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EXCEPTIONAL SETS FOR SUMS OF FIVE AND SIX ALMOST EQUAL PRIME CUBES

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Abstract. We investigate the exceptional sets of natural number n which can be represented as sums of five and six cubes of almost equal primes, i.e. $n = p_1^3 + \cdots + p_s^3$ (s=5,6). It is established that almost all natural numbers n subject to certain congruence conditions have the above representation with $|p_j - (n/s)^{\frac{1}{3}}| \le n^{\theta_s/3+\varepsilon}$ $(1 \le j \le s)$, where $\theta_5 = 8/9 + \varepsilon$ and $\theta_6 = 5/6 + \varepsilon$.

1. Introduction

The Waring–Goldbach problem is to study the representation of positive integers as sums of powers of prime numbers. In this paper, we shall focus on the cubic Waring–Goldbach problem. This topic can be traced back to the work of Hua [1]. He proved that almost all integers satisfying certain congruence conditions can be written as s cubes of primes, where $s = 5, 6, 7, 8$, and the above-mentioned congruence conditions are

$$\mathcal{N}_5 = \left\{ n \in \mathbb{N} : n \equiv 1 \pmod 2, n \not\equiv 0, \pm 2 \pmod 9,\ n \not\equiv 0 \pmod 7 \right\},$$
$$\mathcal{N}_6 = \left\{ n \in \mathbb{N} : n \equiv 0 \pmod 2, n \not\equiv \pm 1 \pmod 9 \right\},$$
$$\mathcal{N}_7 = \left\{ n \in \mathbb{N} : n \equiv 1 \pmod 2, n \not\equiv 0 \pmod 9 \right\},$$
$$\mathcal{N}_8 = \left\{ n \in \mathbb{N} : n \equiv 0 \pmod 2 \right\}.$$

If $E_s(N)$ denotes the number of positive numbers n not exceeding N, satisfying the above congruence conditions but cannot be represented as s cubes of primes, Hua showed that for any $A > 0$,

$$E_s(N) \ll N(\log N)^{-A}. \tag{1.1}$$

Key words and phrases: Waring–Goldbach problem, a Hardy–Littlewood method, exceptional set, almost equal primes.

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In addition, it is worth mentioning that Hua [1] also proved that all sufficiently large odd positive integers are sums of nine cubes of primes. Afterwards, the classical result of Hua (1.1) has been improved to

$$E_s(N) \ll N^{1-\varrho_s},$$

where $\varrho_s > 0$ $(5 \le s \le 8)$. And over the years, a lot of specialists were engaged in improving the values of ϱ_s, and the following results have been achieved,

Ren [11], $\varrho_5 = \frac{1}{153} - \varepsilon$,
Wooley [16], $\varrho_5 = \frac{1}{36} - \varepsilon$, $\varrho_6 = \frac{1}{18} - \varepsilon$, $\varrho_7 = \frac{13}{36} - \varepsilon$, $\varrho_8 = \frac{25}{36} - \varepsilon$,
Kumchev [5], $\varrho_5 = \frac{5}{84}$, $\varrho_6 = \frac{4}{35}$, $\varrho_7 = \frac{11}{28}$, $\varrho_8 = \frac{61}{84}$,
Kawada and Wooley [3], $\varrho_6 = \frac{5}{28}$, $\varrho_7 = \frac{19}{42}$, $\varrho_8 = \frac{11}{14}$,
Zhao [18], $\varrho_5 = \frac{1}{12} - \varepsilon$, $\varrho_6 = \frac{1}{4} - \varepsilon$, $\varrho_7 = \frac{1}{2} - \varepsilon$, $\varrho_8 = \frac{5}{6} - \varepsilon$.

Hua's theorem concerning nine cubes of prime integers has also been developed by Leung [7], Liu [9], and Zhao [19], who investigated the small prime solutions of the cubic equation $a_1p_1^3 + \cdots + a_9p_9^3 = b$, where $a_1, \ldots, a_9$, b represent integers satisfying certain necessary conditions. In particular, Zhao [19] proved that the above cubic equation is solvable with $p_j \ll \max\{|a_1|, \ldots, |a_9|\}^2 + |b|^{1/3}$.

Throughout, we assume that N is a large natural number, and define

$$x_s = (N/s)^{\frac{1}{3}}, \quad y_s = x_s^{\theta_s}, \tag{1.2}$$

where $0 < \theta_s < 1$ is a constant and $5 \le s \le 8$. We shall explore the number of natural numbers n that satisfy the above congruence conditions respectively and make the expression

$$n = p_1^3 + \cdots + p_s^3, \quad |p_j - x_s| \le y_s \quad (j = 1, \cdots, s) \tag{1.3}$$

fail. As usual, let $\mathcal{E}_s(x_s, y_s)$ denote the set of integers n satisfying the congruence condition $\mathcal{N}_s$, with $|n - N| \le s x_s^2 y_s$, such that (1.3) has no solutions. And we use $E_s(x_s, y_s)$ to represent the cardinality of $\mathcal{E}_s(x_s, y_s)$. Our purpose of this paper is to show that

$$E_s(x_s, y_s) \ll x_s^{2-\varepsilon} y_s, \tag{1.4}$$

for any $\varepsilon > 0$ and θ_s is as small as possible.

In this topic, the first breakthrough was made by Liu and Sun [10]. Soon afterwards, Wang [14] and Li [8] reduced these bounds successively. Ren and Yao [12] by applying new estimates on the minor arcs in Zhao [18] improved this result. Wei and Wooley [15] gave a substitute for a Weyl-type estimate in [4] and used Vinogradov's mean value theorem to give an analogue of Hua's lemma. This made it possible for them to improve θ_7 and θ_8. Huang

[2] made further efforts on Weyl sums in short intervals, so that he further reduced the bounds for θ_7 and θ_8. Recently, Kumchev and Liu [6] obtained improvement on the bounds for θ_7 and θ_8 by making use of a sieve method. The above mentioned θ_s ($5 \le s \le 8$) such that (1.4) holds are listed as follows,

Liu and Sun [10], $\theta_5 = \frac{33}{34} + \varepsilon$, $\theta_6 = \frac{17}{18} + \varepsilon$, $\theta_7 = \frac{35}{38} + \varepsilon$, $\theta_8 = \frac{9}{10} + \varepsilon$,
Wang [14], $\theta_5 = \frac{24}{25} + \varepsilon$, $\theta_6 = \frac{14}{15} + \varepsilon$, $\theta_7 = \frac{32}{35} + \varepsilon$, $\theta_8 = \frac{9}{10} + \varepsilon$,
Li [8], $\theta_5 = \frac{15}{16} + \varepsilon$, $\theta_6 = \frac{11}{12} + \varepsilon$, $\theta_7 = \frac{10}{11} + \varepsilon$, $\theta_8 = \frac{19}{21} + \varepsilon$,
Ren and Yao [12], $\theta_5 = \frac{80}{87} + \varepsilon$, $\theta_6 = \frac{48}{53} + \varepsilon$, $\theta_7 = \frac{100}{111} + \varepsilon$, $\theta_8 = \frac{168}{187} + \varepsilon$,
Wei and Wooley [15], $\theta_7 = \theta_8 = \frac{4}{5} + \varepsilon$,
Huang [2], $\theta_7 = \theta_8 = \frac{19}{24} + \varepsilon$,
Kumchev and Liu [6], $\theta_7 = \theta_8 = \frac{31}{40} + \varepsilon$.

In this paper we shall improve θ_5 and θ_6 in [12].

THEOREM 1.1. *Let* $\theta_5 = 8/9 + \varepsilon$ *and* $\theta_6 = 5/6 + \varepsilon$. *Then* (1.4) *holds for* $s = 5, 6$.

We shall prove Theorem 1.1 by means of the Hardy–Littlewood method. The treatment of the integrals on the major arcs is standard, and we will focus on the treatment of the integrals on the minor arcs. As usual, an argument of Wooley [16] yields an upper bound for $E_s(x_s, y_s)$ which involves the integrals of $2s$-th powers of the generating function on the minor arcs. Unlike [12], we apply the idea of [18] for the above integrals directly. Besides, we apply an estimate appeared in [15] to substitute the traditional use of Hua's lemma when we consider the case of $s = 6$.

NOTATION. Throughout the paper, ε denotes a sufficiently small positive number, and c denotes a positive constant. We need to point out that both ε and c are allowed to change at different occurrences. With or without subscript, p denotes a prime number. And as usual, we write L for $\log N$, $e(x)$ for $e^{2\pi ix}$.

2. Preliminaries

As usual, when n is a natural number with $|n - N| \le s x_s^2 y_s$, where x_s and y_s are defined in (1.2) and θ_s is given in Theorem 1.1, we denote by $R_s(n)$ the weighted number of solutions of $n = p_1^3 + \cdots + p_s^3$ with $|p_j - x_s| \le y_s$ $(1 \le j \le s)$ given by

$$R_s(n) = \sum_{\substack{n = p_1^3 + \cdots + p_s^3 \\ |p_j - x_s| \le y_s}} (\log p_1) \cdots (\log p_s).$$

In order to apply the Hardy–Littlewood method, we define the generating function as

$$f_s(\alpha) = \sum_{x_s - y_s \le p \le x_s + y_s} (\log p) e(p^3 \alpha).$$

Then it follows from orthogonality that

$$R_s(n) = \int_0^1 f_s(\alpha)^s e(-n\alpha)\, d\alpha.$$

For the purpose of introducing the major arcs and the minor arcs, we write

$$P_s = y_s^2 x_s^{-19/12-\varepsilon} \quad \text{and} \quad Q_s = x_s^{31/12+\varepsilon}. \tag{2.1}$$

Then we denote by $\mathfrak{M}_s$ and $\mathfrak{m}_s$ the major arcs and minor arcs respectively as follows

$$\mathfrak{M}_s = \bigcup_{q \le P_s} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \{\alpha : |q\alpha - a| \le Q_s^{-1}\} \tag{2.2}$$

and

$$\mathfrak{m}_s = [0,1)/\mathfrak{M}_s. \tag{2.3}$$

In the remaining part of this section we shall introduce some useful lemmas. To begin with, we define the multiplicative function $w(q)$ by

$$w(p^{3u+v}) = \begin{cases} 3p^{-u-\frac{1}{2}}, & u \le 0,\ v = 1, \\ p^{-u-1}, & u \le 0,\ v = 2, 3. \end{cases} \tag{2.4}$$

LEMMA 2.1 ([4, Lemma 2.2] or [17, Lemma 2]). *Assume that* $0 < \rho \le 1/4$, $y \le x$, $x^3 \le y^{4-2\rho}$ *and* ℓ *is a subinterval of* $(x, x+y]$. *Then either*

$$\sum_{n \in \ell} e(n^3 \alpha) \ll y^{1-\rho+\varepsilon},$$

or there exist integers a *and* q *such that*

$$1 \le q \le y^{3\rho}, \quad (a,q) = 1, \quad |q\alpha - a| \le x^{-2} y^{3\rho - 1}$$

and

$$\sum_{n \in \ell} e(n^3 \alpha) \ll \frac{y w(q)}{1 + x^2 y |\alpha - a/q|} + x^{3/2+\varepsilon} y^{-1}.$$

Combining [17, Lemma 1] and [17, Lemma 3] we get the following result.

LEMMA 2.2. *Suppose that x, y and ρ satisfy the conditions of Lemma* 2.1. *Denote by $\mathfrak{M}$ the union of the intervals*

$$\mathfrak{M}(q,a) = \{\alpha : |q\alpha - a| \le x^{-2}y^{3\rho-1}\}$$

with $1 \le a \le q \le y^{3\rho}$ and $(a,q)=1$. Assume that $G(\alpha)$ and $h(\alpha)$ are integrable functions of period one and define

$$g(\alpha) = \sum_{n\in\mathcal{A}} (\log n)e(n^3\alpha),$$

where $\mathcal{A}$ is any subinterval of $(x, x+y]$. Then for any measurable set $\mathfrak{m} \subseteq [0,1)$, one finds that

$$\int_{\mathfrak{m}} G(\alpha)h(\alpha)g(\alpha)\,d\alpha$$
$$\ll y^{5/4}x^{-1/2+\varepsilon}\Big(\int_{\mathfrak{m}} |G(\alpha)|^2\,d\alpha\Big)^{1/4}\mathcal{J}(\mathfrak{m})^{1/2} + y^{1-\rho/2+\varepsilon}\mathcal{J}(\mathfrak{m}),$$

where

$$\mathcal{J}(\mathfrak{m}) = \int_{\mathfrak{m}} |G(\alpha)h(\alpha)|\,d\alpha.$$

At the end of this section, we are going to expound the major arcs contribution in the form of a proposition.

PROPOSITION 2.3 [12, Proposition 1]. *Let P_s and Q_s be defined by* (2.1) *and let $\mathfrak{M}_s$ be defined by* (2.2). *When n belongs to $[N - sx_s^2y_s, N + sx_s^2y_s] \cap \mathcal{N}_s$, for any $A>0$ we have*

$$\int_{\mathfrak{M}_s} f_s(\alpha)^s e(-n\alpha)\,d\alpha = \frac{1}{3^s}\,\mathfrak{S}_s(n)\mathfrak{J}_s(n) + O(y_s^{s-1}x_s^{-2}L^{-A}),$$

where

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty}\Phi(q)^{-s}\sum_{\substack{1\le a\le q\\(a,q)=1}}\Big(\sum_{1\le h\le q (h,q)=1} e\Big(\frac{ah^3}{q}\Big)\Big)^s e\Big(-\frac{an}{q}\Big)$$

is the singular series which is absolutely convergent and satisfies $\mathfrak{S}_s(n) \gg 1$, and

$$\mathfrak{J}_s(n) = \sum_{\substack{m_1+\cdots+m_s=n\\(x_s-y_s)^3\le m_j\le (x_s+y_s)^3}} (m_1\cdots m_s)^{-2/3} \asymp y_s^{s-1}x_s^{-2}$$

is the singular integral.

3. Proof of Theorem 1.1

For the purpose of proving Theorem 1.1, we firstly introduce Kumchev's result [4, Theorem 2] in estimates for exponential sums over primes in short intervals.

LEMMA 3.1. *Let θ be a real number with $4/5 < \theta \le 1$ and assume that*

$$0 < \mu \le \min\big((2\theta-1)/14, (14\theta-11)/30, (5\theta-4)/6\big).$$

For a given positive number P, we suppose that

$$\mathfrak{M}(P) = \bigcup_{q \le P} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \big\{ |q\alpha - a| \le x^{-3+2(1-\theta)} P \big\},$$

and write $\mathfrak{m}(P) = [0,1) \setminus \mathfrak{M}(P)$ for the complementary set to the set $\mathfrak{M}(P)$. Then for any fixed $\varepsilon > 0$, we have

$$\sup_{\alpha \in \mathfrak{m}(P)} \Big| \sum_{x < n \le x+y} \Lambda(n) e(n^3\alpha) \Big| \ll x^{\theta-\mu+\varepsilon} + x^{\theta+\varepsilon} P^{-1/2}.$$

In order to proceed further, we need to define some symbols. Let

$$\rho_5 = 1/4 \quad \text{and} \quad \rho_6 = 1/5. \tag{3.1}$$

LEMMA 3.2. *Assume that*

$$\mathcal{I}(t) = \int_{\mathfrak{m}_5} |f_5(\alpha)|^t \, d\alpha.$$

Then we have

$$\mathcal{I}(10) \ll y_5^{63/8} x_5^{-1+\varepsilon} + y_5^{27/4+\varepsilon}.$$

PROOF. By taking $x = x_5$ and $y = y_5$ then applying Lemma 2.2 with $G(\alpha) = |f_5(\alpha)|^8$, $g(\alpha) = f_5(\alpha)$, $h(\alpha) = f_5(-\alpha)$ and $\rho = \rho_5$ which is defined in (3.1), one finds that

$$\mathcal{I}(10) \ll y_5^{5/4} x_5^{-1/2+\varepsilon} \mathcal{I}(16)^{1/4} \mathcal{I}(9)^{1/2} + y_5^{7/8+\varepsilon} \mathcal{I}(9). \tag{3.2}$$

We next take Cauchy's inequality to $\mathcal{I}(9)$ to obtain

$$\mathcal{I}(9) \le \mathcal{I}(8)^{1/2} \mathcal{I}(10)^{1/2}.$$

An argument similar to the proof of Hua's lemma (e.g., see [13, Lemma 2.5]) yields

$$\mathcal{I}(8) \le \int_0^1 |f_5(\alpha)|^8 \, d\alpha \ll y_5^{5+\varepsilon}. \tag{3.3}$$

Thus it follows from the above two inequalities that

$$\mathcal{I}(9) \ll y_5^{5/2+\varepsilon} \mathcal{I}(10)^{1/2}. \tag{3.4}$$

On the other hand, one can find that

$$\mathcal{I}(16) \le \sup_{\alpha\in\mathfrak{m}_5} |f_5(\alpha)|^6 \mathcal{I}(10). \tag{3.5}$$

Now denote $P=P_5$, $x=x_5$ and $x^\theta=y_5$ in Lemma 3.1. Then $\mathfrak{m}(P)=\mathfrak{m}_5$ and

$$\sup_{\alpha\in\mathfrak{m}_5} |f_5(\alpha)| \ll y_5 x_5^{-\mu+\varepsilon} + y_5^{1+\varepsilon} P_5^{-1/2}.$$

By taking $\mu = 1/27$, (1.2), (2.1) and $\theta_5 = 8/9 + \varepsilon$ which is defined in Theorem 1.1 yields that

$$\sup_{\alpha\in\mathfrak{m}_5} |f_5(\alpha)| \ll y_5 x_5^{-1/27+\varepsilon} + x_5^{19/24+\varepsilon} \ll y_5^{23/24+\varepsilon}. \tag{3.6}$$

Combining (3.5) and (3.6), we can obtain

$$\mathcal{I}(16) \ll y_5^{23/4+\varepsilon} \mathcal{I}(10). \tag{3.7}$$

We therefore deduce from (3.2), (3.4) and (3.7) that

$$\mathcal{I}(10) \ll y_5^{63/16} x_5^{-1/2+\varepsilon} \mathcal{I}(10)^{1/2} + y_5^{27/8+\varepsilon} \mathcal{I}(10)^{1/2} \ll y_5^{63/8} x_5^{-1+\varepsilon} + y_5^{27/4+\varepsilon}. \qquad \square$$

LEMMA 3.3. *Assume that*

$$\mathcal{K}(t) = \int_{\mathfrak{m}_6} |f_6(\alpha)|^t \, d\alpha.$$

Then we have

$$\mathcal{K}(12) \ll y_6^{107/10} x_6^{-7/4+\varepsilon} + y_6^{52/5} x_6^{-3/2+\varepsilon}.$$

PROOF. Setting $x = x_6$, $y = y_6$, $G(\alpha) = |f_6(\alpha)|^{10}$, $g(\alpha) = f_6(\alpha)$, $h(\alpha) = f_6(-\alpha)$ and recalling $\rho_6 = 1/5$ in (3.1), Lemma 2.2 yields

$$\mathcal{K}(12) \ll y_6^{5/4} x_6^{-1/2+\varepsilon} \mathcal{K}(20)^{1/4} \mathcal{K}(11)^{1/2} + y_6^{9/10+\varepsilon} \mathcal{K}(11). \tag{3.8}$$

An application of Hölder's inequality leads to the bound

$$\mathcal{K}(11) \ll \mathcal{K}(8)^{1/4} \mathcal{K}(12)^{3/4}. \tag{3.9}$$

Similarly to (3.3) we can obtain

$$\mathcal{K}(8) \ll y_6^{5+\varepsilon}.$$

Besides, [15, Propositon 2.2] yields

$$\mathcal{K}(12) \ll y_6^{11} x_6^{-2+\varepsilon}. \tag{3.10}$$

We therefore substitute the above two bounds into (3.9) to get

$$\mathcal{K}(11) \ll y_6^{19/2} x_6^{-3/2+\varepsilon}. \tag{3.11}$$

Now we turn to the contribution from $\mathcal{K}(20)$. Indeed,

$$\mathcal{K}(20) \ll \sup_{\alpha \in \mathfrak{m}_6} |f_6(\alpha)|^8 \mathcal{K}(12). \tag{3.12}$$

It follows from Lemma 3.1 with $\mu = 1/48$, $P = P_6$, $x = x_6$ and $x^\theta = y_6$ that

$$\sup_{\alpha \in \mathfrak{m}_6} |f_6(\alpha)| \ll y_6 x_6^{-1/48+\varepsilon} + x_6^{19/24+\varepsilon}.$$

Recalling that $y_6 = x_6^{\theta_6}$ in (1.2) and $\theta_6 = 5/6 + \varepsilon$, one obtains

$$\sup_{\alpha \in \mathfrak{m}_6} |f_6(\alpha)| \ll y_6^{39/40+\varepsilon}. \tag{3.13}$$

Substitute (3.10) and (3.13) into (3.12), one has

$$\mathcal{K}(20) \ll y_6^{94/5} x_6^{-2+\varepsilon}. \tag{3.14}$$

Thus, it follows from (3.8), (3.11) and (3.14) that

$$\begin{aligned}\mathcal{K}(12) &\ll y_6^{5/4} x_6^{-1/2+\varepsilon} (y_6^{94/5} x_6^{-2+\varepsilon})^{1/4} (y_6^{19/2} x_6^{-3/2+\varepsilon})^{1/2} + y_6^{9/10+\varepsilon} y_6^{19/2} x_6^{-3/2+\varepsilon} \\ &\ll y_6^{107/10} x_6^{-7/4+\varepsilon} + y_6^{52/5} x_6^{-3/2+\varepsilon}. \quad \square\end{aligned}$$

We shall finish our proof by means of an argument of Wooley (see [16]), Lemma 3.2 and Lemma 3.3.

PROOF OF THEOREM 1.1. On recalling the definition of $\mathcal{E}_s(x_s, y_s)$ in Section 1, one finds that

$$0 = \sum_{n \in \mathcal{E}_s(x_s, y_s)} \int_0^1 f_s(\alpha)^s e(-n\alpha)\, d\alpha$$

$$= \sum_{n\in\mathcal{E}_s(x_s,y_s)} \int_{\mathfrak{M}_s} f_s(\alpha)^s e(-n\alpha)\,d\alpha + \sum_{n\in\mathcal{E}_s(x_s,y_s)} \int_{\mathfrak{m}_s} f_s(\alpha)^s e(-n\alpha)\,d\alpha$$

$$= \sum_{n\in\mathcal{E}_s(x_s,y_s)} \int_{\mathfrak{M}_s} f_s(\alpha)^s e(-n\alpha)\,d\alpha + \int_{\mathfrak{m}_s} f_s(\alpha)^s \sum_{n\in\mathcal{E}_s(x_s,y_s)} e(-n\alpha)\,d\alpha,$$

and it follows that

$$\Big|\int_{\mathfrak{m}_s} f_s(\alpha)^s \sum_{n\in\mathcal{E}_s(x_s,y_s)} e(-n\alpha)\,d\alpha\Big| = \Big|\sum_{n\in\mathcal{E}_s(x_s,y_s)} \int_{\mathfrak{M}_s} f_s(\alpha)^s e(-n\alpha)\,d\alpha\Big|. \tag{3.15}$$

By applying Proposition 2.3, one has

$$\Big|\sum_{n\in\mathcal{E}_s(x_s,y_s)} \int_{\mathfrak{M}_s} f_s(\alpha)^s e(-n\alpha)\,d\alpha\Big| \gg x_s^{-2} y_s^{s-1} E_s, \tag{3.16}$$

where we use E_s to represent $E_s(x_s, y_s)$, and thus we deduce from (3.15), (3.16) and the triangle inequality that

$$\int_{\mathfrak{m}_s} \Big|f_s(\alpha)^s \sum_{n\in\mathcal{E}_s(x_s,y_s)} e(-n\alpha)\Big|\,d\alpha \gg x_s^{-2} y_s^{s-1} E_s. \tag{3.17}$$

By taking Cauchy's inequality to the left-hand side of (3.17), we obtain

$$\begin{aligned} &\int_{\mathfrak{m}_s} \Big|f_s(\alpha)^s \sum_{n\in\mathcal{E}_s(x_s,y_s)} e(-n\alpha)\Big|\,d\alpha \\ &\le \Big(\int_{\mathfrak{m}_s} |f_s(\alpha)|^{2s}\,d\alpha\Big)^{1/2} \Big(\int_0^1 \Big|\sum_{n\in\mathcal{E}_s(x_s,y_s)} e(-n\alpha)\Big|^2 d\alpha\Big)^{1/2} \\ &\le \Big(\int_{\mathfrak{m}_s} |f_s(\alpha)|^{2s}\,d\alpha\Big)^{1/2} E_s^{1/2}. \end{aligned} \tag{3.18}$$

Consequently, the estimates (3.17) and (3.18) together lead to

$$E_s \ll x_s^4 y_s^{2(1-s)} \int_{\mathfrak{m}_s} |f_s(\alpha)|^{2s}\,d\alpha. \tag{3.19}$$

We consider the case $s = 5$ firstly. Indeed, (3.19) with $s = 5$ yields

$$E_5 \ll x_5^4 y_5^{-8} \mathcal{I}(10).$$

Then by applying Lemma 3.2 one has

$$E_5 \ll x_5^{3+\varepsilon} y_5^{-1/8} + x_5^{4+\varepsilon} y_5^{-5/4}.$$

On recalling $\theta_5 = 8/9 + \varepsilon$, we therefore obtain

$$E_5 \ll x_5^{2-\varepsilon} y_5.$$

We secondly calculate the case $s = 6$. One can deduce from (3.19) with $s = 6$ that

$$E_6 \ll x_6^4 y_6^{-10} \mathcal{K}(12).$$

We substitute Lemma 3.3 into the above expression to get

$$E_6 \ll x_6^{9/4+\varepsilon} y_6^{7/10} + x_6^{5/2+\varepsilon} y_6^{2/5}.$$

Thus our choice of $\theta_6 = 5/6 + \varepsilon$ yields to

$$E_6 \ll x_6^{2-\varepsilon} y_6. \quad \square$$

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