also because later we will refer to the steps of the proof.

2. Proof of Theorem 1 and Corollary 1

Proof of Theorem 1. Fig. 2 walks the reader through a simple area reduction process. We start with an arbitrary polygon satisfying the conditions of Hajós Lemma, and then we change the polygon using the following steps.

Step 1. Using intersection points of the polygon and the unit circle find an inscribed polygon with smaller area.

Step 2. Do.

Step 3. Rearrange the circular sectors so that in clockwise order the central angles decrease: $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

Step 4. Starting with index $i = 1$, whenever side V_iV_{i+1} does not touch the unit circle we relocate V_{i+1} on the circle in clockwise direction until V_iV_{i+1} becomes tangent to the inner circle or V_{i+1} coincides with V_{i+2}.

Step 5. Until all sides of the polygon, with the exception of at most one, are tangent to the smaller circle.

Each step in the above process decreases the area and at the end we get the alleged minimum area polygon. □

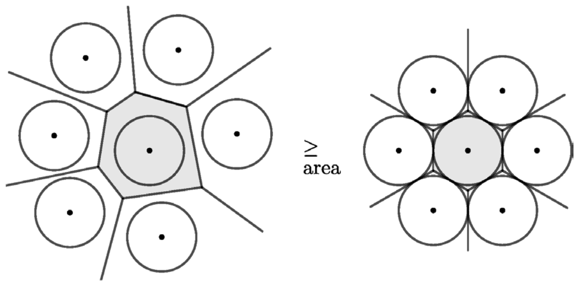

Fig. 3: Corollary 1, where points represent pharmacy sites

PROOF OF COROLLARY 1. In order to use Hajós Lemma in the stated form we prove the statement of the Corollary for circles of radius $\sqrt{3}/2$. Let us start with the nearest point subdivision of the plane determined by the centers of the circles. In particular, we assign to a center, say to center O those points of the plane which are closer to O, than to any other center. This leads to a subdivision of the plane into polygons (Fig. 3). A simple elementary geometric argument shows that no vertex of such polygons can be closer than 1 to the center of that circle which they contain. According to Hajós Lemma in case of $r = \sqrt{3}/2$ the smallest area polygon is the regular one and its area is equal to $3\sqrt{3}/2$. This, in different terms means that the portion of the plane which can be occupied by circles cannot be more than

$$\frac{3\pi/4}{3\sqrt{3}/2} = \frac{\pi}{\sqrt{12}} \approx 0.907. \quad \square$$

3. New results

Several generalizations of Hajós Lemma were considered before. In [1] Böröczky considered a pair of concentric circles and wanted to find the minimum area polygon among those polygons which contain the smaller circle and whose vertices, with the exception of two adjacent vertices, are on the larger circle. In [2] Böröczky et. al. considered a d-dimensional generalization of Hajós Lemma. They wanted to find the minimum d-volume d-polyhedron among those d-polyhedrons whose k-faces lie at least at distance r_k $(k = 0, \dots, d-1)$ from a fixed point.

We generalize Hajós Lemma in the following two ways.

3.1. Hajós Lemma for non concentric circles.

Theorem 2. *Assume a unit circle contains a given non concentric smaller circle. Consider all polygons which are inscribed in the unit circle and contain the smaller circle. The smallest area polygon in this family has all of its sides, with the exception of at most one side, tangent to the smaller circle.*

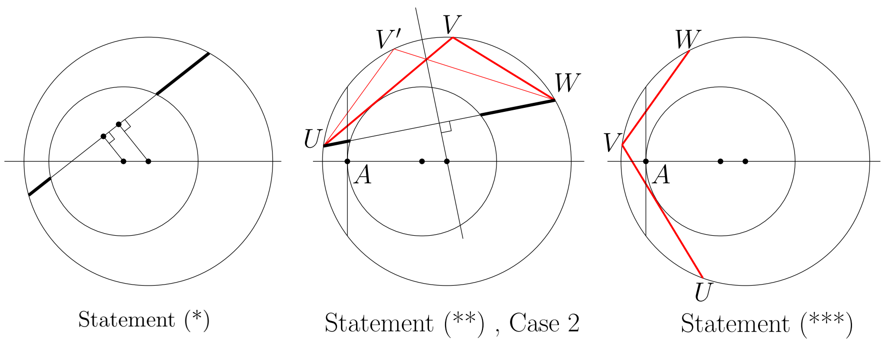

Fig. 4: Proof of Theorem 2

Proof of Theorem 2. For orientation, assume that the centers of the circles are on a horizontal line and the center on the right is the center of the unit circle (Fig. 4). We will say that a point (segment resp.) is *to the right* from another point (segment resp.), if its projection on the horizontal line is to the right from the projection of the other point (segment resp.). The relation *to the left* can be defined similarly. Consider now a non vertical transversal, which intersects both circles. In particular, consider the midpoints of the cords of intersection on the transversal. Notice that the midpoint of the chord of the unit circle is to the right from the midpoint of the chord of the smaller circle. This immediately implies that

$(*)$ *No vertical transversals of the smaller circle intersects the annulus two segments so that the segment on the right is longer than the one on the left.*

Let A be the most left point of the smaller circle (Fig. 4). Our proof continues as a standard minimal choice proof. Let $\mathcal{P}$ be one of the smallest area polygons satisfying the conditions of Theorem 2. Start by the simple fact that polygon $\mathcal{P}$ can not have two non tangent adjacent sides. This simple statement was also proved in the original Hajós Lemma: Indeed assume UV and VW are two non tangent sides. One can reposition vertex V along the unit circle so that no tangency of sides is maintained while the area of the triangle UVW decreases and thus the minimal choice of $\mathcal{P}$ is contradicted. For Theorem 2 it is enough to show the following two statements:

($**$) *No tangent sides of* $\mathcal{P}$ *are completely left from* A *or their horizontal projection contains* A.

($***$) *Polygon* $\mathcal{P}$ *has at most one non tangent side which is completely left from* A *or whose horizontal projection contains* A.

Indeed ($**$) characterizes the non tangent sides while ($***$) claims that there is only one such side.

PROOF OF ($**$). Assume ($**$) does not hold and try to decrease the area of $\mathcal{P}$. Let VW be a non tangent side of $\mathcal{P}$ so that both V and W are to the right from A (Fig. 4/Statement ($**$)). Let UV be the adjacent side of VW in $\mathcal{P}$. The side UV is tangent to the smaller circle otherwise $\mathcal{P}$ would have two non tangent adjacent sides. We distinguish two cases.

Case 1: $UV \leq VW$. In this case modify the polygon $\mathcal{P}$, by moving vertex V along the unit circle toward U. If the move is sufficiently small both UV and VW became non-tangent. Along this change the area of triangle UVW decreases, producing a new polygon with area smaller than that of $\mathcal{P}$, a contradiction.

Case 2: $UV > VW$. In this case replace vertex V in $\mathcal{P}$ by its reflected image V' along the perpendicular bisector of UW to get a new polygon $\mathcal{P}'$. Polygons $\mathcal{P}$ and $\mathcal{P}'$ have equal areas, moreover UW is a non vertical line so that W is to the right from U. Closer look reveals that ($*$) implies that both UV' and $V'W$ are non tangent segments. Thus, the area of $\mathcal{P}$ cannot be the smallest. $\square$

PROOF OF ($***$). Indirectly assume ($***$) is false and $\mathcal{P}$ has at least two non tangent sides. If both are left from A, then there are two adjacent such sides and by repositioning by their common vertex one can decrease the area of $\mathcal{P}$.

Let VW be a non tangent side of $\mathcal{P}$ so that its horizontal projection contains A (Fig. 4/ Statement ($***$)). Let again UV be the adjacent side of VW in $\mathcal{P}$. We may assume that UV is tangent to $\mathcal{P}$. If U is to the left from A, then it is also left from W and thus just like in ($**$) we can reposition V and decrease the area of polygon $\mathcal{P}$. If U is right over A, then the method of ($**$) produces a new polygon whose area is less than the area of $\mathcal{P}$. If U is right from A, then any other non tangent side would have both of its vertices right from A, which contradicts ($**$). $\square$

In contrast to the original concentric circle version of Hajós Lemma, in the non concentric case Theorem 2 does not describe uniquely the minimizing polygon. Theorem 2 reduces the problem to finding the minimum area within a one parameter family. This is a computational problem unless the minimizing polygons have a nice geometric property. For example, we hoped that the minimizing polygon is always symmetrical to the line connecting the centers of the circles. Theorem 3 is a numerical evidence that this is not always the case.

THEOREM 3. *Assume the unit circle of center* $O(0,0)$ *contains a given non concentric smaller circle of radius* r *and center* $O_2(-c,0)$. *Consider all polygons which are inscribed in the unit circle and contain the smaller circle. If* $0 < c \le r \le 0.24$, *then* $(1,0)$ *is a vertex of the smallest area polygon. If* $c = 0.6$, $r = 0.2$, *then* $(1,0)$ *is not a vertex of the smallest area polygon.*

Theorem 3 settles a special case, where the smaller circle is so small that the one parameter family consists of triangles only. Depending on the relative location of the smaller circle the optimizing triangle sometimes is isosceles, sometimes is not.

Section 4 at the end of this paper contains the short version of the computational proof of Theorem 3.

3.2. Hajós Lemma for disc polygons. An *R-disc polygon* (or shortly *disc polygon*) is the intersection of (finitely many) congruent discs of radius R. The circular arcs on the boundary of a disc polygon are called *sides*, and common points of adjacent sides are called *vertices*. We say that a side of a disc polygon is *tangent* to a circle if the circular arc of the disc polygon is tangent to the circle. We say that an R-disc polygon with $R > 1$ is inscribed in a unit circle, if the vertices of P lie on the unit circle.

To illustrate the use of terminology we refer to Fig. 5 which shows a 2-disc polygon inscribed in the circle $\mathcal{C}_1$ of radius 1. The side A_2A_3 of the 2-disc polygon is tangent to the circle $\mathcal{C}_2$.

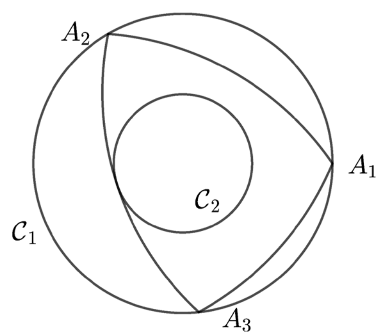

Fig. 5: A 2-disc polygon in $\mathcal{C}_1$

THEOREM 4. *Let* $0 < r < 1$ *and* $R > 1$. *Among all* R*-disc polygons which contain the circle of radius* r *and have no vertices inside of the concentric* *unit circle, the one which is inscribed in the unit circle so that all sides (with the exception of at most one) are tangent to the smaller circle is of minimum area.*

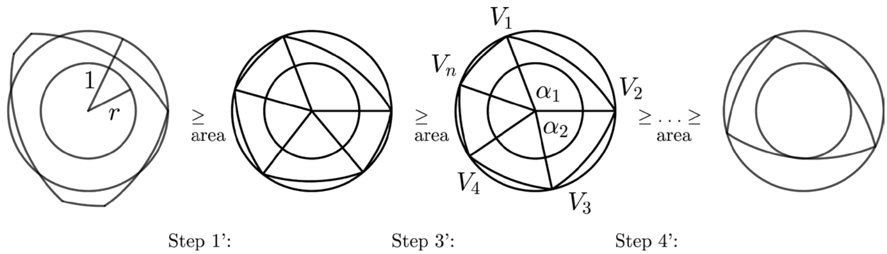

Fig. 6: Proof of a generalization of Hajós Lemma

PROOF OF THEOREM 4. Fig. 6 walks the reader through a simple area reduction process. We start with an arbitrary R-disc polygon satisfying the conditions of Theorem 4, next we change the polygon using the following steps.

Step $1'$. Using intersection points of the disc polygon and the unit circle find an inscribed polygon with smaller area.

Step $2'$. Do

Step $3'$. Rearrange the circular sectors so that in clockwise order the central angles decrease: $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$.

Step $4'$. Starting with index $i = 1$, whenever side $\widehat{V_iV_{i+1}}$ does not touch the unit circle we relocate V_{i+1} on the circle in clockwise direction until $\widehat{V_iV_{i+1}}$ becomes tangent to the inner circle or V_{i+1} coincides with V_{i+2}.

Step $5'$. Until all sides of the disc polygon, with the exception of at most one, are tangent to the smaller circle. □

At first it looks like we have again a straightforward proof. First by cutting off parts, next rearranging its pieces and finally moving some of its vertices, one at a time, it seems that we keep decreasing the area and at the end we get the alleged minimum area disc polygon. But closer look reveals that proving that at Step $3'$ the area indeed decreases is not so simple. The following lemmas are going to be the key.

We start with two elementary properties of circles:

LEMMA 1. *Let c be a semicircle of diameter AB and center O. Let C be a point on AB such that $CB < AC$. Let us move a point along the semicircle from B to A and parameterize the position by P_θ so that $\theta = \angle BCP_\theta$. Along the motion both distances CP_θ and $P_\theta P_{\theta+\gamma}$, where γ is a fixed angle, increase.*

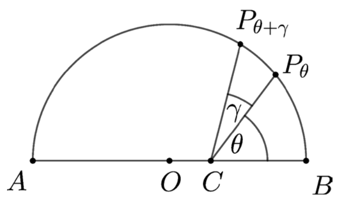

Fig. 7: Lemma 1

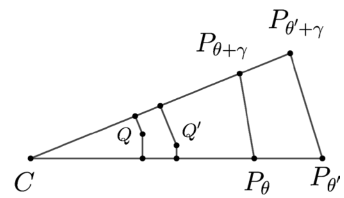

Fig. 8: Proof of Lemma 1

Proof of Lemma 1. We will compare instances corresponding to $0 < \theta < \theta' < \pi$. The inequality $CP_\theta < CP_{\theta'}$ follows from the fact that the perpendicular bisector of $P_\theta P_{\theta'}$ passes through O. To see that the inequality $P_\theta P_{\theta+\gamma} < P_{\theta'} P_{\theta'+\gamma}$ holds, it is enough to see that the circumradius of the triangle $CP_\theta P_{\theta+\gamma}$ is less than that of triangle $CP_{\theta'} P_{\theta'+\gamma}$. Let us superimpose the two triangles (Fig. 8) and add the perpendicular bisectors of the sides emanating from C to get the circumcenters Q and Q'. We already know that distance CP_θ increases as θ increases, thus the relative positions of Q and Q' is exactly as shown on Fig. 8. Such relative positions imply $OQ < OQ'$. $\square$

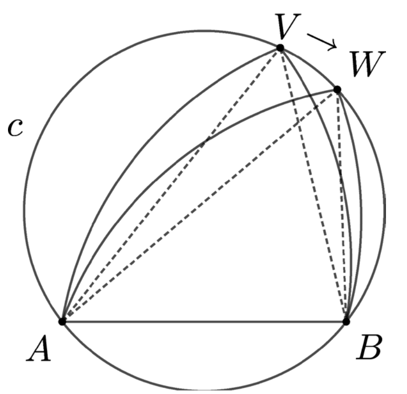

Fig. 9: Lemma 2

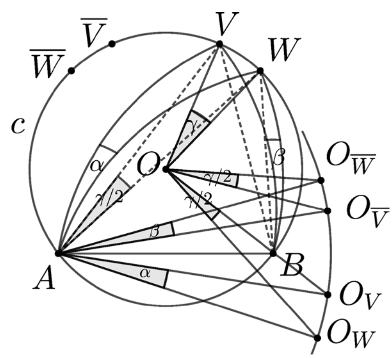

Fig. 10: Proof of Lemma 2

Next we show that small replacement of a vertex of disc polygons and also of a regular polygons along their circumcircle changes their areas exactly in the same way:

Lemma 2. *Let A, B and V be three points on a unit circle c, so that V is closer to B than to A. Moving point V along the circle c toward B decreases both the area of triangle ABV and the area of the convex region bounded by the segment AB and circular arcs $\widehat{AV}$ and $\widehat{BV}$ of radii $R > 1$ provided O lies in the mentioned convex region.*

Proof of Lemma 2. The statement concerning the area of triangle ABV is trivial, we included it in Lemma 2 mainly to stress analogy. Assume the orientation is so that AB is horizontal (Fig. 9). Let W be the new position of V and let γ be the central angle $\angle VOW$. Denote by α (β resp.) the angle between the circular arcs $\widehat{AV}$ and $\widehat{AW}$ (and between the circular arcs $\widehat{BV}$ and $\widehat{BW}$ resp.). Reflect V and W along the vertical line through center O to get $\overline{V}$ and $\overline{W}$. Let $O_{\overline{W}}$ ($O_{\overline{V}}$, O_V and O_W resp.) be the center of the circle containing the arc $\widehat{A\overline{W}}$ ($\widehat{A\overline{V}}$, $\widehat{AV}$ and $\widehat{AW}$ resp.). These centers lie on a circle of radius R and center A. Observe that the angle $\angle VAW$ is inscribed in the circle c and is equal to $\gamma/2$. Since the triangle AO_WW (AO_VV resp.) is isosceles, we have that the segment O_WO (O_VO, resp.) is perpendicular to the chord AW (AV resp.). Thus $\angle O_WOO_V = \gamma/2$. By a similar reason, we get that $\angle O_{\overline{V}}OO_{\overline{W}} = \gamma/2$. According to Lemma 1 we get the distance inequality $O_{\overline{W}}O_{\overline{V}} < O_VO_W$, which in term of the central angles is the same as $\alpha = \angle O_{\overline{W}}AO_{\overline{V}} < \angle O_VAO_W = \beta$. Once we established $\alpha < \beta$ Lemma 2 follows immediately from the observation that i) it is enough to show lemma for small γ, and ii) in case of small γ it is obvious that the region what we gain can be placed inside of the region what we loose, thus the total area decreases. $\square$

4. Proof of Theorem 3

First we prove if $0 < c \le r \le 0.24$, then the smallest area polygon in this family has a vertex with coordinates $(1, 0)$.

Let $\mathcal{C}_1$ ($\mathcal{C}_2$ resp.) be the circle of radius 1 (r ($r < 1$) resp.) and center O (O_2 resp.). Let $P(\cos\varphi, \sin\varphi)$ be a point on the circle $\mathcal{C}_1$. Let T_1 and T_2 be two different points on the circle $\mathcal{C}_2$ such that PT_1 and PT_2 be the tangent lines of the circle $\mathcal{C}_2$ as in Fig. 11.

Let Q_1 (Q_2 resp.) be the intersection point of the line through the points P and T_1 (T_2 resp.) as in Figure 11. Let α_1 (α_2 resp.) be the convex angle $\angle POQ_1$ ($\angle POQ_2$ resp.). Observe, the straight line segment Q_1Q_2 does not intersects $\mathcal{C}_2$. If we search the minimum of the area of the triangle PQ_1Q_2, then by Statement $(**)$ of Theorem 2, the point P lies on the circular arc of $\mathcal{C}_1$ between the points when T_1 coincides with A and T_2 coincides with A.

Let S be the sum of the areas of the triangles POQ_1 and POQ_2. Let β (ε resp.) be the angle $\angle O_2PT_2$ ($\angle O_2PO$ resp.). First we find the minimum of S. We have

$$S = \frac{1}{2}\sin\alpha_1 + \frac{1}{2}\sin\alpha_2 = \frac{1}{2}\sin(2\beta + 2\varepsilon) + \frac{1}{2}\sin(2\beta - 2\varepsilon)$$
$$= \sin(2\beta)\cos(2\varepsilon) = 2\sin\beta\cos\beta(2\cos^2\varepsilon - 1).$$

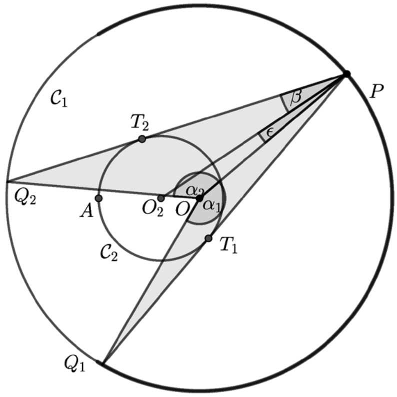

Fig. 11: The circles C_1 and C_2

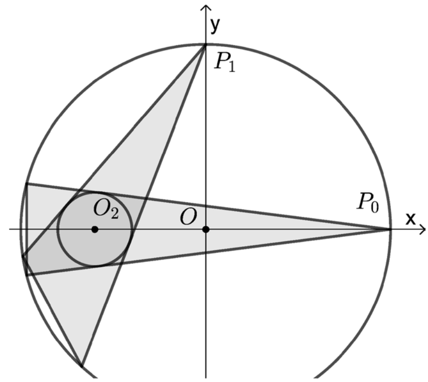

Fig. 12: The non symmetric case

From the triangle OO_2P (T_2O_2P resp.) we have

$$\varepsilon = \arccos \frac{1 + (c + \cos\varphi)^2 + \sin^2\varphi - c^2}{2\sqrt{(c + \cos\varphi)^2 + \sin^2\varphi}}$$

$$\left(\beta = \arcsin \frac{r}{\sqrt{(c + \cos\varphi)^2 + \sin^2\varphi}} \text{ resp.}\right).$$

Since $0 < \beta < \pi/2$ and $0 < \varepsilon < \pi/2$,

$$S = 2\frac{r}{\sqrt{(c + \cos\varphi)^2 + \sin^2\varphi}} \sqrt{1 - \left(\frac{r}{\sqrt{(c + \cos\varphi)^2 + \sin^2\varphi}}\right)^2}$$

$$\times \left(2\left(\frac{1 + (c + \cos\varphi)^2 + \sin^2\varphi - c^2}{2\sqrt{(c + \cos\varphi)^2 + \sin^2\varphi}}\right)^2 - 1\right).$$

Let us assume that the area S is a function of φ. The derivative of S is

$$S'(\varphi) = \frac{-2cr\sin\varphi}{(2c\cos\varphi + c^2 + 1)^3\sqrt{2c\cos\varphi + c^2 - r^2 + 1}} f(\varphi),$$

where

$$f(\varphi) = 4c^3\cos^3\varphi + (10c^4 + 6c^2)\cos^2\varphi + (4c^5 + 16c^3 - 4c^3r^2)\cos\varphi$$
$$+ 5c^4 + 4c^2 - 6c^2r^2 + 2r^2 - 1.$$

The sign of $S'(\varphi)$ depends on the sign of $f(\varphi)$ only. Let

$$g(x) = 4c^3x^3 + (10c^4 + 6c^2)x^2 + (4c^5 + 16c^3 - 4c^3r^2)x$$
$$+ 5c^4 + 4c^2 - 6c^2r^2 + 2r^2 - 1.$$

Observe $g(\cos(\varphi)) = f(\varphi)$. We have

$$g'(x) = 12c^3x^2 + (20c^4 + 12c^2)x + 4c^5 + 16c^3 - 4c^3r^2.$$

The zeros of g' are

$$x_1 = \frac{-3 - 5c^2 + \sqrt{9 - 18c^2 + 13c^4 + 12r^2c^2}}{6c}$$

and

$$x_2 = \frac{-3 - 5c^2 - \sqrt{9 - 18c^2 + 13c^4 + 12r^2c^2}}{6c}.$$

We show $x_2 < -1$. After performing the equivalent changes on the inequality

$$\frac{-3 - 5c^2 - \sqrt{9 - 18c^2 + 13c^4 + 12r^2c^2}}{6c} < -1$$

we have

$$\sqrt{9 - 18c^2 + 13c^4 + 12r^2c^2} \geq 0 > -5c^2 + 6c - 3$$

which is true for for all $r \in [0, 1]$ and $c \in [0, 1]$.

We show $-1 < x_1$. Let us assume that x_1 is a function of c and r. Let us denote this function by $F(c, r)$. The function $h(c) = F(c, 0)$ is a function of one variable and it is decreasing on the interval $[0, 1]$ and its minimum is -1. The partial derivative

$$F'_r(c, r) = \frac{2cr}{\sqrt{9 - 18c^2 + 13c^4 + 12r^2c^2}}$$

is positive if $c > 0$ and $r > 0$.

We show $x_1 < 0$. After performing the equivalent changes on the inequality

$$\frac{-3 - 5c^2 + \sqrt{9 - 18c^2 + 13c^4 + 12r^2c^2}}{6c} < 0$$

we have $r^2 < 4 + c^2$ which is true for all $r \in [0, 1]$ and $c \in [0, 1]$. Thus $g'(x) > 0$ if $x_1 < x \leq 1$. If $g(1) \leq 0$, then $g(x) < 0$ for $x \in (x_1, 1)$.

Let us assume that $g(1)$ is a function of c and r which is denoted by $G(c, r)$. Thus

$$G(c, r) = 4c^5 + 15c^4 + (20 - 4r^2)c^3 + (10 - 6r^2)c^2 + 2r^2 - 1.$$

We will show that $G(c, r) < 0$ if $0 < c \leq 0.24$ and $0 < r \leq 0.24$. Since

$$G'_c(c, r) = 4c(c + 1)(5c^2 + 10c - 3r^2 + 5) > 0$$

for $0 < c \leq 0.24$, $0 < r \leq 0.24$,

$$G'_r(c,r) = 4r(1-2c)(c+1)^2 > 0$$

for $0 < c \leq 0.24$ and $0 < r \leq 0.24$,

$$g(1) = G(c,r) \leq G(c,0.24) \leq G(0.24,0.24) = -0.002 < 0.$$

Thus $g(x) < 0$ if $x_1 < x \leq 1$. If $x_1 < x \leq 1$ and $\cos\varphi = x$, then $0 < \varphi < \varphi_0$ where $\varphi_0 > \pi/2$. Moreover $f(\varphi) < 0$ if $0 < \varphi < \varphi_0$. Therefore $S'(\varphi) > 0$ if $0 < \varphi < \varphi_0$ but $S'(0) = 0$.

Observe

$$g(1) - g(-1) = 8c^5 + 40c^3 - 8c^3r^2 = 8c^3(c^2 + 5 - r^2) > 0$$

for all $r \in [0,1]$ and $c \in [0,1]$. Since $g'(x) < 0$ if $-1 < x < x_1$ and $g'(x) > 0$ if $x_1 < x < 1$ and $0 > g(1) > g(-1)$, $g(x) < 0$ if $x \in [-1,1]$. Therefore $S'(\varphi) > 0$ if $0 < \varphi < \pi$, $S'(0) = 0$ and $S'(\pi) = 0$. Moreover S is the smallest if $\varphi = 0$ and the greatest if $\varphi = \pi$.

Now we find the smallest area of the triangle OQ_1Q_2. Observe $4\beta < \pi/2$. The area of the triangle OQ_1Q_2 is the smallest if β is the smallest. Thus the area of the triangle OQ_1Q_2 is the smallest if the coordinates of P are $(1,0)$. Therefore the area of the triangle PQ_1Q_2 is the smallest if the coordinates of P are $(1,0)$.

Next we prove if $c = 0.6$, $r = 0.2$, then the smallest area polygon in this family does not have the vertex with coordinates $(1,0)$.

If $c = 0.6$, $r = 0.2$ and $P_0(1,0)$, then the area of the triangle $P_0Q_1Q_2$ is 0.4883. If $c = 0.6$, $r = 0.2$ and $P_1(0,1)$, then the area of the triangle $P_1Q_1Q_2$ is 0.4771. Thus if $c = 0.6$ and $r = 0.2$, then the minimum of the area of the triangle does not attain in the symmetric case (Fig. 12). $\square$

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