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SUBLINEAR OPERATORS ON RADIAL REARRANGEMENT-INVARIANT QUASI-BANACH FUNCTION SPACES

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Abstract. This paper establishes the boundedness of the disc multiplier, the spherical means, the Bochner–Riesz means, the rough singular integral operators, the universal maximal operator and the directional operators on the radial subspace of rearrangement-invariant quasi-Banach function spaces.

1. Introduction

This paper aims to study the mapping properties of the disc multiplier, the spherical means, the Bochner–Riesz means, the rough singular integral operators, the universal maximal operator and the directional operators on rearrangement-invariant quasi-Banach function spaces.

These operators have two common features: they are sublinear and roughly speaking, they are not bounded on the entire L^p, $1 < p < \infty$, while they are bounded operators from the radial Lebesgue space L^p_{rad} to L^p where L^p_{rad} denotes the subspace of L^p consisting of radial functions.

Our study is mainly motivated by the disc multiplier. It is a celebrated result from Fefferman [10] which shows that the disc multiplier is bounded on L^p if and only if $p = 2$. On the other hand, Herz [12] proved that the disc multiplier is bounded from L^p_{rad} to L^p when $\frac{4}{3} < p < 4$. The result from Herz shows that the mapping properties of operators on radial Lebesgue spaces are different from the entire Lebesgue spaces.

We see that a number of operators share this property. For instance, the Bochner–Riesz means [18,19], the Hankel multipliers [11], the rough singular integral operators [24], the universal maximal operator [3,4] and the directional operators [6].

Key words and phrases: radial function, rearrangement-invariant, quasi-Banach function space, disc multiplier, spherical mean, Bochner–Riesz mean, universal maximal operator, directional operator.

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The above results motivate us to investigate the mapping properties of the above mentioned operators on some extensions of Lebesgue spaces such as the Lorentz spaces, the Orlicz spaces and the Lorentz–Karamata spaces. The above mentioned function spaces are members of rearrangement-invariant quasi-Banach function spaces.

We have a powerful tool to obtain the mapping properties of sublinear operators on rearrangement-invariant quasi-Banach function spaces, namely, the Boyd's interpolation theorem [1, Ch. 3, Theorem 5.16]. On the other hand, roughly speaking, the Boyd's interpolation theorem just assures the boundedness of a given operator whenever this operator is bounded on the entire Lebesgue spaces while the disc multiplier does not fulfill this criterion.

Therefore, in this paper, we modify the original Boyd's interpolation theorem and extend it to the case where the operator is bounded on radial Lebesgue spaces. With this extended Boyd's interpolation theorem, we apply it to the Bochner–Riesz means, the Hankel multipliers, the rough singular integral operators, the universal maximal operator and the directional operators and obtain the corresponding mapping properties on rearrangement-invariant quasi-Banach function spaces.

This paper is organized as follows. Section 2 presents some definitions and preliminary results on rearrangement-invariant quasi-Banach function spaces. A general interpolation result for operator on radial function spaces is established in Section 3.2. The mapping properties of the disc multiplier, the spherical means, the Bochner–Riesz means, the rough singular integral operators, the universal maximal operator and the directional operators are given in Section 3.2 also.

2. Preliminaries and definitions

Denote $\mathcal{M}(\mathbb{R}^n)$ and L^p, $0 < p \leq \infty$, the set of Lebesgue measurable functions and the Lebesgue spaces on $\mathbb{R}^n$, respectively.

We recall the definition of a rearrangement-invariant quasi-Banach function space from [14]. For any $f \in \mathcal{M}(\mathbb{R}^n)$ and $s > 0$, write

$$d_f(s) = \big|\{x \in \mathbb{R}^n : |f(x)| > s\}\big|$$

and

$$f^*(t) = \inf\big\{ s > 0 : d_f(s) \leq t\big\}, \quad t > 0.$$

For any $f, g \in \mathcal{M}(\mathbb{R}^n)$, f and g are equimeasurable if $d_f(s) = d_g(s)$ for all $s > 0$.

We say that the function $f\colon [1,\infty) \to (0,\infty)$ is equivalent to a function $g\colon [1,\infty) \to (0,\infty)$ if there exist constants $B, C > 0$ such that

$$Cg(t) \leq f(t) \leq Bg(t), \quad t \geq 1.$$

We restate the definition of rearrangement-invariant quasi-Banach function space(r.i.q.B.f.s.) from [14, Definition 4.1].

Definition 2.1. A quasi-Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is a rearrangement-invariant quasi-Banach function space if there exists a unique quasi-norm $\rho_X : \mathcal{M}(0,\infty) \to [0,\infty]$ satisfying

(1) $\rho_X(f) = 0 \Leftrightarrow f = 0$ a.e.,
(2) $|g| \le |f|$ a.e. $\Rightarrow \rho_X(g) \le \rho_X(f)$,
(3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho_X(f_n) \uparrow \rho_X(f)$,
(4) $\chi_E \in \mathcal{M}(0,\infty)$ and $|E| < \infty \Rightarrow \rho_X(\chi_E) < \infty$,

so that

$$\|f\|_X = \rho_X(f^*), \quad \forall f \in X. \tag{2.1}$$

Write

$$\bar{X} = \left\{ g \in \mathcal{M}(0,\infty) : \rho_X(g) < \infty \right\}.$$

Obviously, $\bar{X}$ is a r.i.q.B.f.s. on $(0,\infty)$.

A Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is a Banach function space if $\|\cdot\|_X$ is a norm and fulfills items (1)–(3),

$$\chi_E \in \mathcal{M}(\mathbb{R}^n) \text{ and } |E| < \infty \quad \Rightarrow \quad \chi_E \in X \tag{2.2}$$

and

$$\chi_E \in \mathcal{M}(\mathbb{R}^n) \text{ and } |E| < \infty \quad \Rightarrow \quad \int_E f\,dx \le C_E \|f\|_X, \tag{2.3}$$

for some $C_E > 0$.

Moreover, X is a rearrangement-invariant Banach function space (r.i.B.f.s.) if X is a Banach function space and for any equimeasurable functions f and g, $\|f\|_X = \|g\|_X$.

Let X be a r.i.B.f.s. The Luxemburg representation theorem [1, Ch. 2, Theorem 4.10] guarantees the existence of ρ_X for X. The reader is referred to [1, p. 64] for the uniqueness of ρ_X.

The Lebesgue spaces, the Lorentz spaces, the Orlicz spaces, the Lorentz–Zygmund spaces and the Lorentz–Karamata spaces are members of r.i.q.B.f.s.

We recall the definition of Lorentz–Karamata spaces $L^{p,q,b}$ in the following as we will study the mapping properties of the disc multiplier, the spherical means,the Bochner–Riesz means, the universal maximal operator and the directional operators on $L^{p,q,b}_{\mathrm{rad}}$.

We recall the notion of slowly varying function from [9, Definition 3.4.32].

Definition 2.2. A Lebesgue measurable function $b\colon [1,\infty) \to (0,\infty)$ is called a slowly varying function if for any given $\varepsilon > 0$, $t^{\varepsilon} b(t)$ and $t^{-\varepsilon} b(t)$ are equivalent to a non-decreasing function and a non-increasing function, respectively.

For any slowly varying function b, define $\gamma_b\colon (0,\infty)\to(0,\infty)$ by

$$\gamma_b(t) = b(\max\{t, 1/t\}), \quad t > 0.$$

We now define the Lorentz–Karamata space [9, Definition 3.4.38].

Definition 2.3. Let $1 < p, q < \infty$ and b be a slowly varying function. The Lorentz–Karamata space $L^{p,q,b}$ consists of those Lebesgue measurable functions f satisfying

$$\|f\|_{L^{p,q,b}} = \left(\int_0^\infty (t^{1/p}\gamma_b(t) f^*(t))^q \, \frac{dt}{t}\right)^{1/q} < \infty.$$

The Lorentz–Karamata space $L^{p,q,b}$ is a r.i.q.B.f.s., see [9, Theorem 3.4.41]. When $b \equiv 1$, $L^{p,q,b}$ reduces to the Lorentz space. For the studies of the Hausdorff-Young inequality, the Hankel transform and the oscillatory integrals on Lorentz–Karamata spaces, the reader is referred to [14,15]. Furthermore, we refer the reader to [8,9,22] for the details of Lorentz–Karamata spaces.

We need to use the following indices in our study. The Boyd's indices give control on the operator norm of the dilation operators.

For any $s \geq 0$ and $f \in \mathcal{M}(0,\infty)$, define $(D_s f)(t) = f(st)$, $t \in (0,\infty)$. Let $\|D_s\|_{\bar{X}\to\bar{X}}$ be the operator norm of D_s on $\bar{X}$. We recall the definition of Boyd's indices for r.i.q.B.f.s. from [14, Definition 2.2].

Definition 2.4. Let X be a r.i.q.B.f.s. on $\mathbb{R}^n$. Define the lower Boyd index of X, p_X, and the upper Boyd index of X, q_X, by

$$p_X = \sup\left\{p > 0 : \exists C > 0 \text{ such that } \forall s \in [0,1),\ \|D_s\|_{\bar{X}\to\bar{X}} \leq Cs^{-1/p}\right\},$$

$$q_X = \inf\left\{q > 0 : \exists C > 0 \text{ such that } \forall s \in [1,\infty),\ \|D_s\|_{\bar{X}\to\bar{X}} \leq Cs^{-1/q}\right\},$$

respectively.

It is easy to see that the Boyd's index for the Lebesgue space L^p, $0 < p < \infty$, is p. Furthermore, according to [15, Proposition 6.1], we have

$$p_{L^{p,q,b}} = q_{L^{p,q,b}} = p. \tag{2.4}$$

The other index is related to the triangle inequality satisfied by the quasi-norm $\|\cdot\|_X$. Let X be a r.i.q.B.f.s. The Aoki–Rolewicz theorem [17, Theorem 1.3] offers a constant $0 < \kappa_X \leq 1$ such that $\rho_X^{\kappa_X}$ is sub-additive. That is,

$$\rho_X^{\kappa_X}(f+g) \leq \rho_X^{\kappa_X}(f) + \rho_X^{\kappa_X}(g).$$

Next, we restate the p-convexification of r.i.q.B.f.s. in the following. Let X be a r.i.q.B.f.s. For any $0 < p < \infty$, the p-convexification of X,

X^p is defined by $X^p = \{f : |f|^p \in X\}$. We equip X^p with the quasi-norm $\|f\|_{X^p} = \||f|^p\|_X^{1/p}$, see [20, Vol. II, p. 53].

Since for any $0 < p < \infty$, $(|f|^p)^* = (f^*)^p$ [1, Ch. 1, (1.20)], we have $\rho_{X^p}(f^*) = \rho_X(|f^*|^p)^{\frac{1}{p}}$.

We recall the definition of sublinear operators, see [16].

Definition 2.5. Let (R_0, μ_0) and (R_1, μ_1) be totally σ-finite measure spaces. Let T be an operator with domain being a linear subspace of $\mathcal{M}_0(\mu_0)$ and taking values in $\mathcal{M}(\mu_1)$ where $\mathcal{M}(\mu_i)$, $i = 0, 1$ denote the sets of μ_i-measurable functions. We say that T is sublinear if

$$|T(f+g)| \le (|Tf| + |Tg|), \quad |T(\lambda f)| = |\lambda||T(f)| \quad \mu_1 \text{ a.e.}$$

for all f and g in the domain of T and all scalar λ.

3. Main result

We obtain the extended Boyd's interpolation theorem in this section. As applications of this theorem, we establish the mapping properties of the Bochner–Riesz means, the Hankel multipliers, the rough singular integral operators, the universal maximal operator and the directional operators on rearrangement-invariant quasi-Banach function spaces.

For any r.i.q.B.f.s. X, define

$$X_{\mathrm{rad}} = \left\{ f \in X : f(x) = f(y),\ |x| = |y|,\ x, y \in \mathbb{R}^n \right\}.$$

We call X_{rad} a radial rearrangement-invariant quasi-Banach function space. In particular, L^p_{rad} consists of radial functions f belonging to L^p.

We first establish an estimate on the decreasing rearrangement of $T(f)$.

Proposition 3.1. *Let $0 < p_0 < p_1 < \infty$ and X be a r.i.q.B.f.s. Suppose that T is sublinear and $T\colon L^{p_i}_{\mathrm{rad}} \to L^{p_i}$, $i = 0, 1$, are bounded. If $p_0 < p_X \le q_X < p_1$, then there exist constants $C_0, C_1 > 0$ such that for any $f \in X_{\mathrm{rad}}$,*

$$(Tf)^*(t) \le C_0 \bigg(\int_0^1 (f^*(tu))^{p_0}\, du \bigg)^{\frac{1}{p_0}} + C_1 \bigg(\int_1^\infty (f^*(tu))^{p_1}\, du \bigg)^{\frac{1}{p_1}}, \quad t > 0.$$

Proof. Let $f \in X_{\mathrm{rad}}$. For any $t > 0$, define

$$f_{1,t}(x) = \min(|f(x)|, f^*(t)) \operatorname{sgn} f(x),$$
$$f_{0,t}(x) = (|f(x)| - f^*(t))^+ \operatorname{sgn} f(x) = f(x) - f_1(x)$$

where $\operatorname{sgn} f(x) = \frac{f(x)}{|f(x)|}$ when $f(x) \neq 0$ and $\operatorname{sgn} f(x) = 0$ otherwise.

For any $s > 0$,

$$f_{1,t}^*(s) = \min(f^*(s), f^*(t)), \quad f_{0,t}^*(s) = (f^*(s) - f^*(t))^+.$$

Since f^* is non-increasing,

$$\begin{aligned}\|f_{1,t}\|_{L^{p_1}}^{p_1} &= \int_0^\infty (f_{1,t}^*(s))^{p_1}\,ds = \int_0^t (f_{1,t}^*(s))^{p_1}\,ds + \int_t^\infty (f_{1,t}^*(s))^{p_1}\,ds \\ &= t(f^*(t))^{p_1} + \int_t^\infty (f^*(s))^{p_1}\,ds.\end{aligned}$$

Consequently, there is a constant $C > 0$ such that for any fixed $t > 0$

$$\|f_{1,t}\|_{L^{p_1}} \le C\bigg(t^{\frac{1}{p_1}} f^*(t) + \Big(\int_t^\infty (f^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}\bigg). \tag{3.1}$$

By using the inequality $(a-b)^{p_0} \le a^{p_0} - b^{p_0}$, $a \ge b \ge 0$, we obtain

$$\begin{aligned}\|f_{0,t}\|_{L^{p_0}}^{p_0} &= \int_0^\infty (f_{0,t}^*(s))^{p_0}\,ds = \int_0^t (f_{0,t}^*(s))^{p_0}\,ds = \int_0^t (f^*(s) - f^*(t))^{p_0}\,ds \\ &\le \int_0^t (f^*(s))^{p_0}\,ds - \int_0^t (f^*(t))^{p_0}\,ds = \int_0^t (f^*(s))^{p_0}\,ds - t(f^*(t))^{p_0}.\end{aligned}$$

The inequality $(a-b)^{\frac{1}{p_0}} \le 2a^{\frac{1}{p_0}} - b^{\frac{1}{p_0}}$, $a \ge b \ge 0$ yields

$$\|f_{0,t}\|_{L^{p_0}} \le 2\Big(\int_0^t (f^*(s))^{p_0}\,ds\Big)^{\frac{1}{p_0}} - t^{\frac{1}{p_0}}(f^*(t)). \tag{3.2}$$

Obviously, when $|x| = |y|$, $f_{0,t}(x) = f_{0,t}(y)$ and $f_{1,t}(x) = f_{1,t}(y)$, (3.1) and (3.2) assure that $f_{1,t} \in L_{\mathrm{rad}}^{p_1}$ and $f_{0,t} \in L_{\mathrm{rad}}^{p_0}$.

Since T is sublinear, according to [1, Ch. 1, (1.16)], we have

$$(Tf)^*(t) \le (Tf_{0,t})^*(t/2) + (Tf_{1,t})^*(t/2).$$

As $T\colon L_{\mathrm{rad}}^{p_i} \to L^{p_i}$, $i = 0, 1$ are bounded and $f_{i,t} \in L_{\mathrm{rad}}^{p_i}$, $i = 0, 1$, we have

$$(Tf)^*(t) \le (t/2)^{-\frac{1}{p_0}} M_0 \|f_{0,t}\|_{L^{p_0}} + (t/2)^{-\frac{1}{p_1}} M_1 \|f_{1,t}\|_{L^{p_1}} \tag{3.3}$$

for some $M_0, M_1 > 0$.

Consequently, (3.1) and (3.2) yield $C_0, C_1 > 0$ such that

$$(Tf)^*(t) \le C_0 t^{-\frac{1}{p_0}} \Big(\int_0^t (f^*(s))^{p_0}\,ds\Big)^{\frac{1}{p_0}} + C_1 t^{-\frac{1}{p_1}} \Big(\int_t^\infty (f^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}$$

$$= C_0\Big(\int_0^1 (f^*(tu))^{p_0}\,du\Big)^{\frac{1}{p_0}} + C_1\Big(\int_1^\infty (f^*(tu))^{p_1}\,du\Big)^{\frac{1}{p_1}}$$

where we use the substitution $s = tu$ at the last equality. □

THEOREM 3.2. *Let $0 < p_0 < p_1 < \infty$ and X be a r.i.q.B.f.s. Suppose that T is sublinear and $T\colon L^{p_i}_{\mathrm{rad}} \to L^{p_i}$, $i = 0,1$ are bounded. If $p_0 < p_X \le q_X < p_1$, then, $T\colon X_{\mathrm{rad}} \to X$ is bounded.*

PROOF. The Aoki–Rolewicz theorem yields κ_0 and κ_1 so that $\rho^{\kappa_i}_{X^{\frac{1}{p_i}}}(\cdot)$, $i = 0,1$ are sub-additive.

Since ρ_X is a quasi-norm, in view of Proposition 3.1, we have

$$\rho_X((Tf)^*(t)) \le C\rho_X\Big(\int_0^1 (f^*(tu))^{p_0}\,du\Big)^{\frac{1}{p_0}}$$

$$+ C\rho_X\Big(\int_1^\infty (f^*(tu))^{p_1}\,du\Big)^{\frac{1}{p_1}} = I + II$$

for some $C > 0$.

Since f^* is non-increasing and $\rho^{\kappa_0}_{X^{\frac{1}{p_0}}}(\cdot)$ is sub-additive, we get

$$I^{p_0\kappa_0} = C\rho^{\kappa_0}_{X^{\frac{1}{p_0}}}\Big(\int_0^1 (f^*(tu))^{p_0}\,du\Big)$$

$$\le C\rho^{\kappa_0}_{X^{\frac{1}{p_0}}}\Big(\sum_{j=-\infty}^{0} 2^{j-1}(f^*(2^{j-1}t))^{p_0}\Big) \le C\sum_{j=-\infty}^{0} 2^{(j-1)\kappa_0}\rho^{\kappa_0}_{X^{\frac{1}{p_0}}}((f^*(2^{j-1}t))^{p_0})$$

$$= C\sum_{j=-\infty}^{0} 2^{(j-1)\kappa_0}\rho^{\kappa_0}_{X^{\frac{1}{p_0}}}((D_{2^{j-1}}f^*(t))^{p_0}) = C\sum_{j=-\infty}^{0} 2^{(j-1)\kappa_0}\rho_X(D_{2^{j-1}}f^*(t))^{p_0\kappa_0}$$

Select an $\varepsilon > 0$ so that $p_0 < p_X - \varepsilon$, the definition of Boyd's indices yields a constant $C > 0$ such that

$$I \le C\Big(\sum_{j=-\infty}^{0} 2^{(j-1)\kappa_0}2^{-\frac{(j-1)p_0\kappa_0}{p_X-\varepsilon}}\rho_X(f^*(t))^{p_0\kappa_0}\Big)^{\frac{1}{p_0\kappa_0}} \tag{3.4}$$

$$\le C\rho_X(f^*(t)) = C\|f\|_X.$$

Similarly, as f^* is non-increasing and $\rho^{\kappa_1}_{X^{\frac{1}{p_1}}}(\cdot)$ is sub-additive, we obtain

$$II^{p_1\kappa_1} \le \sum_{j=0}^{\infty} 2^{j\kappa_1}\rho_X(D_{2^j}f^*(t))^{p_1\kappa_1}.$$

Take an $\varepsilon > 0$ so that $q_X + \varepsilon < p_1$. The definition of Boyd's indices guarantees that

$$(3.5)\quad II \le C\Big(\sum_{j=0}^{\infty} 2^{j\kappa_1} 2^{-\frac{jp_1\kappa_1}{q_X+\varepsilon}} \rho_X(f^*(t))^{p_1\kappa_1}\Big)^{\frac{1}{p_1\kappa_1}} \le C\rho_X(f^*(t)) = C\|f\|_X$$

for some $C > 0$. Therefore, (3.4) and (3.5) yield the boundedness of $T\colon X_{\mathrm{rad}} \to X$. $\square$

If Tf is a radial function whenever f is radial, then the above theorem shows that T is bounded on X_{rad}.

3.1. Disc multipliers and spherical means. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$.

For any $R > 0$ and $f \in L^2$, define

$$S_R f(x) = (2\pi)^{-n/2} \int_{|\xi|<R} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

and

$$S^* f(x) = \sup_{R>0} |S_R f(x)|, \quad x \in \mathbb{R}^n.$$

Write $S = S_1$. In view of [10], we find that when $n \ge 2$, S is bounded on L^p if and only if $p = 2$.

On the other hand, Herz [12] obtained the following boundedness result for S on the radial Lebesgue spaces.

THEOREM 3.3. *Let $n \ge 2$. If $\frac{2n}{n+1} < p < \frac{2n}{n-1}$, then S is bounded on L^p_{rad}.*

This result has been extended to S^* in [18].

THEOREM 3.4. *Let $n \ge 2$. If $\frac{2n}{n+1} < p < \frac{2n}{n-1}$, then S^* is bounded on L^p_{rad}.*

Theorem 3.2 gives the subsequent boundedness result for S and S^* on radial rearrangement-invariant quasi-Banach function spaces.

THEOREM 3.5. *Let $n \ge 2$ and X be a r.i.q.B.f.s. on $\mathbb{R}^n$. If $\frac{2n}{n+1} < p_X \le q_X < \frac{2n}{n-1}$, then S and S^* are bounded on X_{rad}.*

Theorems 3.4 and 3.5 have also been generalized to the Bochner–Riesz means and the Hankel multipliers.

Let $0 < \alpha < \frac{n-1}{2}$. For any $R > 0$, the Bochner–Riesz mean is defined as

$$S_R^\alpha f(x) = (2\pi)^{-n/2} \int_{|\xi|<R} \Big(1 - \frac{|\xi|^2}{R^2}\Big)^\alpha \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \quad f \in \mathcal{S}'(\mathbb{R}^n),\ x \in \mathbb{R}^n,$$

and the corresponding maximal operator is given by

$$S_*^\alpha f(x) = \sup_{R>0} |S_R^* f(x)|.$$

Theorem 3.6 (Kanjin [19]). *Let $n \geq 2$ and $0 < \alpha < \frac{n-1}{2}$. If $\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha}$, then S_*^α is bounded on L^p_{rad}.*

Therefore, Theorem 3.2 yields the corresponding result for radial rearrangement-invariant quasi-Banach function spaces.

Theorem 3.7. *Let $n \geq 2$, $0 < \alpha < \frac{n-1}{2}$ and X be a r.i.q.B.f.s. on $\mathbb{R}^n$. If $\frac{2n}{n+1+2\alpha} < p_X \leq q_X < \frac{2n}{n-1-2\alpha}$, then S_*^α is bounded on X_{rad}.*

We now turn to the Hankel multipliers. Let $K \in \mathcal{S}'(\mathbb{R}^n)$ be a radial convolution kernel. For any $t > 0$, write $K_t(x) = t^{-n} K(t^{-1}x)$, $x \in \mathbb{R}^n$. Define the Hankel multiplier associated with K, T_K by $T_K f = K * f$.

Theorem 3.8 (G. Garrigós and A. Seeger [11]). *Let $n > 1$, $1 < p < \frac{2n}{n+1}$ and K be a radial convolution kernel. Suppose that $\hat{K}$ is locally square integrable. If*

$$\sup_{t>0} \|\Phi * K_t\|_{L^p} < \infty \tag{3.6}$$

for a radial Schwartz function Φ whose Fourier transform is compactly supported in $\mathbb{R}^n \backslash \{0\}$, then T_K is bounded on L^p_{rad}.

By using Theorem 3.2, we obtain the mapping properties for the Hankel multipliers T_K on radial rearrangement-invariant quasi-Banach function spaces.

Theorem 3.9. *Let $n > 1$, K be a radial convolution kernel and X be a r.i.q.B.f.s. on $\mathbb{R}^n$. Suppose that $\hat{K}$ is locally square integrable. If K satisfies (3.6) for some p_0, p_1 with $1 < p_0 < p_X \leq q_X < p_1 < \frac{2n}{n+1}$, then the Hankel multiplier T_K is bounded from X_{rad} to X_{rad}.*

In view of (2.4), we have the following result for the disc multiplier, the Bochner–Riesz mean and the Hankel multipliers on $L^{p,q,b}$.

Corollary 3.10. *Let $n \geq 2$, $0 < \alpha < \frac{n-1}{2}$, $1 \leq p, q < \infty$ and b be a slowly varying function.*

(1) *If $\frac{2n}{n+1} < p < \frac{2n}{n-1}$, then S and S^* are bounded on $L^{p,q,b}_{\mathrm{rad}}$.*

(2) *If $\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha}$, then S_*^α is bounded on $L^{p,q,b}_{\mathrm{rad}}$.*

(3) *If $1 < p < \frac{2n}{n+1}$ and K satisfies (3.6) and $\hat{K}$ is locally square integrable, then T_K is bounded on $L^{p,q,b}_{\mathrm{rad}}$.*

3.2. Rough singular integrals. Let $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega(\omega)\, d\omega = 0$. The rough singular integral operator is defined as

$$T_\Omega f(x) = \lim_{\substack{\varepsilon\to 0 \\ R\to\infty}} \int_{\varepsilon<|y|<R} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy.$$

The corresponding maximal singular integral operator is given by

$$T_\Omega^* f(x) = \sup_{0<\varepsilon<R<\infty} \left| \int_{\varepsilon<|y|<R} \frac{\Omega(y)}{|y|^n} f(x-y) dy \right|.$$

[24, Theorem 2] asserts the boundedness of the T_Ω and T_Ω^* on radial Lebesgue spaces.

THEOREM 3.11. *Let $1 < p < \infty$ and $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega\, d\omega = 0$. The rough singular integral operator T_Ω and the maximal singular integral operator T_Ω^* are bounded from L^p_{rad} to L^p.*

The boundedness of the rough singular integral operator on the entire L^p requires some extra conditions imposed on Ω, see [5,23].

Theorem 3.2 assures the boundedness of T_Ω and T_Ω^* on radial rearrangement-invariant quasi-Banach function spaces.

THEOREM 3.12. *Let $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega(\omega)\, d\omega = 0$ and X be a r.i.g.B.f.s. on $\mathbb{R}^n$. If $1 < p_X \le q_X < \infty$, then the rough singular integral operator T_Ω and the maximal singular integral operator T_Ω^* are bounded from X_{rad} to X.*

In particular, if $1 < p, q < \infty$, b be a slowly varying function and $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega(\omega)\, d\omega = 0$, then $T_\Omega \colon L^{p,q,b}_{\mathrm{rad}} \to L^{p,q,b}$ and $T_\Omega^* \colon L^{p,q,b}_{\mathrm{rad}} \to L^{p,q,b}$ are bounded.

3.3. Universal maximal operator. Let f be a locally integrable function. The universal maximal operator $\mathcal{M}$ is defined as

$$\mathcal{M} f(x) = \sup_R \frac{1}{|R|} \int_R |f(y)|\, dy$$

where the supremum is taken over all rectangles R in $\mathbb{R}^n$ containing x with arbitrary directions.

THEOREM 3.13. *Let $p > n$. The universal maximal operator $\mathcal{M}$ is bounded on L^p_{rad}.*

For the proof of the above result, the reader is referred to [3,4]. From [3,4], it is also known that $\mathcal{M}$ is not bounded on L^p for all $1 \le p < \infty$. The reader is referred to [7] for some extensions of the above result on l^q radial functions.

The preceding theorem and Theorem 3.2 yields the boundedness of the universal maximal operator on radial rearrangement-invariant quasi-Banach function spaces.

THEOREM 3.14. *Let X be a r.i.q.B.f.s. If $n < p_X \le q_X < \infty$, then the universal maximal operator $\mathcal{M}$ is bounded from X_{rad} to X.*

As a consequence of the preceding theorem, the universal maximal operator $\mathcal{M}$ is bounded from $L^{p,q,b}_{\mathrm{rad}}$ to $L^{p,q,b}$ when $n < p < \infty$, $1 \le q < \infty$ and b is a slowing varying function.

3.4. Directional operators. We recall the definitions of directional operators from [6]. Let $0 < \theta < 2\pi$. For any $f \in L^2(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ we define

$$M_\theta f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - te^{i\theta})|\, dt,$$

$$H_\theta f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t|>\varepsilon} \frac{f(x - te^{i\theta})}{t}\, dt = \lim_{\varepsilon \to 0} H_{\varepsilon,\theta} f(x),$$

$$H^*_\theta f(x) = \sup_{\varepsilon>0} |H_{\varepsilon,\theta} f(x)|.$$

Let S^1 be the unit circle in $\mathbb{R}^2$ and E be a closed subset of S^1. The maximal directional operators associated with E are defined by

$$\mathcal{M}_E f(x) = \sup_{\theta \in E} |M_\theta f(x)|,$$

$$\mathcal{H}_E f(x) = \sup_{\theta \in E} |H_\theta f(x)|, \quad \mathcal{H}^*_E f(x) = \sup_{\theta \in E} |H^*_\theta f(x)|.$$

Whenever E has positive Lebesgue measure, $\mathcal{M}_E$ is unbounded on $L^p(\mathbb{R}^2)$, $1 \le p < \infty$. If $E = \{\theta_j\}_{j\in\mathbb{N}}$ is a lacunary set, then $\mathcal{M}_E$ is bounded on $L^p(\mathbb{R}^2)$, $1 < p \le \infty$, see [21].

For any $E \subset S^1$, define

$$d(E) = \lim_{\delta \to 0^+} \sup \frac{\log N(\delta)}{-\log \delta}$$

where $N(\delta)$ is the minimum number of closed intervals of length δ needed to cover E. For instance, when E is a Cantor ternary set, then $d(E) = \frac{\log 2}{\log 3}$.

In view of [6], we have the following boundedness result for $L^p_{\mathrm{rad}}(\mathbb{R}^2)$.

THEOREM 3.15. *Let $p > 1 + d(E)$. Then*
(1) *$\mathcal{M}_E$ is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^2)$.*
(2) *If $d(E) < 1$, then $\mathcal{H}_E$ and $\mathcal{H}^*_E$ is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^2)$.*

Theorem 3.2 gives the corresponding result for radial rearrangement-invariant quasi-Banach function spaces.

THEOREM 3.16. *Let X be a r.i.q.B.f.s. on $\mathbb{R}^2$ with $1 + d(E) < p_X \leq q_X < \infty$. Then*
(1) *$\mathcal{M}_E : X_{\mathrm{rad}} \to X$ is bounded.*
(2) *If $d(E) < 1$, then $\mathcal{H}_E\colon X_{\mathrm{rad}} \to X$ and $\mathcal{H}^*_E\colon X_{\mathrm{rad}} \to X$ are bounded.*

When E is a Cantor ternary set, the preceding theorem assures that $\mathcal{M}_E\colon L^{p,q,b}_{\mathrm{rad}} \to L^{p,q,b}$, $\mathcal{H}_E\colon L^{p,q,b}_{\mathrm{rad}} \to L^{p,q,b}$ and $\mathcal{H}^*_E\colon L^{p,q,b}_{\mathrm{rad}} \to L^{p,q,b}$ are bounded provided that $1 + \frac{\log 2}{\log 3} < p < \infty$, $1 \leq q < \infty$ and b is a slowly varying function.

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References

[1] C. Bennett and R. Sharpley, *Interpolations of Operators*, Academic Press (1988).
[2] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag (1976).
[3] A. Carbery, E. Hernández and F. Soria, The behaviour on radial functions of maximal operators along arbitrary directions and the Kakeya maximal operator, *Tohoku Math. J.*, **41** (1989), 647–656. Erratum: "The behaviour on radial functions of maximal operators along arbitrary directions and the Kakeya maximal operator", *Tohoku Math. J.*, **42** (1990), 287.
[4] A. Carbery, E. Hernández and F. Soria, Estimates for the Kakeya maximal operator on radial functions in $\mathbb{R}^n$, in: *Harmonic Analysis*, ICM-90 Satell. Conf. Proc., Sendai, 1990, Springer (Tokyo, 1991), pp. 41–50.
[5] W. C. Connett, Singular integrals near L^1, Part I, *Proc. Symposia Pure Math.*, **35** (1979), 163–165.
[6] J. Duoandikoetxea and A. Vargas, Directional operators and radial functions on the plane, *Ark. Mat.*, **33** (1995), 281–291.
[7] J. Duoandikoetxea and V. Naibo, The universal maximal on special classes of functions, *Indiana Univ. Math. J.*, **54** (2005), 1351–1369.
[8] D. E. Edmunds, R. Kerman and L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, *J. Funct. Anal.*, **170** (2000), 307–355.
[9] D. E. Edmunds and W. D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer (2004).
[10] C. Fefferman, The multiplier problem for the ball, *Ann. of Math.*, **94** (1971), 330–336.
[11] G. Garrigós and A. Seeger, Characterizations of Hankel multipliers, *Math. Ann.*, **342** (2008), 31–68.
[12] C. S. Herz, On the mean inversion of Fourier and Hankel transforms, *Proc. Nat. Acad. Sci. USA.*, **40** (1954), 996–999.
[13] T. Holmstedt, Interpolation of quasi-normed spaces, *Math. Scand.*, **26** (1970), 177–199.

[14] K.-P. Ho, Fourier integrals and Sobolev embedding on rearrangement-invariant quasi-Banach function spaces, *Ann. Acad. Sci. Fenn. Math.*, **41** (2016), 897–922.

[15] K.-P. Ho, Fourier type transforms on rearrangement-invariant quasi-Banach function spaces, *Glasgow Math. J.*, **61** (2019), 231–248.

[16] K.-P. Ho, Interpolation of sublinear operators which map into Riesz spaces and applications, *Proc. Amer. Math. Soc.*, doi.org/10.1090/proc/14506 (published online).

[17] N. Kalton, N. Peck and J. Roberts, *An F-space Sampler*, London Mathematical Society, Lecture Note Series, 89, Cambridge University Press (1984).

[18] Y. Kanjin, Convergence and divergence almost everywhere of spherical means for radial functions, *Proc. Amer. Math. Soc.*, **103** (1988), 1063–1069.

[19] Y. Kanjin, Convergence almost everywhere of Bochner–Riesz means for radial functions, *Ann. Sci. Kanazawa Univ.*, **25** (1988), 11–25.

[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, I, II, Springer (1996).

[21] A. Nagel, E. Stein and S. Wainger, Differentiation in lacunary directions, *Proc. Nat. Acad. Sci. USA*, **75** (1978), 1060–1062.

[22] J. Neves, Lorentz–Karamata spaces, Bessel and Riesz potentials and embeddings, *Dissertationes Math. (Rozprawy Mat.)*, **405** (2002), 1–46.

[23] F. Ricci and G. Weiss, A characterization of $H^1(\sum_{n-1})$, Part I, *Proc. Symposia Pure Math.*, **35** (1979), 289–294.

[24] S. Sjögren and F. Soria, Rough maximal functions and rough singular integral operators applied to integrable radial functions, *Rev Mat. Iberoam.*, **13** (1997), 1–18.