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ALMOST UNIFORM AND STRONG CONVERGENCES IN ERGODIC THEOREMS FOR SYMMETRIC SPACES

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Abstract. Let (Ω,μ) be a σ-finite measure space, and let $X\subset L^1(\Omega)+L^\infty(\Omega)$ be a fully symmetric space of measurable functions on (Ω,μ). If $\mu(\Omega)=\infty$, necessary and sufficient conditions are given for almost uniform convergence in X (in Egorov's sense) of Cesàro averages $M_n(T)(f)=\frac{1}{n}\sum_{k=0}^{n-1}T^k(f)$ for all Dunford–Schwartz operators T in $L^1(\Omega)+L^\infty(\Omega)$ and any $f\in X$. If (Ω,μ) is quasi-non-atomic, it is proved that the averages $M_n(T)$ converge strongly in X for each Dunford–Schwartz operator T in $L^1(\Omega)+L^\infty(\Omega)$ if and only if X has order continuous norm and $L^1(\Omega)$ is not contained in X.

1. Introduction

Let $(\Omega,\mathcal{A},\mu)$ be a complete σ-finite measure space. Denote by $L^0=L^0(\Omega)$ the algebra of equivalence classes of almost everywhere (a.e.) finite real-valued measurable functions on Ω. Let $L^p=L^p(\Omega)\subset L^0$, $1\le p\le\infty$, be the L^p-space equipped with the standard norm $\|\cdot\|_p$.

Let $T\colon L^1+L^\infty\to L^1+L^\infty$ be a Dunford–Schwartz operator (writing $T\in DS$), that is, T is linear and

$$\|T(f)\|_1\le\|f\|_1 \text{ for all } f\in L^1 \quad\text{and}\quad \|T(f)\|_\infty\le\|f\|_\infty \text{ for all } f\in L^\infty.$$

If $T\in DS$ is positive, that is, $T(f)\ge 0$ whenever $f\ge 0$, then we shall write $T\in DS^+$.

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The Dunford–Schwartz individual ergodic theorem states that for every $T \in DS$ and $f \in L^p$, $1 \leq p < \infty$, the averages

$$M_n(T)(f) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(f) \tag{1}$$

converge a.e. to some $\widehat{f} \in L^p$ (see, for example, [6, Ch. VIII, §6, Theorem VIII.6.6]).

In the case $\mu(\Omega) < \infty$, it follows from Egorov's theorem that a.e. convergence coincides with the almost uniform (a.u.) convergence, thus, the Dunford–Schwartz individual ergodic theorem asserts a.u. convergence of the averages $M_n(T)(f)$ for each $f \in L^p$, $1 \leq p < \infty$, and all $T \in DS$. If $\mu(\Omega) = \infty$, then it is clear that a.u. convergence (in Egorov's sense) is generally stronger than a.e. convergence, so it is interesting to see if there exist functions $f \in L^1 + L^\infty$ such that the ergodic averages (1) converge a.u. for every $T \in DS$.

Thus, if $\mu(\Omega) = \infty$, there is the problem of describing the largest subspace of $L^1 + L^\infty$ for which a.u. convergence in the Dunford–Schwartz individual ergodic theorem holds. To this end, let

$$\mathcal{R}_\mu = \left\{ f \in L^1 + L^\infty : \mu\{|f| > \lambda\} < \infty \text{ for all } \lambda > 0 \right\}. \tag{2}$$

In Section 3 we prove (Theorem 3.1) that for each $f \in \mathcal{R}_\mu$ and any $T \in DS$ the averages $M_n(T)(f)$ converge a.u. to some $\widehat{f} \in \mathcal{R}_\mu$. It should be pointed out that, by virtue of Lemma 3.1, the proof of a.u. convergence in the Dunford–Schwartz ergodic theorem is noticeably simpler than the proof of a.e. convergence. We also show that $\mathcal{R}_\mu$ is the largest subspace of $L^1 + L^\infty$ for which the convergence takes place: if $f \in (L^1 + L^\infty) \setminus \mathcal{R}_\mu$, then there exists $T \in DS$ such that the sequence $\{M_n(T)(f)\}$ does not converge a.u. (Theorem 3.5).

A well-known mean ergodic theorem asserts (see, for example, [6, Ch. VIII, §5]) that the averages $M_n(T)$ converge strongly in a reflexive Banach space $(X, \|\cdot\|_X)$ for every linear contraction T of X, that is, given $x \in X$, there exists $\widehat{x} \in X$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) - \widehat{x} \right\|_X \to 0 \quad \text{as } n \to \infty.$$

Important examples illustrating this ergodic theorem are the reflexive spaces L^p, $1 < p < \infty$. In particular, the averages $M_n(T)$ converge strongly in L^p for any $T \in DS$. For the spaces L^1 and L^∞, the mean ergodic theorem is false, in general.

It is known that if $T \in DS$, then $T(X) \subset X$ for any exact interpolation (for the Banach pair (L^1, L^∞)) symmetric space X of real-valued measurable functions on (Ω, μ). In addition, $\|T\|_{X \to X} \leq 1$ (see, for example, [11, Ch. II, §4, Sec. 2]). Recall also that the class of exact interpolation symmetric spaces for the Banach pair (L^1, L^∞) coincides with the class of fully symmetric spaces on (Ω, μ) [11, Ch. II, §4, Theorem 4.3]. Therefore, there is the problem of describing the class of fully symmetric spaces X for which the mean ergodic theorem with respect to the action of an arbitrary $T \in DS$ is valid.

If X is a separable symmetric space on the non-atomic measure space $((0,a), \nu)$, where $0 < a < \infty$ and ν is the Lebesgue measure, then the averages $M_n(T)$ converge strongly in X for every $T \in DS$ (see [21,22]; also [23, Ch. 2, §2.1, Theorem 2.1.3]). At the same time, if X is a non-separable fully symmetric space, then for each $f \in X \setminus \overline{L^\infty(0,a)}^{\|\cdot\|_X}$ there exist $T \in DS$ and a function $\widetilde{f}$, equimeasurable with f, such that the sequence $\{A_n(T)(\widetilde{f})\}$ does not converge strongly in X [22]. Note also that, for the separable symmetric space $L^1((0,\infty), \nu)$, there exists $T \in DS$ such that the averages $M_n(T)$ do not converge strongly in $L^1((0,\infty), \nu)$.

The main result of Section 4 is Theorem 4.5, which gives a criterion for a fully symmetric space X to satisfy the following: the averages $M_n(T)(f)$ converge strongly in X for every $f \in X$ and $T \in DS$.

In Section 5 we discuss some (classes of) fully symmetric spaces for which Dunford–Schwartz-type ergodic theorems hold/fail.

2. Preliminaries

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $L^0 = L^0(\Omega)$ be the algebra of (classes of) a.e. finite real-valued measurable functions on $(\Omega, \mathcal{A}, \mu)$. Let L^0_μ be the subalgebra in L^0 consisting of the functions $f \in L^0$ such that $\mu\{|f| > \lambda\} < \infty$ for some $\lambda > 0$.

In what follows t_μ will stand for the *measure topology* in L^0, that is, the topology given by the following system of neighborhoods of zero:

$$\mathcal{N}(\varepsilon, \delta) = \big\{ f \in L^0 : \ \mu\{|f| > \delta\} \leq \varepsilon \big\}, \quad \varepsilon > 0, \ \delta > 0.$$

It is well-known (see, for example, [7, Ch. IV, §27, Theorem 5]) that (L^0, t_μ) is a complete metrizable vector space. Since L^0_μ is a closed linear subspace of (L^0, t_μ) (see, for example, [11, Ch. II, §2]), (L^0_μ, t_μ) is also a complete metrizable vector space.

Denote by $L^p = L^p(\Omega) \subset L^0_\mu$, $1 \leq p \leq \infty$, the classical Banach space equipped with the standard norm $\|\cdot\|_p$.

If $f \in L^0_\mu$, then the *non-increasing rearrangement* of f is defined as

$$f^*(t) = \inf \left\{ \lambda > 0 : \ \mu\{|f| > \lambda\} \le t \right\}, \quad t > 0,$$

(see [11, Ch. II, §2]).

Consider the σ-finite measure space $((0,\infty),\nu)$, where ν is the Lebesgue measure. A non-zero linear subspace $X \subset L^0_\nu$ with a Banach norm $\|\cdot\|_X$ is called *symmetric (fully symmetric)* on $((0,\infty),\nu)$ if

$$f \in X, \quad g \in L^0_\nu, \quad g^*(t) \le f^*(t) \quad \text{for all } t > 0$$

respectively,

$$f \in X, \ \ g \in L^0_\nu, \ \ \int_0^s g^*(t)\,dt \le \int_0^s f^*(t)\,dt \text{ for all } s > 0 \text{ (writing } g \prec\prec f)$$

implies that $g \in X$ and $\|g\|_X \le \|f\|_X$.

Let $(X, \|\cdot\|_X)$ be a symmetric (fully symmetric) space on $((0,\infty),\nu)$. Define

$$X(\Omega) = \left\{ f \in L^0_\mu : f^*(t) \in X \right\}$$

and set

$$\|f\|_{X(\Omega)} = \|f^*(t)\|_X, \quad f \in X(\Omega).$$

It is shown in [10] (see also [17, Ch. 3, Section 3.5]) that $(X(\Omega), \|\cdot\|_{X(\Omega)})$ is a Banach space and the conditions $f \in X(\Omega)$, $g \in L^0_\mu$, $g^*(t) \le f^*(t)$ for every $t > 0$ (respectively, $g \prec\prec f$) imply that $g \in X(\Omega)$ and $\|g\|_{X(\Omega)} \le \|f\|_{X(\Omega)}$. In such a case, we say that $(X(\Omega), \|\cdot\|_{X(\Omega)})$ is a symmetric (fully symmetric) space on $(\Omega, \mathcal{A}, \mu)$ *generated* by the symmetric (fully symmetric) space $(X, \|\cdot\|_X)$. It is clear that if $f, g \in X(\Omega)$ and $f^* = g^*$, then $\|f\|_{X(\Omega)} = \|g\|_{X(\Omega)}$.

In what follows, if it does not cause confusion, we will write $(X, \|\cdot\|_X)$, or simply X, instead of $(X(\Omega), \|\cdot\|_{X(\Omega)})$.

Immediate examples of fully symmetric spaces are $L^1 \cap L^\infty$ with the norm

$$\|f\|_{L^1 \cap L^\infty} = \max \left\{ \|f\|_{L^1}, \|f\|_{L^\infty} \right\}$$

and $L^1 + L^\infty$ with the norm

$$\|f\|_{L^1 + L^\infty} = \inf \left\{ \|g\|_1 + \|h\|_\infty : f = g + h, \ g \in L^1, \ h \in L^\infty \right\} = \int_0^1 f^*(t)\,dt$$

(see [11, Ch. II, §4]).

It is known that a symmetric space X is fully symmetric if and only if X is an exact interpolation space for the Banach couple (L^1, L^∞) (see, for example, [1, Ch. 3, §2, Theorem 2.2], [11, Ch. II, §4, Theorem 4.3]). Consequently, $T(X) \subset X$ and $\|T\|_{X \to X} \leq 1$ for any $T \in DS$.

We need the following property of embeddings of symmetric spaces [20, Ch. 6, §6.1, Proposition 6.1.1].

Proposition 2.1. *If $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ are symmetric spaces with $X_1 \subset X_2$, then there is a constant $c > 0$ such that $\|f\|_{X_2} \leq c\|f\|_{X_1}$ for all $f \in X_1$.*

It follows from Proposition 2.1 that if $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are symmetric spaces, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Given a symmetric space $(X, \|\cdot\|_X)$, in view of the embeddings

$$L^1 \cap L^\infty \subset X \subset L^1 + L^\infty$$

[1, Ch. 2, §6, Theorem 6.6], it follows from Proposition 2.1 that there exist constants $c_1 > 0$ and $c_2 > 0$ such that $\|f\|_X \leq c_1\|f\|_{L^1 \cap L^\infty}$ for all $f \in L^1 \cap L^\infty$ and $\|f\|_{L^1 + L^\infty} \leq c_2\|f\|_X$ for all $f \in X$.

A symmetric space $(X, \|\cdot\|_X)$ is said to have *order continuous norm* if $\|f_\alpha\|_X \downarrow 0$ whenever $f_\alpha \in X$ and $f_\alpha \downarrow 0$. It is clear that a symmetric space X has (respectively, has no) order-continuous norm if and only if a symmetric space on $((0, \infty), \nu)$, that generates X, has (respectively, has no) order-continuous norm. Besides, a symmetric space on $((0, \infty), \nu)$ has order continuous norm if and only if it is separable [20, II, Ch. 6, §6.5, Theorem 6.5.3.]. In addition, by [11, Ch. II, §4, Theorem 4.10] and [20, II, Ch. 6, §6.5, Theorem 6.5.3], every separable symmetric space X on $((0, \infty), \nu)$ is an exact interpolation space for the Banach pair (L^1, L^∞). Hence, in this case, X is a fully symmetric space [11, Ch. II, §4, Theorem 4.3].

If $(X, \|\cdot\|_X)$ is a symmetric space on $((0, \infty), \nu)$, then the Köthe dual $X^\times$ is defined as

$$X^\times = \left\{ f \in L^0_\nu : fg \in L^1 \text{ for all } g \in X \right\},$$

and

$$\|f\|_{X^\times} = \sup \left\{ \left| \int_0^\infty fg \, d\nu \right| : g \in X, \ \|g\|_X \leq 1 \right\} \quad \text{if } f \in X^\times.$$

It is known that $(X^\times, \|\cdot\|_{X^\times})$ is a fully symmetric space (see, for example, [11, Ch. II, §4, Theorem 4.9], [20, II, Ch. 7, §7.2, Theorem 7.2.2]). In addition,

$$X \subset X^{\times\times}, \quad (L^1)^\times = L^\infty, \quad (L^\infty)^\times = L^1;$$

$$(L^1 + L^\infty, \|\cdot\|_{L^1 + L^\infty})^\times = (L^1 \cap L^\infty, \|\cdot\|_{L^1 \cap L^\infty});$$

$$(L_1 \cap L^\infty, \|\cdot\|_{L^1 \cap L^\infty})^\times = (L^1 + L^\infty, \|\cdot\|_{L^1 + L^\infty})$$

(see [20, II, Ch. 7]).

Note that

$$X^{\times}(\Omega) = \left\{ f \in L^0_\mu : fg \in L^1(\Omega) \text{ for all } g \in X(\Omega) \right\},$$

and

$$\|f\|_{X^\times(\Omega)} = \sup\left\{ \left| \int_\Omega fg \, d\mu \right| : g \in X(\Omega), \ \|g\|_{X(\Omega)} \le 1 \right\}, \quad f \in X^\times(\Omega).$$

The fully symmetric space $(X^\times(\Omega), \|\cdot\|_{X^\times(\Omega)})$ is called the *Köthe dual space* of the symmetric space $(X(\Omega), \|\cdot\|_{X(\Omega)})$.

A symmetric space $(X, \|\cdot\|_X)$ is said to possess the *Fatou property* if the conditions

$$0 \le f_n \in X, \quad f_n \le f_{n+1} \text{ for all } n, \quad \text{and} \quad \sup_n \|f_n\|_X < \infty$$

imply that $f = \sup_n f_n \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$ exist.

If $X = X^{\times\times}$, then the symmetric space X possesses the Fatou property (see, for example, [14, Vol.II, Ch. I, §1b]); in particular, the fully symmetric space $(L^1 + L^\infty, \|\cdot\|_{L^1+L^\infty})$ possesses the Fatou property. In addition, in any symmetric space $(X, \|\cdot\|_X)$ with the Fatou property the conditions $f_n \in X$, $\sup_n \|f_n\|_X \le \alpha$, $f \in L^0$ and $f_n \to f$ in t_μ imply that $f \in X$ and $\|f\|_X \le \alpha$ (see, for example, [9, Ch. IV, §3, Lemma 5]).

Define

$$\mathcal{R}_\mu = \left\{ f \in L^1 + L^\infty : f^*(t) \to 0 \text{ as } t \to \infty \right\}.$$

It is clear that $\mathcal{R}_\mu$ admits a more direct description (2).

Note that if $\mu(\Omega) < \infty$, then $\mathcal{R}_\mu$ is simply L^1. However, we will be concerned with infinite measure spaces.

By [11, Ch. II, §4, Lemma 4.4], $(\mathcal{R}_\mu, \|\cdot\|_{L^1+L^\infty})$ is a symmetric space. In addition, $\mathcal{R}_\mu$ is the closure of $L^1 \cap L^\infty$ in $L^1 + L^\infty$ (see [11, Ch. II, §3, Section 1]). In particular, $(\mathcal{R}_\mu, \|\cdot\|_{L^1+L^\infty})$ is a fully symmetric space.

Let χ_E be the characteristic function of a set $E \in \mathcal{A}$. Denote $\mathbf{1} = \chi_\Omega$. The following gives a necessary and sufficient condition for the embedding of a symmetric space into $\mathcal{R}_\mu$.

Proposition 2.2. *If $\mu(\Omega) = \infty$, then a symmetric space $X \subset L^0_\mu$ is contained in $\mathcal{R}_\mu$ if and only if $\mathbf{1} \notin X$.*

We will also need the next property of the fully symmetric space $\mathcal{R}_\mu$.

Proposition 2.3. *For every $f \in \mathcal{R}_\mu$ and $\varepsilon > 0$ there exist $g_\varepsilon \in L^1$ and $h_\varepsilon \in L^\infty$ such that*

$$f = g_\varepsilon + h_\varepsilon \quad \textit{and} \quad \|h_\varepsilon\|_\infty \le \varepsilon.$$

For proofs of Propositions 2.2 and 2.3, see [2, Propositions 2.1, 2.2].

3. Almost uniform convergence in the Dunford–Schwartz pointwise ergodic theorem

A sequence $\{f_n\} \subset L^0$ is said to converge *almost uniformly* to $f \in L^0$ if for every $\varepsilon > 0$ there exists a set $E \subset \Omega$ such that $\mu(\Omega \setminus E) \leq \varepsilon$ and $\|(f - f_n)\chi_E\|_\infty \to 0$. The main goal of this section is to prove the following extension of the classical Dunford–Schwartz pointwise ergodic theorem.

THEOREM 3.1. *Assume that $(\Omega, \mathcal{A}, \mu)$ is an arbitrary measure space, and let X be a fully symmetric space on $(\Omega, \mathcal{A}, \mu)$ such that $\mathbf{1} \notin X$. If $T \in DS$ and $f \in X$, then the averages* (1) *converge a.u. to some $\widehat{f} \in X$. In particular, $M_n(T)(f) \to \widehat{f} \in \mathcal{R}_\mu$ a.u. for all $f \in \mathcal{R}_\mu$.*

REMARK 3.1. In proving Theorem 3.1, we can and will assume that $(\Omega, \mathcal{A}, \mu)$ is σ-finite. Indeed, if $f \in X$ and $\mathbf{1} \notin X$, then $f \in \mathcal{R}_\mu$ by Proposition 2.2, which implies that $\{T^k(f)\}_{k=0}^{\infty} \subset \mathcal{R}_\mu$. Therefore, the set $\Omega_f = \bigcup_k \{T^k(f) \neq 0\}$ is σ-finite, and one can replace Ω by Ω_f.

In view of Propositions 2.2 and 2.3, the proof of Theorem 3.1 can be easily reduced to the case $X = L^1$, so we shall treat this case first.

Let $(X, \|\cdot\|_X)$ be a Banach space, and let $M_n \colon X \to L^0$ be a sequence of linear maps. Denote

$$M^\star(f) = \sup_n |M_n(f)|,$$

the *maximal function* of $f \in X$. If $M^\star(f) \in L^0$ for all $f \in X$, then the function

$$M^\star \colon X \to L^0, \quad f \in X,$$

is called the *maximal operator* of the sequence $\{M_n\}$.

REMARK 3.2. (1) If $\mu(\Omega) < \infty$, then the Banach principle implies that if $M^\star(f) \in L^0$ for all $f \in X$, then $M^\star \colon (X, \|\cdot\|_X) \to (L^0, t_\mu)$ is continuous at zero, which is not the case when μ is not finite; see [16, Sec. 2].

(2) If $f \in X$ and $E \in \mathcal{A}$ are such that $\{M_n(f)\chi_E\} \subset L^\infty$, then it is easy to see that

$$\|M^\star(f)\chi_E\|_\infty = \sup_n \|M_n(f)\chi_E\|_\infty$$

[15, Proposition 1.1]. Therefore, the continuity of $M^\star : (X, \|\cdot\|_X) \to (L^0, t_\mu)$ at zero can be expressed as follows: given $\varepsilon > 0$, $\delta > 0$, there exists a $\gamma > 0$ such that for every $f \in X$ with $\|f\|_X \leq \gamma$ it is possible to find $E \subset \Omega$ satisfying the conditions

$$\mu(\Omega \setminus E) \leq \varepsilon \quad \text{and} \quad \sup_n \|M_n(f)\chi_E\|_\infty \leq \delta.$$

Proposition 3.1. *The algebra L^0_μ is complete with respect to a.u. convergence.*

Proof. Assume that $\{f_n\} \subset L^0_\mu$ is a.u. Cauchy. Then it is clearly Cauchy with respect to t_μ. Since L^0_μ is complete relative to t_μ, there exists an $f \in L^0_\mu$ such that $f_n \to f$ in measure.

To show that $f_n \to f$ a.u., fix $\varepsilon > 0$. Since $f_n \in L^0_\mu$ for every n and $\{f_n\}$ is a.u. Cauchy, it is possible to construct $E \subset \Omega$ such that $\mu(\Omega \setminus E) \le \varepsilon$, $f_n \chi_E \in L^\infty$ for every n, and

$$\|(f_m - f_n)\chi_E\|_\infty \to 0 \quad \text{as } m, n \to \infty.$$

This implies that there exists an $\widehat{f} \in L^\infty$ such that $\|\widehat{f} - f_n\chi_E\|_\infty \to 0$, hence $f_n\chi_E \to \widehat{f}$ in measure. But $f_n \to f$ in measure implies that $f_n\chi_E \to f\chi_E$ in measure, hence $\widehat{f} = f\chi_E$ and so

$$\|(f - f_n)\chi_E\|_\infty = \|\widehat{f} - f_n\chi_E\|_\infty \to 0.$$

Therefore, the sequence $\{f_n\}$ is a.u. convergent in L^0_μ, that is, L^0_μ is a.u. complete. $\square$

Lemma 3.1. *If the maximal operator $M^\star : (X, \|\cdot\|_X) \to (L^0, t_\mu)$ of a sequence $M_n \colon X \to L^0_\mu$ of linear maps is continuous at zero, then the set*

$$X_c = \big\{ f \in X : \{M_n(f)\} \text{ converges a.u.} \big\}$$

is closed in X.

Proof. Let $X_c \ni f_k \to f$ in the norm $\|\cdot\|_X$. Fix $\varepsilon > 0$, $\delta > 0$. In view of Remark 3.2(2) and since $M^\star$ is continuous at zero, there exist f_{k_0} and $E_0 \subset \Omega$ such that

$$\mu(\Omega \setminus E_0) \le \frac{\varepsilon}{2} \quad \text{and} \quad \sup_n \big\| M_n(f - f_{k_0})\chi_{E_0} \big\|_\infty \le \frac{\delta}{3}.$$

Next, since the sequence $\{M_n(f_{k_0})\}$ converges a.u., there exist $E_1 \subset \Omega$ and $N \in \mathbb{N}$ such that

$$\mu(\Omega \setminus E_1) \le \frac{\varepsilon}{2} \quad \text{and} \quad \big\| (M_m(f_{k_0}) - M_n(f_{k_0}))\chi_{E_1} \big\|_\infty \le \frac{\delta}{3} \quad \text{for all } m, n \ge N.$$

Therefore, setting $E = E_0 \cap E_1$, we arrive at $\mu(\Omega \setminus E) \le \varepsilon$ and

$$\big\| (M_m(f) - M_n(f))\chi_E \big\|_\infty \le \big\| M_m(f - f_{k_0})\chi_E \big\|_\infty + \big\| M_n(f - f_{k_0})\chi_E \big\|_\infty + \big\| (M_m(f_{k_0}) - M_n(f_{k_0}))\chi_E \big\|_\infty \le \delta$$

for all $m, n \geq N$. This means that the sequence $\{M_n(f)\}$ is a.u. Cauchy, which, by Proposition 3.1, entails that $\{M_n(f)\}$ converges a.u., hence $f \in X_c$, and we conclude that X_c is closed in $(X, \|\cdot\|_X)$. $\square$

Therefore, since $T(L^1) \subset L^1 \subset L^0_\mu$ for a given $T \in DS$, in order to prove that the averages (1) converge a.u. for every $f \in X = L^1$, it is sufficient to show that

(A) the maximal operator $M(T)^\star : (L^1, \|\cdot\|_1) \to (L^0, t_\mu)$ is continuous at zero;

(B) there exists a dense subset $\mathcal{D}$ of L^1 such that the sequence $\{M_n(T)(f)\}$ converges a.u. whenever $f \in \mathcal{D}$.

Here is our main tool, Hopf's maximal ergodic theorem [8]; see also [19, Theorem 1.1, p.75]:

THEOREM 3.2. *If $T \colon L^1 \to L^1$ is a positive linear contraction and $f \in L^1$, then*

$$\int_{\{M(T)^\star(f)>0\}} f \, d\mu \geq 0.$$

We shall prove the following maximal inequality for $T \in DS$ acting in L^p, $1 \leq p < \infty$. Note that, in order to establish Theorem 3.1, we will only need it for $p = 1$ and $p = 2$.

THEOREM 3.3. *If $T \in DS$ and $1 \leq p < \infty$, then*

$$\mu\{M(T)^\star(|f|) > \lambda\} \leq \Big(2\frac{\|f\|_p}{\lambda}\Big)^p \quad \text{for all } f \in L^p,\ \lambda > 0. \tag{3}$$

PROOF. Assume first that $T \in DS^+$. Fix $f \in L^1$ and $\lambda > 0$. Pick $F \subset \Omega$ such that $\mu(F) < \infty$ and let $f_{\lambda,F} = |f| - \lambda\chi_F$. Then, since $\lambda\chi_F \in L^\infty$, we have $\|T(\lambda\chi_F)\|_\infty \leq \lambda$, hence $T(\lambda\chi_F) \leq \lambda \cdot \mathbf{1}$. Therefore

$$T(f_{\lambda,F}) \geq T(|f|) - \lambda \cdot \mathbf{1},$$

and we derive

$$M(T)^\star(f_{\lambda,F}) \geq M(T)^\star(|f|) - \lambda \cdot \mathbf{1}.$$

By Theorem 3.2,

$$\int_{\{M(T)^\star(f_{\lambda,F})>0\}} f_{\lambda,F} \geq 0,$$

implying that

$$\|f\|_1 \geq \int_{\{M(T)^\star(f_{\lambda,F})>0\}} |f| = \int_{\{M(T)^\star(f_{\lambda,F})>0\}} (f_{\lambda,F} + \lambda\chi_F)$$

$$\geq \int_{\{M(T)^\star(f_{\lambda,F})>0\}} \lambda\chi_F = \lambda\mu\big(\{M(T)^\star(f_{\lambda,F})>0\}\cap F\big)$$
$$\geq \lambda\mu\big(\{M(T)^\star(|f|)>\lambda\}\cap F\big).$$

Therefore, we have

$$\mu\big(\{M(T)^\star(|f|)>\lambda\}\cap F\big) \leq \frac{\|f\|_1}{\lambda}$$

for every $F\subset\Omega$ with $\mu(F)<\infty$. Since (Ω,μ) is σ-finite, we arrive at the following maximal inequality for $T\in DS^+$ acting in L^1:

$$\mu\big\{M(T)^\star(|f|)>\lambda\big\} \leq \frac{\|f\|_1}{\lambda} \quad \text{for all } f\in L^1,\ \lambda>0. \tag{4}$$

Now, fix $1<p<\infty$, $f\in L^p$, and $\lambda>0$. Since $t\geq\frac{\lambda}{2}$ implies $t\leq(\frac{2}{\lambda})^{p-1}t^p$, we have

$$|f(\omega)|\leq\Big(\frac{2}{\lambda}\Big)^{p-1}|f(\omega)|^p \quad \text{whenever } |f(\omega)|\geq\frac{\lambda}{2}.$$

Then, denoting $A_\lambda=\{|f|<\lambda/2\}$ and $g_\lambda=|f|\chi_{A_\lambda}$, we obtain

$$|f|\leq g_\lambda+\Big(\frac{2}{\lambda}\Big)^{p-1}|f|^p.$$

Since $g_\lambda\in L^\infty$, we have $\|T(g_\lambda)\|_\infty\leq\|g_\lambda\|_\infty\leq\frac{\lambda}{2}$, and it follows that

$$M(T)^\star(|f|)\leq\frac{\lambda}{2}\cdot\mathbf{1}+\Big(\frac{2}{\lambda}\Big)^{p-1}M(T)^\star(|f|^p).$$

As $|f|^p\in L^1$, employing (4), we obtain a maximal inequality for $T\in DS^+$ acting in L^p, $1\leq p<\infty$:

$$\mu\big\{M(T)^\star(|f|)>\lambda\big\} \leq \mu\Big\{\Big(\frac{2}{\lambda}\Big)^{p-1}M(T)^\star(|f|^p)>\frac{\lambda}{2}\Big\} \tag{5}$$
$$=\mu\Big\{(M(T)^\star(|f|^p)>\Big(\frac{\lambda}{2}\Big)^p\Big\}\leq\Big(2\frac{\|f\|_p}{\lambda}\Big)^p, \quad f\in L^p,\ \lambda>0.$$

Finally, let $T\in DS$. If $|T|\colon L^p\to L^p$ is the linear modulus of $T\colon L^p\to L^p$, then $|T|\in DS^+$ and $|T^k(f)|\leq|T|^k(|f|)$ for all $f\in L^p$ and $1\leq p\leq\infty$, $k=0,1,\dots$ (see, for example, [18, Ch. 1, §1.3], [13, Ch. 4, §4.1, Theorem 1.1]). Therefore, given $f\in L^p$, $1\leq p<\infty$, we have

$$\big|M_n(T)(|f|)\big|\leq\frac{1}{n}\sum_{k=0}^{n-1}\big|T^k(|f|)\big|\leq\frac{1}{n}\sum_{k=0}^{n-1}|T|^k(|f|)=M_n(|T|)(|f|).$$

Thus, applying inequality (5) to $|T| \in DS^+$, we obtain (3):

$$\mu\{M(T)^\star(|f|)>\lambda\} \le \mu\{M(|T|)^\star(|f|)>\lambda\} \le \Big(2\frac{\|f\|_p}{\lambda}\Big)^p, \ f \in L^p, \ \lambda > 0. \ \square$$

Proof of Theorem 3.1. Show first that the sequence $\{M_n(f)\}$ converges a.u. whenever $f \in L^1$. In view of Theorem 3.3, the maximal operator $M(T)^\star\colon (L^p, \|\cdot\|) \to (L^0, t_\mu)$ is continuous at zero for every $1 \le p < \infty$. This, by Lemma 3.1, implies that the set

$$\mathcal{C}_p = \big\{ f \in L^p : \{M_n(f)\} \text{ converges a.u.} \big\}$$

is closed in L^p, $1 \le p < \infty$.

In particular, the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed in L^1 and L^2, respectively. Therefore, taking into account that the set $L^1 \cap L^2$ is dense in L^1, it is sufficient to show that the sequence $\{M_n(T)(f)\}$ converges a.u. for each f in a dense subset of L^2.

Denote by $(\cdot,\cdot)$ the standard inner product in L^2. Let

$$N = \big\{ T(h) - h : \ h \in L^2 \cap L^\infty \big\}.$$

If $L^2 \ni g \in N^\perp$, then, as $L^2 \cap L^\infty$ is dense in L^2, we have

$$0 = (g, T(h) - h) = (T^*(g) - g, h), \quad h \in L^2,$$

so $T^*(g) = g$. Recalling that T is a contraction in L^2, we obtain

$$\|T(g) - g\|_2^2 = (T(g) - g, T(g) - g) = \tag{6}$$

$$\|T(g)\|_2^2 - (g, T^*(g)) - (T^*(g), g) + \|g\|_2^2 = \|T(g)\|_2^2 - \|g\|_2^2 \le 0,$$

so $T(g) = g$ as well, hence $N^\perp \subset L = \{g \in L^2 : T(g) = g\}$. Conversely, if $g \in L$, then, since T^* is also a contraction in L^2, replacing T by T^* in (6), we obtain $T^*(g) = g$, which implies that $g \in N^\perp$. Therefore $N^\perp = L$, hence $\overline{N} \oplus L = L^2$, and we conclude that the set

$$\mathcal{D} = \big\{ g + (T(h) - h) : g \in L^2, \ T(g) = g; \ h \in L^2 \cap L^\infty \big\}$$

is dense in L^2. Because $h \in L^\infty$, it is clear that the sequence $\{M_n(f)\}$ converges a.u. for every $f \in \mathcal{D}$, and we conclude that this sequence converges a.u. for all $f \in L^1$.

Now, let $X \subset L^1 + L^\infty$ be a fully symmetric space such that $\mathbf{1} \notin X$, and let $f \in X$. By Proposition 2.2, $f \in \mathcal{R}_\mu$. Fix $\varepsilon > 0$ and $\delta > 0$. In view of Proposition 2.3, there exist $g \in L^1$ and $h \in L^\infty$ such that

$$f = g + h, \quad g \in L^1, \quad \text{and} \quad \|h\|_\infty \le \frac{\delta}{3}.$$

Since $g \in L^1$, there exist $E \subset \Omega$ and $N \in \mathbb{N}$ satisfying the conditions

$$\mu(\Omega \setminus E) \le \varepsilon \quad \text{and} \quad \big\| (M_m(g) - M_n(g))\chi_E \big\|_\infty \le \frac{\delta}{3} \text{ for all } m, n \ge N.$$

Then, given $m, n \ge N$, we have

$$\begin{aligned} &\big\| (M_m(f) - M_n(f))\chi_E \big\|_\infty \\ &= \big\| (M_m(g) - M_n(g))\chi_E \big\|_\infty + \big\| (M_m(h) - M_n(h))\chi_E \big\|_\infty \\ &\le \frac{\delta}{3} + \|M_m(h)\|_\infty + \|M_n(h)\|_\infty \le \frac{\delta}{3} + 2\|h\|_\infty \le \delta, \end{aligned}$$

implying, by Proposition 3.1, that the sequence $\{M_n(f)\}$ converges a.u. to some $\widehat{f} \in L^0_\mu$.

Since $L^1 + L^\infty$ possesses the Fatou property and

$$M_n(f) \in L^1 + L^\infty, \ \sup_n \|M_n(f)\|_{L^1+L^\infty} \le \|f\|_{L^1+L^\infty}, \ M_n(f) \to \widehat{f} \text{ in } t_\mu,$$

it follows that $\widehat{f} \in L^1 + L^\infty$ [9, Ch. IV, §3, Lemma 5]. In addition,

$$M_n(f)^*(t) \to \widehat{f}^*(t) \quad \text{a.e. on } (0, \infty)$$

(see, for example, [11, Ch. II, §2, Property 11°]). Since $T \in DS$, it follows that $M_n(f)^*(t) \prec\prec f^*(t)$ for all n (see, for example, [11, Ch. II, §3, Section 4]). Consequently, by the Fatou Theorem,

$$\int_0^s \widehat{f}^*(t)\,dt \le \sup_n \int_0^s M_n(f)^*(t)\,dt \le \int_0^s f^*(t)\,dt \quad \text{for all } s > 0,$$

that is, $\widehat{f}^*(t) \prec\prec f^*(t)$. Since X is a fully symmetric space and $f \in X$, it follows that $\widehat{f} \in X$. □

Now we shall show that, for a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, the space $E = \mathcal{R}_\mu$ is the largest fully symmetric subspace of $L^1 + L^\infty$ for which Theorem 3.1 is valid. We will utilize the following result obtained recently by Kunszenti-Kovács [12] (cf. [3, Theorem 4.1]).

Theorem 3.4. *Let $(\Omega, \mathcal{A}, \mu)$ be σ-finite infinite measure space. If $f \in (L^1 + L^\infty) \setminus \mathcal{R}_\mu$, then there exists $T \in DS$ such that the sequence $M_n(T)(f)(\omega)$ fails to converge for almost every $\omega \in \Omega$.*

Theorem 3.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite infinite measure space. Given a fully symmetric space $X \subset L^1 + L^\infty$, the following conditions are equivalent:*

(i) $X \subseteq \mathcal{R}_\mu$.

(ii) $\mathbf{1} \notin X$.

(iii) *For every* $f \in X$ *and* $T \in DS$ *the averages* (1) *converge a.u. to some* $\widehat{f} \in X$.

PROOF. Implications (i) $\Leftrightarrow$ (ii) is Proposition 2.2, while (ii) $\Rightarrow$ (iii) is Theorem 3.1. By Theorem 3.4, given $f \in (L^1 + L^\infty) \setminus \mathcal{R}_\mu$, there exists $T \in DS$ such that the averages (1) do not converge a.e., hence a.u., and implication (iii) $\Rightarrow$ (i) follows. $\square$

Now we shall present some examples of fully symmetric spaces X such that $\mathbf{1} \notin X$ or $\mathbf{1} \in X$. Recall that it is assumed that $\mu(\Omega) = \infty$.

1. Let Φ be an *Orlicz function*, that is, $\Phi\colon [0,\infty) \to [0,\infty)$ is a left-continuous, convex, increasing function such that $\Phi(0) = 0$ and $\Phi(u) > 0$ for some $u \neq 0$ (see, for example [4, Ch. 2, §2.1]). Let

$$L^\Phi = \left\{ f \in L^0_\mu : \int_\Omega \Phi\Big(\frac{|f|}{a}\Big) d\mu < \infty \text{ for some } a > 0 \right\}$$

be the corresponding *Orlicz space*, and let

$$\|f\|_\Phi = \inf \left\{ a > 0 : \int_\Omega \Phi\Big(\frac{|f|}{a}\Big) d\mu \leq 1 \right\}$$

be the *Luxemburg norm* in L^Φ. It is well-known that $(L^\Phi, \|\cdot\|_\Phi)$ is a fully symmetric space.

Since $\mu(\Omega) = \infty$, if $\Phi(u) > 0$ for all $u \neq 0$, then $\int_\Omega \Phi(\frac{1}{a} \cdot \mathbf{1})\, d\mu = \infty$ for each $a > 0$, hence $\mathbf{1} \notin L^\Phi$. If $\Phi(u) = 0$ for all $0 \leq u < u_0$, then $\mathbf{1} \in L^\Phi$.

2. If X is a symmetric space with order continuous norm, then $\mu\{|f| > \lambda\} < \infty$ for all $f \in X$ and $\lambda > 0$, so $X \subset \mathcal{R}_\mu$; in particular, $\mathbf{1} \notin X$.

3. Let φ be a concave function on $[0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, and let

$$\Lambda_\varphi = \left\{ f \in L^0_\mu : \ \|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(t)\, d\varphi(t) < \infty \right\}$$

be the corresponding *Lorentz space*.

It is well-known that $(\Lambda_\varphi, \|\cdot\|_{\Lambda_\varphi})$ is a fully symmetric space; in addition, if $\varphi(\infty) = \infty$, then $\mathbf{1} \notin \Lambda_\varphi$ and if $\varphi(\infty) < \infty$, then $\mathbf{1} \in \Lambda_\varphi$.

Let φ be as above, and let

$$M_\varphi = \left\{ f \in L^0_\mu : \ \|f\|_{M_\varphi} = \sup_{0<s<\infty} \frac{1}{\varphi(s)} \int_0^s f^*(t)\, dt < \infty \right\}$$

be the corresponding *Marcinkiewicz space*. It is known that $(M_\varphi, \|\cdot\|_{M_\varphi})$ is a fully symmetric space such that $\mathbf{1} \notin M_\varphi$ if and only if $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$.

4. On strong convergence of Cesàro averages

In this section we give a characterization of fully symmetric spaces for which the mean ergodic theorem is valid.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. If we consider the complete Boolean algebra $\nabla_\mu = \{e = [E] : E \in \mathcal{A}\}$ of equivalence classes of μ-a.e. equal sets in $\mathcal{A}$ (that is, when $E, G \in \mathcal{A}$ and $\mu(E \Delta G) = 0$), then $\mu(e) := \mu(E)$ is a strictly positive measure on ∇_μ. Denote by $\nabla_\nu(0, a) = \{[E] : E \in \mathcal{A}_\nu\}$ the complete Boolean algebra of equivalence classes of ν-a.e. equal sets in $((0, a), \nu)$, $0 < a \leq \infty$.

A Boolean subalgebra ∇_0 in ∇_μ is called *regular* if $\sup D \in \nabla_0$ for every subset $D \subseteq \nabla_0$. If ∇_0 is a regular subalgebra in ∇_μ, then clearly $\mathcal{A}_0 = \{E \in \mathcal{A} : [E] \in \nabla_0\}$ is a σ-subalgebra in $\mathcal{A}$ and $\nabla_0 = \{[E] : E \in \mathcal{A}_0\}$.

It is known that there exists $e \in \nabla_\mu$ such that $e \cdot \nabla_\mu$ is *non-atomic*, that is, the Boolean algebra $e \cdot \nabla_\mu$ has no atoms, and $(\mathbf{1} - e) \cdot \nabla_\mu$ is a *totally atomic* Boolean algebra, that is, $\mathbf{1} - e = \sup_n q_n$, where $\{q_n\}$ is the set of atoms in the Boolean algebra ∇_μ (see, for example, [24, I, Ch. 2, §2]).

Let ∇_μ be a non-atomic Boolean algebra. In view of [1, Ch. 2, Corollary 7.6], we have the following.

Proposition 4.1. *There exist a regular subalgebra ∇_0 in ∇_μ and a Boolean isomorphism $\varphi \colon \nabla_\nu(0, \mu(\Omega)) \to \nabla_0$ onto such that $\mu(\varphi(e)) = \nu(e)$ for all $e \in \nabla_\nu(0, \mu(\Omega))$.*

Utilizing Proposition 4.1 and [3, Theorem 2.4], we obtain the following.

Corollary 4.1. *Let $0 \neq e_0 \in \nabla_\mu$ be such that $e_0 \cdot \nabla_\mu$ is non-atomic, and let ∇_0 and φ be as in Proposition 4.1 (with respect to the Boolean algebra $e_0 \cdot \nabla_\mu$). Then there exists a unique algebraic isomorphism $\Phi \colon L^0((0, \mu(\Omega)), \nu) \to L^0(\Omega, \mathcal{A}_0, \mu)$ such that*

(i) $\Phi(e) = \varphi(e)$ for all $e \in \nabla_\nu(0, \mu(\Omega))$;

(ii) $\Phi \colon L^1((0, \mu(\Omega)), \nu) \to L^1(\Omega, \mathcal{A}_0, \mu)$ and $\Phi \colon L^\infty((0, \mu(\Omega)), \nu) \to L^\infty(\Omega, \mathcal{A}_0, \mu)$ are bijective linear isometries.

In what follows, $T \in DS(\Omega, \mathcal{A}, \mu)$ will mean that T is a Dunford–Schwartz operator in $L^1(\Omega, \mathcal{A}, \mu) + L^\infty(\Omega, \mathcal{A}, \mu)$.

If $E \in \mathcal{A}$ and $\mathcal{A}_E = \{A \cap E : A \in \mathcal{A}\}$, then it is clear that $(E, \mathcal{A}_E, \mu)$ is a σ-finite measure space. The next property of Dunford–Schwartz operators can be found in [3, Corollary 2.1].

Theorem 4.1. *Let $0 \neq e = [E] \in \nabla_\mu$, and let ∇_0 be a regular subalgebra in $e \cdot \nabla_\mu$ such that $(\Omega, \mathcal{A}_0, \mu)$ is a σ-finite measure space. If $T \in DS(E, \mathcal{A}_0, \mu)$, then there exists $\widehat{T} \in DS(\Omega, \mathcal{A}, \mu)$ such that*

$$\widehat{T}(g) = T(g) \quad \text{and} \quad M_n(\widehat{T})(g) = M_n(T)(g)$$

for all $g \in L^1(E, \mathcal{A}_0, \mu) + L^\infty(E, \mathcal{A}_0, \mu)$ and $n \in \mathbb{N}$.

We say that a fully symmetric space X possesses the *mean ergodic theorem property* (writing $X \in$ (MET)) if the averages $M_n(T)$ converge strongly in X for any $T \in DS$. A measure space will be called *quasi-non-atomic* if it has finitely many atoms or its atoms have equal measures.

THEOREM 4.2. *If $(\Omega, \mathcal{A}, \mu)$ is quasi-non-atomic, $\mu(\Omega) = \infty$, and X is a fully symmetric space such that $X \subset L^1$, then $X \notin$ (MET).*

PROOF. Assume first $(\Omega, \mathcal{A}, \mu) = ((0,\infty), \nu)$ and consider the operator $T \in DS$ defined by

$$T(f)(t) = \begin{cases} f(t-1) & \text{if } t > 1, \\ 0 & \text{if } t \in (0,1]. \end{cases}$$

Since

$$\begin{aligned} &\big\| M_{2n}(T)(\chi_{(0,1]}) - M_n(T)(\chi_{(0,1]}) \big\|_1 \\ &= \Big\| \frac{1}{2n}\chi_{(0,2n]} - \frac{1}{n}\chi_{(0,n]} \Big\|_1 = n\Big(\frac{1}{n} - \frac{1}{2n}\Big) + n\frac{1}{2n} = 1, \end{aligned}$$

it follows that the averages $M_n(T)(\chi_{(0,1]})$ do not converge in the norm $\|\cdot\|_1$. Therefore $L^1 \notin$ (MET).

Using the inclusion $X \subset L^1$ and Proposition 2.1, we conclude that there is a constant $c > 0$ such that $\|f\|_1 \le c\|f\|_X$ for all $f \in X$. Consequently, the sequence $\{M_n(T)(\chi_{(0,1]})\}$ cannot converge strongly in X, hence $X \notin$ (MET).

Assume now that $(\Omega, \mathcal{A}, \mu)$ is non-atomic. By Corollary 4.1, there exist a regular subalgebra ∇_0 in ∇_μ and an algebraic isomorphism $\Phi\colon L^0((0,\infty),\nu) \to L^0(\Omega, \mathcal{A}_0, \mu)$ such that $\mu(\Phi(e)) = \nu(e)$ for all $e \in \nabla_\nu(0, \mu(\Omega))$ and $\Phi: L^1((0,\infty),\nu) \to L^1(\Omega, \mathcal{A}_0, \mu)$ and $\Phi\colon L^\infty((0,\infty),\nu) \to L^\infty(\Omega, \mathcal{A}_0, \mu)$ are bijective linear isometries. Therefore $\widetilde{T} = \Phi \circ T \circ \Phi^{-1} \in DS(\Omega, \mathcal{A}_0, \mu)$, and the sequence $\{M_n(\widetilde{T})(\Phi(\chi_{(0,1]}))\}$ does not converge in the space $\big(X(\Omega, \mathcal{A}_0, \mu), \|\cdot\|_{X(\Omega,\mathcal{A}_0,\mu)}\big)$. Note that $(\Omega, \mathcal{A}_0, \mu)$ is a σ-finite measure space.

By Theorem 4.1, there exists $\widehat{T} \in DS(\Omega, \mathcal{A}, \mu)$ such that $\widehat{T}(g) = \widetilde{T}(g)$ and $M_n(\widehat{T})(g) = M_n(\widetilde{T})(g)$ for all $g \in L^1(\Omega, \mathcal{A}_0, \mu) + L^\infty(\Omega, \mathcal{A}_0, \mu)$ and $n \in \mathbb{N}$. Thus, the sequence $\{M_n(\widehat{T})(\Phi(\chi_{(0,1]}))\}$ does not converge in the space $X = X(\Omega, \mathcal{A}, \mu)$, hence $X \notin$ (MET).

Next, let $(\Omega, \mathcal{A}, \mu)$ be a totally atomic infinite measure space with the atoms of equal measures. In this case $L^1(\Omega) = l^1$, $L^\infty(\Omega) = l^\infty$ and $l^1 \subset X \subset l^\infty$, which, by the assumption, implies that $X = l^1$. Consequently, by Proposition 2.1, the norms $\|\cdot\|_X$ and $\|\cdot\|_1$ are equivalent.

Define $T \in DS$ by $T(\{\xi_n\}_{n=1}^\infty) = \{0, \xi_1, \xi_2, \dots\}$ if $\{\xi_n\}_{n=1}^\infty \in l^\infty$. If $e_1 = \{1, 0, 0, \dots\}$, then we have

$$\big\| M_{2n}(T)(e_1) - M_n(T)(e_1) \big\|_1$$

$$= \Big\| \frac{1}{2n} \{\underbrace{1,1,\dots,1}_{2n},0,0,\dots\} - \frac{1}{n} \{\underbrace{1,1,\dots,1}_{n},0,0,\dots\} \Big\|_1 = 1,$$

implying that the sequence $\{M_n(T)(e_1)\}$ does not converge in the norm $\|\cdot\|_1$. Since the norms $\|\cdot\|_X$ and $\|\cdot\|_1$ are equivalent, it follows that the sequence $\{M_n(T)(e_1)\}$ is not convergent in $\|\cdot\|_X$ as well, hence $X \notin$ (MET).

Assume now that $(\Omega, \mathcal{A}, \mu)$ is an arbitrary quasi-non-atomic infinite measure space. As noted above, there exists $e = [E] \in \nabla_\mu$ such that $e \cdot \nabla_\mu$ is a non-atomic and $(\mathbf{1} - e) \cdot \nabla_\mu$ is a totally atomic Boolean algebra.

Let $\mu(E) < \infty$. Since $\mu(\Omega) = \infty$, it follows that $(\Omega \setminus E, \mathcal{A}_{\Omega\setminus E}, \mu)$ is a totally atomic σ-finite infinite measure space with the atoms of the same measure. According to what has been proved above, we have

$$\big((\Omega \setminus E, \mathcal{A}_{\Omega\setminus E}, \mu), \|\cdot\|_X\big) \notin (\mathrm{MET}).$$

Further, by Theorem 4.1, there exists $\widehat{T} \in DS(\Omega, \mathcal{A}, \mu)$ such that $\widehat{T}(g) = T(g)$ and $M_n(\widehat{T})(g) = M_n(T)(g)$ for all $g \in L^1(\Omega \setminus E, \mathcal{A}_{\Omega\setminus E}, \mu) + L^\infty(\Omega \setminus E, \mathcal{A}_{\Omega\setminus E}, \mu)$ and $n \in \mathbb{N}$. Therefore $X(\Omega, \mathcal{A}, \mu) \notin$ (MET).

If $\mu(E) = \infty$, then, as we have shown, $(X(E, \mathcal{A}_E, \mu), \|\cdot\|_X) \notin$ (MET). In particular, there exist $T \in DS(E, \mathcal{A}_E, \mu)$ and $f \in X(E, \mathcal{A}_E, \mu)$ such that the sequence $\{M_n(T)(f)\}$ is not convergent in the norm $\|\cdot\|_X$. By Theorem 4.1, there exists $\widehat{T} \in DS(\Omega, \mathcal{A}, \mu)$ such that $\widehat{T}(g) = T(g)$ and $M_n(\widehat{T})(g) = M_n(T)(g)$ for all $g \in L^1(E, \mathcal{A}_E, \mu) + L^\infty(E, \mathcal{A}_E, \mu)$ and $n \in \mathbb{N}$. Therefore $X(\Omega, \mathcal{A}, \mu) \notin$ (MET). $\square$

The next theorem gives another condition under which a fully symmetric space $X(\Omega, \mathcal{A}, \mu)$ does not belong to (MET).

THEOREM 4.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a quasi-non-atomic σ-finite infinite measure space. If X is a fully symmetric space generated by a non-separable fully symmetric space $X(0,\infty)$, then $X \notin$ (MET).*

PROOF. Assume first that $(\Omega, \mathcal{A}, \mu) = ((0,\infty), \nu)$. Since $(X(0,\infty), \|\cdot\|_{X(0,\infty)})$ is not separable, it follows that there exists $a > 0$ such that the symmetric space $(X(0,a), \|\cdot\|_{X(0,a)})$ also is not separable [11, Ch. II, §4, Theorem 4.8]. Therefore, by [23, Theorem 2.5.1], there exist a function $f_0 \in X(0,a) \setminus \overline{L^\infty(0,a)}^{\|\cdot\|_{X(0,a)}}$ and a Dunford–Schwartz operator $T_0 \in DS((0,a), \nu)$ such that the averages $M_n(T_0)(f_0)$ do not converge in the norm $\|\cdot\|_{X(0,a)}$.

Define a Dunford–Schwartz operator $T \in DS((0,\infty), \nu)$ by

$$T(g) = T_0(g \cdot \chi_{(0,a)}), \quad g \in L^1(0,\infty) + L^\infty(0,\infty).$$

If we set $f = f_0 \cdot \chi_{(0,a)} + 0 \cdot \chi_{[a,\infty)}$, then $f \in X(0,\infty)$ and $M_n(T)(f) = M_n(T_0)(f_0)$ for every n. Consequently, the sequence $\{M_n(T)(f)\}$ does not converge strongly in $X(0,\infty)$.

Next, let $(\Omega, \mathcal{A}, \mu)$ be non-atomic. By Corollary 4.1, there exist a regular subalgebra ∇_0 in ∇_μ and an algebraic isomorphism $\Phi\colon L^0((0,\infty),\nu) \to L^0(\Omega, \mathcal{A}_0, \mu)$ such that

$$\mu(\Phi(e)) = \nu(e) \quad \text{for all } e \in \nabla_\nu(0, \mu(\Omega))$$

and $\Phi\colon L^1((0,\infty),\nu) \to L^1(\Omega,\mathcal{A}_0,\mu)$ and $\Phi\colon L^\infty((0,\infty),\nu) \to L^\infty(\Omega,\mathcal{A}_0,\mu)$ are bijective linear isometries. In particular, $(\Omega, \mathcal{A}_0, \mu)$ is σ-finite. According to what has been proved above, there exists $T \in DS((0,\infty),\nu)$ such that the averages $M_n(T)$ do not converge strongly in $X(0,\infty)$. It is clear then that $\widetilde{T} = \Phi \circ T \circ \Phi^{-1} \in DS(\Omega, \mathcal{A}_0, \mu)$ and the averages $M_n(\widetilde{T})$ do not converge strongly in $X(\Omega, \mathcal{A}_0, \mu)$.

By Theorem 4.1, there exists $\widehat{T} \in DS(\Omega, \mathcal{A}, \mu)$ such that $\widehat{T}(g) = \widetilde{T}(g)$ and $M_n(\widehat{T})(g) = M_n(\widetilde{T})(g)$ for all $g \in L^1(\Omega, \mathcal{A}_0, \mu) + L^\infty(\Omega, \mathcal{A}_0, \mu)$ and $n \in \mathbb{N}$. It follows then that the averages $M_n(\widehat{T})$ do not converge strongly in $X = X(\Omega, \mathcal{A}, \mu)$, hence $X \notin$ (MET).

Now, let $(\Omega, \mathcal{A}, \mu)$ be a totally atomic infinite measure space with all atoms of equal measure. In this case $l^1 \subseteq X \subseteq l^\infty$, and $\mathcal{R}_\mu = c_0$, the fully symmetric space of sequences $f = \{\xi_n\}_{n=1}^\infty$ of real numbers converging to zero with respect to the norm $\|f\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|$.

If there exists an $f \in X \setminus c_0$, then $f^* \geq \alpha \mathbf{1}$ for some $\alpha > 0$, where $\mathbf{1} = \{1, 1, \ldots\}$, hence $\mathbf{1} \in X$ and $X = l^\infty$. Therefore, if X is a symmetric sequence space, then either $X \subset c_0$ or $X = l^\infty$. Since a.u. convergence in l^∞ and c_0 coincides with the convergence in the norm $\|\cdot\|_\infty$, Theorem 3.5 implies that $l^\infty \notin$ (MET).

Let now $X \subset c_0$. Recall that a fully symmetric space X on $(\Omega, \mathcal{A}, \mu)$ generated by a fully symmetric space $X(0,\infty)$ has order-continuous norm if and only if the space $(X(0,\infty), \|\cdot\|_{X(0,\infty)})$ is separable. Consequently, a symmetric sequence space X has no order-continuous norm. Thus, there exists $f = \{\xi_n\}_{n=1}^\infty = \{\xi_n^*\}_{n=1}^\infty \in X$ such that

$$\xi_n \downarrow 0 \quad \text{and} \quad \big\|\big\{\underbrace{0,0,\ldots,0}_{n}, \xi_{n+1}, \xi_{n+2}, \ldots\big\}\big\|_X \downarrow \alpha > 0. \tag{7}$$

Let $T \in DS$ be defined as $T(\{\eta_n\}_{n=1}^\infty) = \{0, \eta_1, \eta_2, \ldots\}$ whenever $\{\eta_n\} \in l^\infty$. Then $T^k(f) = \{\underbrace{0,0,\ldots,0}_{k}, \xi_1, \xi_2, \ldots\}$, so

$$\{\eta_m^{(n)}\}_{m=1}^\infty := \sum_{k=0}^{n-1} T^k(f) = \big\{\xi_1, \xi_1+\xi_2, \ldots, \xi_1+\xi_2+\ldots\xi_n, \xi_2+\xi_3+\ldots$$

$$+\xi_{n+1}, \xi_3+\xi_4+\cdots+\xi_{n+2}, \ldots, \xi_{m-n+1}+\xi_{m-n+2}+\cdots+\xi_m, \ldots\big\},$$

that is,

$$\eta_m^{(n)} = \xi_{m-n+1} + \xi_{m-n+2} + \dots + \xi_m \quad \text{if } m \geq n$$

and

$$\eta_m^{(n)} = \xi_1 + \xi_2 + \dots \xi_m \quad \text{if } 1 \leq m < n.$$

Since $\xi_n \downarrow 0$, it follows that $\frac{1}{n}\sum_{k=0}^{n-1}\xi_k \to 0$ as $n \to \infty$. Consequently,

$$0 \leq \frac{1}{n}\eta_m^{(n)} = \frac{1}{n}(\xi_{m-n+1} + \xi_{m-n+2} + \dots + \xi_m) \leq \frac{1}{n}\sum_{k=0}^{m}\xi_k \to 0$$

as $n \to \infty$ for any fixed $m \in \mathbb{N}$. Therefore, the sequence $\{M_n(T)(f)\}$ converges to zero coordinate-wise.

Suppose that there exists $\widehat{f} \in X$ such that $\|M_n(T)(f) - \widehat{f}\|_X \to 0$. Then the sequence $\{M_n(T)(f)\}$ converges to $\widehat{f}$ coordinate-wise, implying that $\widehat{f} = 0$. On the other hand, as $\xi_n \downarrow 0$, we have

$$M_n(T)(f) = \Big\{\frac{1}{n}\xi_1, \frac{1}{n}(\xi_1 + \xi_2), \dots, \frac{1}{n}(\xi_1 + \xi_2 + \dots + \xi_n),$$
$$\frac{1}{n}(\xi_2 + \xi_3 + \dots + \xi_{n+1}), \frac{1}{n}(\xi_3 + \xi_4 + \dots + \xi_{n+2}), \dots,$$
$$\frac{1}{n}(\xi_{m-n+1} + \xi_{m-n+2} + \dots + \xi_m), \dots\Big\} \geq \Big\{\underbrace{0, 0, \dots, 0}_{n}, \xi_{n+1}, \xi_{n+2}, \dots\Big\} \geq 0.$$

Thus, by (7),

$$\big\| M_n(T)(f)\big\|_X \geq \big\|\{\underbrace{0, 0, \dots, 0}_{n}, \xi_{n+1}, \xi_{n+2}, \dots\}\big\|_X \geq \alpha > 0,$$

implying that the sequence $\{M_n(T)(f)\}$ is not convergent in the norm $\|\cdot\|_X$, that is, $X \notin$ (MET).

Repeating the ending of the proof of Theorem 4.2, we conclude that $X \notin$ (MET) for any quasi-non-atomic σ-finite infinite measure space $(\Omega, \mathcal{A}, \mu)$. □

Let X be a symmetric space on $((0,\infty), \nu)$. The *fundamental function* of X is defined by $\varphi_X(t) = \|\chi_{(0,t]}\|_X$. It is known that $\varphi_X(t)$ is a *quasi-concave* function (see [11, Ch. II, §4, Theorem 4.7]); in particular, $\varphi_X(t)$ increases, while the function $\frac{\varphi_X(t)}{t}$ decreases [11, Ch. II, §1, Definition 1.1]. Consequently, the limits

$$\alpha(X) = \lim_{t\to\infty}\frac{\varphi_X(t)}{t} \quad \text{and} \quad \beta(X) = \lim_{t\to 0^+}\varphi_X(t) = \varphi_X(+0)$$

exist. Note that

$$\alpha(L^1) = 1, \ \ \beta(L^\infty) = 1, \ \ \alpha(L^p) = 0, \ 1 < p \le \infty, \ \ \beta(L^p) = 0, \ 1 \le p < \infty.$$

We need the following necessary and sufficient conditions for an embedding of a symmetric space X into L^∞ or L^1.

PROPOSITION 4.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite infinite measure space. If X is the symmetric space generated by a symmetric space $X(0,\infty)$ on $((0,\infty),\nu)$, then*

(i) *$X \subseteq L^\infty$ if and only if $\beta(X) > 0$;*
(ii) *$X \subseteq L^1$ if and only if $\alpha(X) > 0$;*
(iii) *$X \subseteq L^1$ if and only if $L^\infty \subseteq X^\times$, where $X^\times$ is the Köthe dual of X.*

PROOF. It is clear that $X(\Omega) \subseteq Y(\Omega)$ if and only if $X(0,\infty) \subseteq Y(0,\infty)$, where $Y(0,\infty)$ is a symmetric space on $((0,\infty),\nu)$ that generated $Y(\Omega)$. Consequently, it is sufficient to prove the proposition in the case $(\Omega, \mathcal{A}, \mu) = ((0,\infty),\nu)$.

(i) If $X \subseteq L^\infty(0,\infty)$, then there exists a $c_0 > 0$ such that $\|f\|_\infty \le c_0 \|f\|_X$ for all $f \in X$ (see Proposition 2.1). Therefore

$$\varphi_X(t) = \|\chi_{(0,t]}\|_X \ge \frac{\|\chi_{(0,t]}\|_\infty}{c_0} = \frac{1}{c_0} \ \text{ and } \ \beta(X) = \lim_{t\to 0^+} \varphi_X(t) \ge \frac{1}{c_0} > 0.$$

If $X \nsubseteq L^\infty(0,\infty)$, then there exists a positive unbounded function f in $X(0,\infty) \setminus L^\infty(0,\infty)$; in particular, $\nu(A_n) > 0$, where $A_n = \{f \ge n\}$, $n \in \mathbb{N}$. Choose a sequence $B_n \subseteq A_n$ such that $B_n \supseteq B_{n+1}$, $0 < \nu(B_n) < \infty$, and $\lim_{n\to\infty} \nu(B_n) = 0$. Then we have

$$n \cdot \beta(X) \le n\|\chi_{B_n}\|_X = \|n \cdot \chi_{B_n}\|_X \le \|f\|_X < \infty \quad \forall n \in \mathbb{N},$$

hence $\beta(X) = 0$.

(ii) If $X \subseteq L^1(0,\infty)$, then there exists a $c_1 > 0$ such that $\|f\|_1 \le c_1 \|f\|_X$ for all $f \in X$ (see Proposition 2.1). Consequently,

$$\frac{\varphi_X(t)}{t} = \frac{\|\chi_{(0,t]}\|_X}{t} \ge \frac{\|\chi_{(0,t]}\|_1}{c_1 \cdot t} = \frac{1}{c_1}$$

and

$$\alpha(X) = \lim_{t\to\infty} \frac{\varphi_X(t)}{t} \ge \frac{1}{c_1} > 0.$$

Assume now that $\alpha(X) > 0$. By [11, Ch. II, §4, inequality (4.6)], we have

$$\|f^* \cdot \chi_{(0,t]}\|_1 \le \frac{t}{\varphi_X(t)} \cdot \|f^* \cdot \chi_{(0,t]}\|_X$$

for all $f \in X$, $t > 0$. Since $\frac{\varphi_X(t)}{t} \geq \alpha(X) > 0$, $t > 0$, it follows that $\frac{t}{\varphi_X(t)} \leq \frac{1}{\alpha(X)}$, implying that

$$\|f^* \cdot \chi_{(0,t]}\|_1 \leq \frac{\|f^* \cdot \chi_{(0,t]}\|_X}{\alpha(X)} \leqslant \frac{\|f^*\|_X}{\alpha(X)},$$

so $\|f\|_1 \leq \frac{\|f\|_X}{\alpha(X)} < \infty$ for all $f \in X$, that is, $X \subseteq L^1(0,\infty)$.

(iii) If $X \subseteq L^1(0,\infty)$, then $\int_0^\infty |\mathbf{1} \cdot f| d\nu = \|f\|_1 < \infty$ for all $f \in X$. Thus $\mathbf{1} \in X^\times$ and $L^\infty(0,\infty) \subseteq X^\times$.

Conversely, if $L^\infty(0,\infty) \subseteq X^\times$, then $\mathbf{1} \in X^\times$, that is, the linear functional

$$\varphi(f) = \int_0^\infty \mathbf{1} \cdot f \; d\nu, \quad f \in X,$$

is bounded on X. Then it follows that

$$\|f\|_1 = \int_0^\infty \mathbf{1} \cdot |f| d\mu = \varphi(|f|) \leq \|\varphi\| \cdot \|f\|_X < \infty$$

for every function $f \in X$, hence $X \subseteq L^1(0,\infty)$. □

Corollary 4.2. *Let* $(\Omega, \mathcal{A}, \mu)$ *and* X *be as in Proposition* 4.2. *Then the following are equivalent*:

(i) $X \not\subseteq L^1$;
(ii) $L^\infty \not\subseteq X^\times$;
(iii) $\mathbf{1} \notin X^\times$.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite infinite measure space. Let $f \in \mathcal{R}_\mu$ and $T \in DS$. By Theorem 3.1, there exists $\widehat{f} \in \mathcal{R}_\mu$ such that the sequence $\{M_n(T)(f)\}$ converges a.u. to $\widehat{f}$. Define the mapping $P\colon \mathcal{R}_\mu \to \mathcal{R}_\mu$ by setting

$$P(f) = \widehat{f} = (\text{a.u.})\text{-}\lim_{n\to\infty} M_n(T)(f), \quad f \in \mathcal{R}_\mu.$$

It is clear that P is linear. Since $(L^1, \|\cdot\|_1)$ possesses the Fatou property and $\|M_n(T)(f)\|_1 \leq \|f\|_1$ for all $f \in L^1$, it follows that $\|P(f)\|_1 \leq \|f\|_1$ for every $f \in L^1$ [9, Ch. IV, §3, Lemma 5], that is, $\|P\|_{L^1 \to L^1} \leq 1$.

Similarly, if $f \in L^1 \cap L^\infty$, then $\|M_n(T)(f)\|_\infty \leqslant \|f\|_\infty$. Therefore, a.u. convergence $M_n(T)(f) \to P(f)$ implies that $\|P(f)\|_\infty \leq |f\|_\infty$.

According to [2, Theorem 3.1], there exists a unique operator $\widehat{P} \in DS$ such that $\widehat{P}(f) = P(f)$ for all $f \in \mathcal{R}_\mu$; in particular, $\|P\|_{\mathcal{R}_\mu \to \mathcal{R}_\mu} \leq 1$.

Additionally, by the classical mean ergodic theorem in the space L^2, we have $\|M_n(T)(f) - P(f)\|_2 \to 0$ as $n \to \infty$ for any $f \in L^2$.

The next theorem is a version of the mean ergodic theorem for the fully symmetric space $(\mathcal{R}_\mu, \|\cdot\|_{L^1+L^\infty})$.

THEOREM 4.4. *If* $T \in DS$, *then*

$$\big\| M_n(T)(f) - P(f)\big\|_{L^1+L^\infty} \to 0 \quad \textit{for all } f \in \mathcal{R}_\mu. \tag{8}$$

PROOF. We have

$$\sup_n \|M_n(T)\|_{L^1+L^\infty \to L^1+L^\infty} \le 1 \quad \text{and} \quad \|P\|_{\mathcal{R}_\mu \to \mathcal{R}_\mu} \le 1.$$

Since $\|M_n(T)(f) - P(f)\|_2 \to 0$, it follows that $\|M_n(T)(f) - P(f)\|_{L^1+L^\infty} \to 0$ for any $f \in L^1 \cap L^\infty \subset L^2$. Using the density of $L^1 \cap L^\infty$ in the Banach space $(\mathcal{R}_\mu, \|\cdot\|_{L^1+L^\infty})$ and the principle of uniform boundedness, we arrive at (8). □

Now we can establish the following important property of the operator P.

PROPOSITION 4.3. $P^2 = P$ *and*

$$TP(f) = P(f) = PT(f) \quad \textit{for all } f \in \mathcal{R}_\mu.$$

PROOF. Since

$$(I-T)M_n(T) = \frac{I-T^n}{n} = \frac{I}{n} + M_n(T) - \frac{n+1}{n} M_{n+1}(T),$$

it follows that

$$P(f) - PT(f) = (\text{a.u.})\text{-}\lim_{n\to\infty} (I-T)M_n(T)(f) = 0,$$

hence $PT(f) = P(f)$, for all $f \in \mathcal{R}_\mu$.

Denote $\|\cdot\|_{L^1+L^\infty}$ by $\|\cdot\|$. Then, By Theorem 4.4, $\big\| M_n(T)(f) - P(f)\big\| \to 0$ for each $f \in \mathcal{R}_\mu$. Consequently,

$$TP(f) = T\big(\|\cdot\| - \lim_{n\to\infty} M_n(T)(f)\big)$$

$$= \|\cdot\| - \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k(f) = \|\cdot\| - \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=0}^{n} T^k(f) - \frac{f}{n}\right)$$

$$= \|\cdot\| - \lim_{n\to\infty} \left(\frac{n+1}{n} M_{n+1}(T)(f) - \frac{f}{n}\right) = P(f).$$

Therefore $TP(f) = P(f) = PT(f)$, $f \in \mathcal{R}_\mu$, hence $M_n(T)P = P$, $n = 1, 2, \dots$, implying that $P^2 = P$. □

We will also need the following property of symmetric spaces [5, Proposition 2.2].

Proposition 4.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite infinite measure space. Let $(X, \|\cdot\|_X)$ be a separable symmetric space on $((0,\infty), \nu)$ such that $X^\times \subseteq \mathcal{R}_\nu$. If $\{f_n\} \subset X(\Omega)$ and $g \in X(\Omega)$ are such that $f_n \prec\prec g$ for all n, then $f_n \to 0$ in measure implies that $\|f_n\|_{X(\Omega)} \to 0$ as $n \to \infty$.*

Theorem 4.5. *Let X be a fully symmetric space on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. If the norm $\|\cdot\|_X$ is order continuous and $L^1 \nsubseteq X$, then the averages $M_n(T)$ converge strongly in X for each $T \in DS$.*

If $(\Omega, \mathcal{A}, \mu)$ is quasi-non-atomic, then strong convergence of the averages $M_n(T)$ for every $T \in DS$ implies that the norm $\|\cdot\|_X$ is order continuous and $L^1 \nsubseteq X$.

Proof. Since the symmetric space $(X(0,\infty), \|\cdot\|_{X(0,\infty)})$ is separable, it follows that $X(0,\infty) \subseteq \mathcal{R}_\nu$, hence $X = X(\Omega) \subseteq \mathcal{R}_\mu$. As $L^1 \nsubseteq X$, we have $L^1(0,\infty) \nsubseteq X(0,\infty)$. Therefore, by Corollary 4.2, $\mathbf{1} \notin X^\times(0,\infty)$, so $X^\times(0,\infty) \subseteq \mathcal{R}_\nu$ by Proposition 2.2.

Since X is a fully symmetric space and $\widehat{P} \in DS$, it follows that $g = f - P(f) \in X \subset \mathcal{R}_\mu$ for any $f \in X$. By Proposition 4.3, $P(g) = 0$. Therefore, in view of Theorem 3.5, $M_n(T)(g) \to P(g) = 0$ in measure. Since $M_n(T)(g) \prec\prec g \in X$ for every n, Proposition 4.4 entails that $\|M_n(T)(g)\|_X \to 0$. Next, by Proposition 4.3,

$$M_n(T)(g) = M_n(T)(f) - M_n(T)(P(f)) = M_n(T)(f) - P(f),$$

implying that $\|M_n(T)(f) - P(f)\|_X \to 0$.

If $(\Omega, \mathcal{A}, \mu)$ is quasi-non-atomic and the averages $M_n(T)$ converge strongly for every $T \in DS$, then Theorems 4.2 and 4.3 entail that the norm $\|\cdot\|_X$ is order continuous and $L^1 \nsubseteq X$. $\square$

Utilizing Theorem 4.5 and Proposition 4.2, we can now state the following.

Corollary 4.3. *Let $(X, \|\cdot\|_X)$ be a fully symmetric space on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. If the norm $\|\cdot\|_X$ is order continuous and $\alpha(X) = 0$, then the averages $M_n(T)$ converge strongly in X for each $T \in DS$.*

If $(\Omega, \mathcal{A}, \mu)$ is quasi-non-atomic, then strong convergence of the averages $M_n(T)$ for every $T \in DS$ implies that the norm $\|\cdot\|_X$ is order continuous and $\alpha(X) = 0$.

5. Ergodic theorems in Orlicz, Lorentz and Marcinkiewicz spaces

In this section we give applications of Theorems 3.1, 3.5, and 4.5, to Orlicz, Lorentz and Marcinkiewicz spaces.

1. Let Φ be an Orlicz function, and let $L^\Phi = (L^\Phi(\Omega), \|\cdot\|_\Phi)$ be the corresponding Orlicz space. As noted in Section 3, if $\Phi(u) > 0$ for all $u \neq 0$, then $\mathbf{1} \notin L^\Phi$; if $\Phi(u) = 0$ for all $0 \leq u < u_0$, then $\mathbf{1} \in L^\Phi$.

Therefore, Theorems 3.1 and 3.5 imply the following.

THEOREM 5.1. *Let* $(\Omega, \mathcal{A}, \mu)$ *be an arbitrary measure space, and let* Φ *be an Orlicz function. If* $\Phi(u) > 0$ *for all* $u > 0$, $T \in DS$, *and* $f \in L^{\Phi}$, *then there exists* $\widehat{f} \in L^{\Phi}$ *such that the averages* (1) *converge a.u. to* $\widehat{f}$.

If $\Phi(u) = 0$ *for all* $0 \le u < u_0$, *then there exist* $T \in DS$ *and* $f \in L^{\Phi}$ *such that the averages* (1) *do not converge a.e., hence a.u.*

It is said that an Orlicz function Φ satisfies the (Δ_2)-*condition at* 0 (at ∞) if there exist $u_0 \in (0,\infty)$ and $k > 0$ such that $\Phi(2u) < k \cdot \Phi(u)$ for all $0 < u < u_0$ (respectively, $u > u_0$). An Orlicz function Φ satisfies the (Δ_2)-condition at 0 and at ∞ if and only if $(L^{\Phi}(0,\infty), \|\cdot\|_{\Phi})$ has order continuous norm [4, Ch. 2, §2.1, Theorem 2.1.17].

By [4, Ch. 2, §2.2, Theorem 2.2.3], $L^{\Phi}(0,\infty) \subseteq L^1(0,\infty)$ if and only if

$$\limsup_{u\to 0} \frac{\Phi(u)}{u} > 0 \quad \text{and} \quad \limsup_{u\to\infty} \frac{\Phi(u)}{u} = 0.$$

Therefore, Theorem 4.5 yields the following.

THEOREM 5.2. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a* σ*-finite measure space, and let an Orlicz function* Φ *satisfy the* (Δ_2)*-condition at* 0 *and at* ∞. *If* $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ *or* $\limsup_{u\to\infty} \frac{\Phi(u)}{u} > 0$, *then the averages* $M_n(T)$ *converge strongly in* L^{Φ} *for all* $T \in DS$.

If $(\Omega, \mathcal{A}, \mu)$ *is a quasi-non-atomic measure space, then strong convergence of the averages* $M_n(T)$ *for every* $T \in DS$ *implies that the Orlicz function* Φ *satisfies the* (Δ_2)*-condition at* 0 *and at* ∞; *in addition,* $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ *or* $\limsup_{u\to\infty} \frac{\Phi(u)}{u} > 0$.

2. Let φ be a concave function on $[0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, and let $\Lambda_{\varphi} = (\Lambda_{\varphi}(\Omega), \|\cdot\|_{\Lambda_{\varphi}})$ be the corresponding Lorentz space. As noted in Section 3, $\varphi(\infty) = \infty$ if and only if $\mathbf{1} \notin \Lambda_{\varphi}$. Therefore, Theorems 3.1 and 3.5 imply the following.

THEOREM 5.3. *If* $(\Omega, \mathcal{A}, \mu)$ *is an arbitrary measure space and* $\varphi(\infty) = \infty$, *then for all* $T \in DS$ *and* $f \in \Lambda_{\varphi}$ *there exists* $\widehat{f} \in \Lambda_{\varphi}$ *such that the averages* (1) *converge a.u. to* $\widehat{f}$.

If $\varphi(\infty) < \infty$, *then there exist* $T \in DS$ *and* $f \in \Lambda_{\varphi}$ *such that the averages* (1) *do not converge a.e., hence a.u.*

It is well-known that the space $(\Lambda_{\varphi}(0,\infty), \|\cdot\|_{\Lambda_{\varphi}})$ is separable if and only if $\varphi(+0) = 0$ and $\varphi(\infty) = \infty$ (see, for example, [11, Ch. II, §5, Lemma 5.1], [20, Ch. 9, §9.3, Theorem 9.3.1]). In addition, the fundamental function satisfies $\varphi_{\Lambda_{\varphi}}(t) = \varphi(t)$. Therefore, Corollary 4.3 entails the following.

THEOREM 5.4. *Let* $(\Omega, \mathcal{A}, \mu)$ *be* σ*-finite, and let* φ *be a concave function on* $[0, \infty)$ *with* $\varphi(0) = 0$*, and* $\varphi(t) > 0$ *for all* $t > 0$*. If* $\varphi(+0) = 0$*,* $\varphi(\infty) = \infty$*, and* $\alpha(\Lambda_\varphi) = \lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$*, then the averages* $M_n(T)$ *converge strongly in* Λ_φ *for each* $T \in DS$*.*

If $(\Omega, \mathcal{A}, \mu)$ *is quasi-non-atomic, then strong convergence of averages* $M_n(T)$ *for every* $T \in DS$ *implies that* φ *satisfies conditions* $\varphi(+0) = 0$*,* $\varphi(\infty) = \infty$*, and* $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$*.*

3. Let φ be as above, and let $M_\varphi = (M_\varphi(\Omega), \|\cdot\|_{M_\varphi})$ be the corresponding Marcinkiewicz space. As noted in Section 3, $\mathbf{1} \notin M_\varphi$ if and only if $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$. Thus, the corresponding version of Theorem 5.3 holds for the Marcinkiewicz space M_φ if we replace condition $\varphi(\infty) = \infty$ by $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$.

If $\varphi(+0) > 0$ and $\varphi(\infty) < \infty$, then $M_\varphi = L^1$ as the sets. In this case, if $(\Omega, \mathcal{A}, \mu)$ is quasi-non-atomic, it follows from Theorem 4.2 that $M_\varphi \notin$ (MET).

Let $\varphi(+0) = 0$ and $\varphi(\infty) = \infty$. If $\lim_{t\to 0} \frac{\varphi(t)}{t} = \infty$, then M_φ is non-separable [11, Ch. II, §5, Lemma 5.4]. Consequently, if $(\Omega, \mathcal{A}, \mu)$ is quasi-non-atomic, then, by Theorem 4.3, $M_\varphi \notin$ (MET).

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