Acta Math. Hungar., **157** (**2**) (2019), 465–477
https://doi.org/10.1007/s10474-018-0899-3
First published online December 6, 2018

ON TWO SIDED α-n-DERIVATION IN 3-PRIME NEAR-RINGS

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(Received July 9, 2018; revised September 2, 2018; accepted September 13, 2018)

Abstract. Let N be a left near-ring and let α be a function of N. We introduce the notion of two sided α-n-derivation and prove that a prime zero symmetric near-ring involving α-n-derivations satisfying certain identities is a commutative ring.Also, some examples are given to shown that the 3-primeness condition in our results is not redundant.

1. Introduction

In this paper, N will denote a zero symmetric left near-ring with multiplicative center $Z(N)$. We will write, for all $x, y \in N$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. N is 2-torsion free, if whenever $2x = 0$ implies $x = 0$. A near ring N is called zero symmetric if $0.x = 0$ for all $x \in N$ (recall that left distributivity yields that $x.0 = 0$). Recall that N is called a 3-prime near-ring, if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. In this paper, unless otherwise specified, we will use the word near-ring to mean zero symmetric left near-ring. An additive mapping $d\colon N \to N$ is said to be a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently, as noted in [17], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$.

An additive mapping $d\colon N \to N$ is called a (α, β)-derivation if there exist functions $\alpha, \beta\colon N \to N$ such that $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in N$, furthermore, an additive mapping $d\colon N \to N$ is called a two-sided

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Key words and phrases: prime near-ring, derivation, two sided α-n-derivation, commutativity.

Mathematics Subject Classification: 16N60, 16W25, 16Y30.

α-derivation if d is an $(\alpha, 1)$-derivation as well as $(1, \alpha)$-derivation (see [16]). Moreover, if d commutes with α, then d is called a semiderivation.

E. C. Posner introduced a technique for investigating rings by the use of derivations, to indicate how strongly a derivation is related to commutativity in [15]. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason [4]. Over the last two decades, there has been a great deal of work concerning commutativity of prime and semiprime rings and near-rings with derivations satisfying certain differential identities (see the references for a partial bibliography).

Let R be a ring and $n \geq 2$ be an integer. A map $D : R \times R \times \cdots \times R \to R$ is said to be an n-derivation of R if it is a derivation in each argument. A 2-derivation of R is said to be a bi-derivation of R. Brešar et al. [8] gave a characterization of bi-derivations of prime rings. More details about bi-derivations and their applications can be found in ([9, Section 3]). Also, the concepts of symmetric bi-derivation, permuting tri-derivations and permuting n-derivations have already been introduced in rings by G. Maksa, M. A. Öztürk and K. H. Park in [10], [11] and [13], respectively. These concepts have been studied for near-rings in [3], [12] and [14]. Motivated by these papers, we define the notion of two sided α-n-derivation and prove that a prime near-ring involving α-n-derivations satisfying certain identities is a commutative ring. Also, some examples are given to shown that the 3-primeness condition in our results is not redundant.

Here we initiate the concept of two sided α-n-derivation as follows:

Definition 1. Let N be a near-ring and let $n \geq 2$ be a fixed positive integer. An n-additive (i.e. additive in each argument) mapping $D\colon \underbrace{N \times N \times \cdots \times N}_{n\text{-times}} \longrightarrow N$ is called two sided α-n-derivation if there is a function $\alpha\colon N \to N$ such that the relations

$$D(x_1x_1', x_2, x_3, \ldots, x_n) = D(x_1, x_2, x_3, \ldots, x_n)\alpha(x_1') + x_1 D(x_1', x_2, x_3, \ldots, x_n)$$
$$= D(x_1, x_2, x_3, \ldots, x_n)x_1' + \alpha(x_1) D(x_1', x_2, x_3, \ldots, x_n),$$
$$D(x_1, x_2x_2', x_3, \ldots, x_n) = D(x_1, x_2, x_3, \ldots, x_n)\alpha(x_2') + x_2 D(x_1, x_2', x_3, \ldots, x_n)$$
$$= D(x_1, x_2, x_3, \ldots, x_n)x_2' + \alpha(x_2) D(x_1, x_2', x_3, \ldots, x_n),$$
$$\cdots$$
$$D(x_1, x_2, x_3, \ldots, x_nx_n')$$
$$= D(x_1, x_2, x_3, \ldots, x_n)\alpha(x_n') + x_n D(x_1, x_2, x_3, \ldots, x_n')$$
$$= D(x_1, x_2, x_3, \ldots, x_n)x_n' + \alpha(x_n) D(x_1, x_2, x_3, \ldots, x_n')$$

hold for all $x_{i,i=1,\ldots,n}, x'_{i,i=1\ldots n} \in N$.

2. Main results

LEMMA 1 [6, Lemma 1.1]. *Let N be a 3-prime near-ring.*

(i) *If $z \in Z(N) \setminus \{0\}$ and $xz \in Z(N)$, then $x \in Z(N)$.*

(ii) *If $N \subseteq Z(N)$, then N is a commutative ring.*

(iii) *If N admits a nonzero derivation d, then $(xd(y)+d(x)y)z = xd(y)z + d(x)yz$ for all $x, y, z \in N$.*

LEMMA 2 [4, Lemma 3(iv)]. *Let N be a 2-torsion free 3-prime near-ring. If N admits a derivation d such that $d^2 = 0$, then $d = 0$.*

In a left near-ring N, right distributive law does not hold in general. However, we can prove the following partial distributive properties in N.

LEMMA 3. *Let D be a two sided α-n-derivation. Then N satisfies the following partial distributive law: for all $x, y, z, x_{i,i=2\dots n} \in N$, we have*

$$\big(D(x, x_2, x_3, \dots, x_n)\alpha(y) + xD(y, x_2, x_3, \dots, x_n)\big)\alpha(z)$$
$$= D(x, x_2, x_3, \dots, x_n)\alpha(yz) + xD(y, x_2, x_3, \dots, x_n)\alpha(z).$$

PROOF. By the defining property of D, we get

$$\begin{aligned} D(xyz, x_2, x_3, \dots, x_n) &= D(xy, x_2, x_3, \dots, x_n)\alpha(z) + xyD(z, x_2, x_3, \dots, x_n) \\ &= \big(D(x, x_2, x_3, \dots, x_n)\alpha(y) + xD(y, x_2, x_3, \dots, x_n)\big)\alpha(z) \\ &\quad + xyD(z, x_2, x_3, \dots, x_n). \end{aligned}$$

On the other hand,

$$\begin{aligned} D(xyz, x_2, x_3, \dots, x_n) &= D(x, x_2, x_3, \dots, x_n)\alpha(yz) + xD(yz, x_2, x_3, \dots, x_n) \\ &= D(x, x_2, x_3, \dots, x_n)\alpha(yz) + xD(y, x_2, x_3, \dots, x_n)\alpha(z) \\ &\quad + xyD(z, x_2, x_3, \dots, x_n). \end{aligned}$$

From the two expressions of $D(xyz, x_2, x_3, \dots, x_n)$, we find that for all $x, y, z, x_{i,i=2\dots n} \in N$

$$\big(D(x, x_2, x_3, \dots, x_n)\alpha(y) + xD(y, x_2, x_3, \dots, x_n)\big)\alpha(z)$$
$$= D(x, x_2, x_3, \dots, x_n)\alpha(yz) + xD(y, x_2, x_3, \dots, x_n)\alpha(z). \quad \square$$

THEOREM 1. *Let N be a 3-prime near-ring. If N admits a nonzero two sided α-n-derivation D such that $D(N, N, \dots, N) \subseteq Z(N)$, then N is a commutative ring.*

Proof. Assume that $D(x_1,x_2,\dots,x_n)\in Z(N)$ for all $x_1,x_2,\dots,x_n\in N$; hence

(1)
$$D(xy,x_2,\dots,x_n)\alpha(z)=\alpha(z)D(xy,x_2,\dots,x_n) \text{ for all } x,y,z,x_{i,i=2\dots n}\in N.$$

In view of Lemma 3, (1) shows that

$$\begin{aligned}(2)\qquad & D(x,x_2,\dots,x_n)\alpha(yz)+xD(y,x_2,\dots,x_n)\alpha(z)\\ &=\alpha(z)D(x,x_2,\dots,x_n)\alpha(y)+\alpha(z)xD(y,x_2,\dots,x_n)\end{aligned}$$

for all $x,y,z,x_{i,i=2\dots n}\in N$. Replacing x by $\alpha(z)$ in (2), we obtain

$$(3)\qquad D(\alpha(z),x_2,\dots,x_n)\alpha(yz)=\alpha(z)D(\alpha(z),x_2,\dots,x_n)\alpha(y)$$

for all $y,z,x_{i,i=2\dots n}\in N$. This equation can be written as

$$(4)\quad D(\alpha(z),x_2,\dots,x_n)(\alpha(yz)-\alpha(z)\alpha(y))=0 \quad\text{for all } y,z,x_{i,i=2\dots n}\in N.$$

In view of $D(\alpha(z),x_2,\dots,x_n)\in Z(N)$, it follows that

(5)
$$D(\alpha(z),x_2,\dots,x_n)N(\alpha(yz)-\alpha(z)\alpha(y))=\{0\}\quad\text{for all } y,z,x_{i,i=2\dots n}\in N.$$

Since N is 3-prime, then (5) shows that

(6)
$$D(\alpha(z),x_2,\dots,x_n)=0 \;\text{ or }\; \alpha(yz)=\alpha(z)\alpha(y)\quad\text{for all } y,z,x_{i,i=2\dots n}\in N.$$

a) Assume that $\alpha(yz)=\alpha(z)\alpha(y)$ for all $y,z\in N$. In this case and by the hypothesis given, equation (2) reduces to $D(y,x_2,\dots,x_n)[x,\alpha(z)]=0$ for all $x,y,z,x_{i,i=2\dots n}\in N$. Now, left multiplying by t, where $t\in N$, and using again $D(y,x_2,\dots,x_n)\in Z(N)$ we get $D(y,x_2,\dots,x_n)t[x,\alpha(z)]=0$ for all $x,y,z,t,x_{i,i=2\dots n}\in N$. Accordingly,

$$D(y,x_2,\dots,x_n)N[x,\alpha(z)]=\{0\}\quad\text{for all } x,y,z,x_{i,i=2\dots n}\in N.$$

In view of the 3-primeness of N and $D\neq 0$, the preceding equation shows that $\alpha(z)\in Z(N)$ for all $z\in N$ and therefore $\alpha(yz)=\alpha(z)\alpha(y)=\alpha(y)\alpha(z)$ for all $y,z\in N$.

For $x,y,z,x_{i,i=2\dots n}\in N$, we have

$$\begin{aligned}D(xyz,x_2,\dots,x_n)&=D(xy,x_2,\dots,x_n)z+\alpha(xy)D(z,x_2,\dots,x_n)\\ &=\big(D(x,x_2,\dots,x_n)y+\alpha(x)D(y,x_2,\dots,x_n)\big)z+\alpha(x)\alpha(y)D(z,x_2,\dots,x_n).\end{aligned}$$

On the other hand,

$$D(xyz,x_2,\dots,x_n)=D(x,x_2,\dots,x_n)yz+\alpha(x)D(yz,x_2,\dots,x_n)$$

$$= D(x, x_2, \ldots, x_n)yz + \alpha(x)\big(D(y, x_2, \ldots, x_n)z + \alpha(y)D(z, x_2, \ldots, x_n)\big)$$
$$= D(x, x_2, \ldots, x_n)yz + \alpha(x)D(y, x_2, \ldots, x_n)z + \alpha(x)\alpha(y)D(z, x_2, \ldots, x_n)$$

Comparing the above two values of $D(xyz, x_2, \ldots, x_n)$, we conclude that

$$D(xy, x_2, \ldots, x_n)z = \big(D(x, x_2, \ldots, x_n)y + \alpha(x)D(y, x_2, \ldots, x_n)\big)z$$
$$= D(x, x_2, \ldots, x_n)yz + \alpha(x)D(y, x_2, \ldots, x_n)z \quad \text{for all } x, y, z, x_{i,i=2\ldots n} \in N.$$

Using the hypothesis of the statement we get

$$D(xy, x_2, \ldots, x_n)z = zD(xy, x_2, \ldots, x_n).$$

Accordingly,

$$(7) \qquad D(x, x_2, \ldots, x_n)yz + \alpha(x)D(y, x_2, \ldots, x_n)z = zD(x, x_2, \ldots, x_n)y + z\alpha(x)D(y, x_2, \ldots, x_n)$$

Since $\alpha(x) \in Z(N)$ and $D(y, x_2, \ldots, x_n) \in Z(N)$ for all $x, y, x_{i,i=2\ldots n} \in N$, then (7) yields

$$(8) \quad D(x, x_2, \ldots, x_n)yz = zD(x, x_2, \ldots, x_n)y \quad \text{for all } x, y, z, x_{i,i=2\ldots n} \in N.$$

Consequently, $D(x, x_2, \ldots, x_n)[y, z] = 0$ for all $x, y, z, x_{i,i=2\ldots n} \in N$ in such a way that

$$(9) \qquad D(x, x_2, \ldots, x_n)N[y, z] = \{0\} \quad \text{for all } x, y, z, x_{i,i=2\ldots n} \in N.$$

Since $D \neq 0$, the 3-primeness of N assures that $N \subseteq Z(N)$. Hence N is a commutative ring by Lemma 1(ii).

b) Assume there exist $y_0, z_0 \in N$ such that $\alpha(y_0 z_0) \neq \alpha(z_0)\alpha(y_0)$, hence by (6) we conclude that $D(\alpha(z_0), x_2, \ldots, x_n) = 0$ for all $x_{i,i=2\ldots n} \in N$. In this case, returning to equation (2) and replacing x by $D(u_1, u_2, \ldots, u_n)$ where $u_{i,i=1\ldots n} \in N$, we obtain

$$D\big(D(u_1, u_2, \ldots, u_n), x_2, \ldots, x_n\big)\left(\alpha(yz) - \alpha(z)\alpha(y)\right) = 0$$

and thus

$$D\big(D(u_1, u_2, \ldots, u_n), x_2, \ldots, x_n\big)N\left(\alpha(yz) - \alpha(z)\alpha(y)\right) = \{0\}$$

for all $y, z, u_{i,i=1\ldots n}, x_{i,i=2\ldots n} \in N$. Since N is a 3-prime, the latter relation forces

$$(10) \quad D\big(D(u_1, u_2, \ldots, u_n), x_2, \ldots, x_n\big) = 0 \ \text{ or } \ \left(\alpha(yz) - \alpha(z)\alpha(y)\right) = 0$$

for all $y, z, u_{i,i=1\dots n}, x_{i,i=2\dots n} \in N$. In light of $\alpha(y_0z_0) \neq \alpha(z_0)\alpha(y_0)$, then (10) reduces to

$$D\big(D(u_1, u_2, \dots, u_n), x_2, \dots, x_n\big) = 0 \quad \text{for all } u_{i,i=1\dots n}, x_{i,i=2\dots n} \in N. \tag{11}$$

Replacing u_1 by u^2 in (11), $u \in N$, we get

$$\begin{aligned} 0 &= D\big(D(uu, u_2, \dots, u_n), x_2, \dots, x_n\big) \\ &= D\big(D(u, u_2, \dots, u_n)\alpha(u) + uD(u, u_2, \dots, u_n), x_2, \dots, x_n\big) \\ &= D\big(D(u, u_2, \dots, u_n)\alpha(u), x_2, \dots, x_n\big) + D\big(uD(u, u_2, \dots, u_n), x_2, \dots, x_n\big) \end{aligned}$$

Setting $I = D\big(D(u, u_2, \dots, u_n)\alpha(u), x_2, \dots, x_n\big)$ and $II = D\big(uD(u, u_2, \dots, u_n), x_2, \dots, x_n\big)$. We have

$$\begin{aligned} I &= D\big(D(u, u_2, \dots, u_n), x_2, \dots, x_n\big)\,\alpha^2(u) \\ &\quad + D(u, u_2, \dots, u_n)D\big(\alpha(u), x_2, \dots, x_n\big) \end{aligned}$$

which by virtue of (11) leads to

$$I = D(u, u_2, \dots, u_n)D\big(\alpha(u), x_2, \dots, x_n\big) \text{ for all } u, u_{i,i=2\dots n}, x_{i,i=2\dots n} \in N. \tag{12}$$

Similarly, one can easily see that

$$II = D(u, x_2, \dots, x_n)D(u, u_2, \dots, u_n) \text{ for all } u, u_{i,i=2\dots n}, x_{i,i=2\dots n} \in N. \tag{13}$$

Adding (12) and (13), we obtain

$$D(u, u_2, \dots, u_n)\big(D(\alpha(u), x_2, \dots, x_n) + D(u, x_2, \dots, x_n)\big) = 0.$$

Replacing u by z_0 in the last equation and using the fact that

$$D(\alpha(z_0), x_2, \dots, x_n) = 0$$

for all $x_{i,i=2\dots n} \in N$, we obtain

$$D(z_0, u_2, \dots, u_n)D(z_0, x_2, \dots, x_n) = 0 \quad \text{for all } u_{i,i=2\dots n}, x_{i,i=2\dots n} \in N$$

so that

$$D(z_0, u_2, \dots, u_n)ND(z_0, u_2, \dots, u_n) = \{0\} \quad \text{for all } u_{i,i=2\dots n} \in N.$$

Since N is 3-prime, the last relation shows that

$$D(z_0, u_2, \dots, u_n) = 0 \quad \text{for all } u_{i,i=2,\dots,n} \in N. \tag{14}$$

Substituting $\alpha(z_0)$ for x in (2), we get

$$\alpha(z_0)D(y, x_2, \dots, x_n)\alpha(z) = \alpha(z)\alpha(z_0)D(y, x_2, \dots, x_n)$$

for all $y, z, x_{i,i=2,\dots,n} \in N$ and thus

$$D(y, x_2, \dots, x_n)[\alpha(z), \alpha(z_0)] = 0 \quad \text{for all } y, z, x_{i,i=2,\dots,n} \in N.$$

Since $D \neq 0$, we conclude that

$$\alpha(z)\alpha(z_0) = \alpha(z_0)\alpha(z) \quad \text{for all } z \in N. \tag{15}$$

Replacing y by z_0 in (2) and using (14), we arrive at

$$\alpha(z_0 z) = \alpha(z)\alpha(z_0) \quad \text{for all } z \in N.$$

Using (15), we find that

$$\alpha(z_0 z) = \alpha(z_0)\alpha(z) \quad \text{for all } z \in N.$$

Replacing z by y_0, we get

$$\alpha(z_0 y_0) = \alpha(z_0)\alpha(y_0). \tag{16}$$

By hypothesis, we have $D(z_0 y_0, x_2, \dots, x_n) \in Z(N)$ for all $x_{i,i=2,\dots,n} \in N$. This implies that $D(z_0, x_2, \dots, x_n)\alpha(y_0) + z_0 D(y_0, x_2, \dots, x_n) \in Z(N)$. Invoking (14), the last relation reduces to

$$z_0 D(y_0, x_2, \dots, x_n) \in Z(N) \quad \text{for all } x_{i,i=2,\dots,n} \in N. \tag{17}$$

In light of Lemma 1(i), equation (17) yields

$$z_0 \in Z(N) \quad \text{or} \quad D(y_0, x_2, \dots, x_n) = 0 \quad \text{for all } x_{i,i=2,\dots,n} \in N.$$

If $z_0 \in Z(N)$ then from (16) we find that $\alpha(z_0 y_0) = \alpha(y_0 z_0) = \alpha(z_0)\alpha(y_0)$ which contradicts $\alpha(y_0 z_0) \neq \alpha(z_0)\alpha(y_0)$. Consequently, $D(y_0, x_2, \dots, x_n) = 0$ for all $x_{i,i=2,\dots,n} \in N$. Replacing y and z by y_0 and z_0 respectively in (2), we arrive at

$$D(x, x_2, \dots, x_n)(\alpha(y_0 z_0) - \alpha(z_0)\alpha(y_0)) = 0 \quad \text{for all } x, x_2, \dots, x_n \in N,$$

which, because of $D \neq 0$, yields

$$\alpha(y_0 z_0) = \alpha(z_0)\alpha(y_0);$$

a contradiction to $\alpha(y_0 z_0) \neq \alpha(z_0)\alpha(y_0)$. Therefore, the condition of the existence of two elements $y_0, z_0 \in N$ such that $\alpha(y_0 z_0) \neq \alpha(z_0)\alpha(y_0)$ cannot occur, and hence N is a commutative ring. $\square$

THEOREM 2. *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d and a nonzero two sided α-n-derivation D such that $[d(x), D(x_1, x_2, \dots, x_n)] = 0$ for all $x, x_{i,i=1,\dots,n} \in N$, then N is a commutative ring.*

PROOF. We are assuming that

(18)
$$d(x)D(x_1, x_2, \dots, x_n) = D(x_1, x_2, \dots, x_n)d(x) \quad \text{for all } x, x_{i,i=1,\dots,n} \in N.$$

Replacing x by xy in (18) and using Lemma 1(iii), we obtain

$$\begin{aligned} &d(x)yD(x_1, x_2, \dots, x_n) + xd(y)D(x_1, x_2, \dots, x_n) \\ &= D(x_1, x_2, \dots, x_n)d(x)y + D(x_1, x_2, \dots, x_n)xd(y) \end{aligned}$$

for all $x, y, x_{i,i=1,\dots,n} \in N$. Writing $d(x)$ for x in the preceding relation and invoking (18), we obtain

(19)
$$d^2(x)yD(x_1, x_2, \dots, x_n) = d^2(x)D(x_1, x_2, \dots, x_n)y \text{ for all } x, y, x_{i,i=1,\dots,n} \in N.$$

Substituting yt for y in (19), we arrive at

$$d^2(x)y[t, D(x_1, x_2, \dots, x_n)] = 0 \text{ for all } x, y, t, x_{i,i=1,\dots,n} \in N,$$

so that

$$d^2(x)N[t, D(x_1, x_2, \dots, x_n)] = \{0\} \quad \text{for all } x, t, x_{i,i=1,\dots,n} \in N.$$

Since N is 3-prime, then the latter relation implies that either

$$d^2(x) = 0 \text{ or } [t, D(x_1, x_2, \dots, x_n)] = 0 \quad \text{for all } x, t, x_{i,i=1,\dots,n} \in N.$$

It follows that $d^2 = 0$ or $D(N, N, \dots, N) \subseteq Z(N)$. Since $d \neq 0$, then by Lemma 2 the first situation cannot occurs; consequently $D(N, N, \dots, N) \subseteq Z(N)$. Applying Theorem 1, we obtain the desired conclusion. □

THEOREM 3. *Let N be a 3-prime near-ring. If N admits a nonzero two sided α-n-derivation D such that $D([N, N], \dots, [N, N]) = 0$, then N is a commutative ring.*

PROOF. We are given that

(20)
$$D\big([x_1, y_1], [x_2, y_2], [x_3, y_3], \dots, [x_n, y_n]\big) = 0 \text{ for all } x_{i,i=1,\dots,n}, y_{i,i=1,\dots,n} \in N.$$

Replacing y_1 by x_1y_1 in (20) we get

$$D\big(x_1, [x_2, y_2], [x_3, y_3], \dots, [x_n, y_n]\big)\,[x_1, y_1] = 0 \text{ for all } x_{i,i=1,\dots,n}, y_{i,i=1,\dots,n} \in N$$

so that,

(21) $$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,x_1y_1$$
$$= D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,y_1x_1 \quad \text{for all } x_{i,i=1,\dots,n}, y_{i,i=1,\dots,n} \in N.$$

Taking $y_1 = y_1t_1$ in (21), we obtain

$$\begin{aligned} &D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,x_1y_1t_1 \\ &= D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,y_1t_1x_1 \\ &= D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,y_1x_1t_1 \end{aligned}$$

for all $t_1, x_{i,i=1,\dots,n}, y_{i,i=1,\dots,n} \in N$ and thus

$$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,y_1[x_1,t_1] = 0$$

for all $t_1, x_{i,i=1,\dots,n}, y_{i,i=1,\dots,n} \in N$. Accordingly,

(22) $$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,N[x_1,t_1] = \{0\}$$

for all $t_1, x_{i,i=1,\dots,n}, y_{i,i=2,\dots,n} \in N$. By the 3-primeness of N, equation (22) shows that

$$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big) = 0 \ \text{ or } \ [x_1,t_1] = 0$$

for all $t_1, x_{i,i=1,\dots,n}, y_{i,i=2,\dots,n} \in N$, which implies that for each fixed $x_1 \in N$, we have

(23) $$x_1 \in Z(N) \ \text{ or } \ D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big) = 0$$

for all $x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N$. Let $x_1, t_1, t_2 \in N$. If $x_1 \in Z(N)$, taking x_1t_1 instead of x_1 in (20) and using (20) we get that

$$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,[t_1,y_1] = 0.$$

Now substituting y_1t_2 for y_1, we obtain

$$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big)\,y_1[t_1,t_2] = 0.$$

Since y_1, t_1, t_2 can be chosen arbitrarily and independently from the other variables, using 3-primeness we have that

$$N \subseteq Z(N) \ \text{ or } \ D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big) = 0$$

for all $x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N$. Suppose that there exists an element $x_1 \in N$ such that $x_1 \notin Z(N)$, then (23) shows that

$$D\big(x_1,[x_2,y_2],[x_3,y_3],\dots,[x_n,y_n]\big) = 0 \quad \text{for all } x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N.$$

Consequently, for each fixed element $x_1 \in N$, we have $N \subseteq Z(N)$ or

$$D\big(x_1, [x_2, y_2], [x_3, y_3], \ldots, [x_n, y_n]\big) = 0 \quad \text{for all } x_{i,i=2,\ldots,n}, y_{i,i=2,\ldots,n} \in N.$$

For the other products $[x_i, y_i]$, $i = 2, \ldots, n$, proceeding as above we establish that

$$D(x_1, x_2, x_3, \ldots, x_n) = 0 \quad \text{for all } x_{i,i=1,\ldots,n} \in N \quad \text{or} \quad N \subseteq Z(N).$$

Since $D \neq 0$ the latter relation reduces to $N \subseteq Z(N)$, and therefore N is a commutative ring by Lemma 1(ii). □

We now consider differential identities involving anti-commutators instead of commutators. Our result is of a different kind, indeed the near-ring cannot be a commutative ring.

THEOREM 4. *Let N be a 2-torsion free 3-prime near-ring. Then N admits no nonzero two sided α-n-derivation D such that $D(x_1 \circ y_1, \ldots, x_n \circ y_n) = 0$ for all $x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N$.*

PROOF. Assume that D is a nonzero two sided α-n-derivation such that

$$D(x_1 \circ y_1, \ldots, x_n \circ y_n) = 0 \quad \text{for all } x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N. \tag{24}$$

Replacing y_1 by $x_1 y_1$ in (24), we obtain

$$D(x_1, x_2 \circ y_2, \ldots, x_n \circ y_n)(x_1 \circ y_1) = 0 \quad \text{for all } x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N$$

which may be written as

$$D(x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) x_1 y_1 = -D(x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) y_1 x_1 \tag{25}$$

for all $x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N$. Writing $y_1 t$ for y_1 in (25), $t \in N$, and using (25), we get for all $t, x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N$

$$D(x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) y_1 (-x_1) t = D(x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) y_1 t (-x_1).$$

Accordingly,

$$D(x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) y_1 [-x_1, t] = 0 \text{ for all } t, x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N. \tag{26}$$

Taking $-x_1$ instead of x_1 in (26), we obtain

$$D(-x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) y_1 [x_1, t] = 0 \quad \text{for all } t, x_{i,i=1,\ldots,n}, y_{i,i=1,\ldots,n} \in N$$

which leads to

$$D(-x_1, x_2 \circ y_2, \ldots, x_n \circ y_n) N [x_1, t] = \{0\} \text{ for all } t, x_{i,i=1,\ldots,n}, y_{i,i=2,\ldots,n} \in N.$$

By view of the 3-primeness, we see that

$$D(-x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \quad \text{or} \quad [x_1, t] = 0 \tag{27}$$

for all $t, x_{i,i=1,\dots,n}, y_{i,i=2,\dots,n} \in N$, but, since D is n-additive, then $D(-x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0$ implies that $D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0$. Therefore, equation (27) reduces to

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \ \text{ or } \ [x_1, t] = 0 \ \text{ for all } t, x_{i,i=1,\dots,n}, y_{i,i=2,\dots,n} \in N.$$

This yields that for each fixed $x_1 \in N$, we have either

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \quad \text{or} \quad x_1 \in Z(N) \tag{28}$$

for all $x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N$. If $x_1 \in Z(N)$, then equation (24) becomes

$$D(x_1 y_1 + x_1 y_1,\ x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \quad \text{for all } x_{i,i=2,\dots,n}, y_{i,i=1,\dots,n} \in N.$$

Using the fact that D is n-additive together with 2-torsion freeness of N, we get

$$D(x_1 y_1,\ x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \quad \text{for all } x_{i,i=2,\dots,n}, y_{i,i=1,\dots,n} \in N,$$

that is

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) y_1 + \alpha(x_1) D(y_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0$$

for all $x_{i,i=2,\dots,n}, y_{i,i=1,\dots,n} \in N$. Now, replacing y_1 by $x_1 y_1$ and using also $x_1 \in Z(N)$, we obtain

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) y_1 x_1 = 0 \quad \text{for all } x_{i,i=2,\dots,n}, y_{i,i=1,\dots,n} \in N.$$

This equation can be written as

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) N x_1 = \{0\} \quad \text{for all } x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N.$$

Hence either $x_1 = 0$ or

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \quad \text{for all } x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N.$$

Therefore, in both the cases we find that $D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0$ for all $x_{i,i=2,\dots,n}, y_{i,i=2,\dots,n} \in N$. Accordingly equation (28) reduces to

$$D(x_1, x_2 \circ y_2, \dots, x_n \circ y_n) = 0 \quad \text{for all } x_{i,i=1,\dots,n}, y_{i,i=2,\dots,n} \in N.$$

Similarly, proceeding as above, it is obvious to verify that

$$D(x_1, x_2, \dots, x_n) = 0 \quad \text{for all } x_{i,i=1,\dots,n} \in N.$$

Therefore $D = 0$; a contradiction. $\square$

The following example demonstrates that the 3-primeness assumption is essential in the hypotheses of the our theorems.

EXAMPLE 1. Let us consider

$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y \in S \right\},$$

where S is a 2-torsion free noncommutative ring. It is easy to verify that N is a non 3-prime near-ring. Define $d, \alpha \colon N \to N$ by

$$d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us define $D \colon \underbrace{N \times N \times \dots \times N}_{n\text{-times}} \to N$ by

$$D(A_1, A_2, \dots, A_n) = \begin{pmatrix} 0 & 0 & \prod x_i \\ 0 & 0 & \prod y_i \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 0 & 0 & x_i \\ 0 & 0 & y_i \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for all } i = 1, \dots, n.$$

It is obvious that d is a nonzero derivation of N, also D is a nonzero two sided α-n-derivation. Moreover

$$D(A_1, A_2, \dots, A_n) \in Z(N), \quad D([A_1, B_1], [A_2, B_2], \dots, [A_n, B_n]) = 0,$$
$$[d(B), D(A_1, A_2, \dots, A_n)] = 0, \quad D(A_1 \circ B_1, A_2 \circ B_2, \dots, A_n \circ B_n) = 0$$

for all $B, A_{i,i=1,\dots,n}, B_{i,i=1,\dots,n} \in N$. However, N is not a commutative ring.

As an application of the previous theorems, we get the following corollaries, whose d, d_1, d_2 are derivations, semiderivations or two sided α-derivations of N, respectively.

COROLLARY 1. *Let N be a 3-prime near-ring and d a nonzero mapping of N.*

(i) *If $d(N) \subseteq Z(N)$, then N is a commutative ring.*
(ii) *If $d([x, y]) = 0$ for all $x, y \in N$, then N is a commutative ring.*

COROLLARY 2. *Let N be a 2-torsion free 3-prime near-ring.*

(i) *If N admits a nonzero mapping d_1 and d_2 such that $[d_1(x), d_2(y)] = 0$ for all $x, y \in N$, then N is a commutative ring.*

(ii) *N admits no nonzero mapping d such that $d(x \circ y) = 0$ for all $x, y \in N$.*

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