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ON PRODUCTS OF CONSECUTIVE ARITHMETIC PROGRESSIONS. II

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Abstract. Let $f(x,k,d) = x(x+d)\cdots(x+(k-1)d)$ be a polynomial with $k \geq 2$, $d \geq 1$. We consider the Diophantine equation $\prod_{i=1}^{r} f(x_i,k_i,d) = y^2$, which is inspired by a question of Erdős and Graham [4, p. 67]. Using the theory of Pellian equation, we give infinitely many (nontrivial) positive integer solutions of the above Diophantine equation for some cases.

1. Introduction

Let us define the polynomial

$$f(x,k,d) = x(x+d)\cdots(x+(k-1)d)$$

with $k \geq 2$, $d \geq 1$. Many authors have studied the Diophantine equation

$$\prod_{i=1}^{r} f(x_i,k_i,d) = y^2, \tag{1.1}$$

where $r \geq 1$, with $x_i + (k_i-1)d < x_{i+1}$ for $i = 1,\dots,r-1$, and $2 \leq k_1 \leq k_2 \leq \cdots \leq k_r$. When $r = 1$, there are many results about Eq. (1.1) and the more general Diophantine equation

$$f(x,k,d) = by^l,$$

where $b > 0$, $l \geq 3$ and the greatest prime factor of b does not exceed k, we can refer to [2,5–9,12,13].

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(1) *The case* $r \geq 2$, $d = 1$. When $r = 2$, $k_i = 3$, Sastry [6] showed that Eq. (1.1) has infinitely many positive integer solutions (x_1, x_2, y), where x_1, x_2 satisfying $x_2 = 2x_1 - 1$ and $(x_1 + 1)(2x_1 - 1)$ is a square.

Erdős and Graham [4, p. 67] asked if Eq. (1.1) has, for fixed $r \geq 1$ and $k_1, k_2, \ldots, k_r$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i - 1 < x_{i+1}$ for $1 \leq i \leq r - 1$. Skałba [14] obtained a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [17] answered the above question of Erdős and Graham in the negative when either $r = 4$, $(k_1, k_2, k_3, k_4) = (4, 4, 4, 4)$ or $r \geq 6$ and $k_i = 4$, $1 \leq i \leq r$. Bauer and Bennett [1] extended this result to the cases $r = 3$ and $r = 5$.

For the case $(r, k_1, k_2) = (2, 4, 4)$, Eq. (1.1) has an integer solution $(x_1, x_2, y) = (33, 1680, 3361826160)$. Luca and Walsh [11] studied this case by using the identity $(x-1)x(x+1)(x+2) = (x^2+x-1)^2 - 1$ to reduce the original problem to a Pellian equation $(x^2 + x - 1)^2 - dy^2 = 1$, where $d > 1$ is a squarefree integer. Tengely [15] provided an upper bound for the size of the solutions and determined all solutions up to some bounds for this case.

Bennett and Van Luijk [3] constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square. Tengely and Ulas [16] studied Eq. (1.1) in further cases, and gave a partial answer to Question 3.2 in [18].

At the end of [1], Bauer and Bennett stated that it is easy to show that Eq. (1.1) has infinitely many positive integer solutions with $r = 2$, $k_1 = 3$, $k_2 = 4$, and $d = 1$. Now we give a proof of this result and show the results for $d = 1$. Noting that the k_i are different, we replace the condition $x_i + (k_i - 1) < x_{i+1}$ for $i = 1, \ldots, r - 1$ with the blocks of consecutive integers are disjoint.

THEOREM 1.1. *For* $d = 1$, *if either* $r = 2$, $k_1 = 3$, $k_2 = 4$, *or* $r = 3$, $k_1 = 3$, $k_2 = 4$, $k_3 = 4$, *or* $r = 3$, $k_1 = 3$, $k_2 = 4$, $k_3 = 5$, *then Eq.* (1.1) *has infinitely many positive integer solutions.*

Combining Theorem 1.1 and the results of [1,17], we have

COROLLARY 1.2. *For* $d = 1$, *if* $r \geq 2$, $k_1 = 3$, $k_i = 4$, $i = 2, \ldots, r$, *then Eq.* (1.1) *has infinitely many positive integer solutions.*

Moreover, we have

THEOREM 1.3. *For* $d = 1$, *if* $r \geq 3$, $k_1 = 3$, $k_2 = 5$, $k_i \geq 5$, $i = 3, \ldots, r$, *then Eq.* (1.1) *has infinitely many positive integer solutions.*

(2) *The case* $r \geq 2$, $d \geq 2$. We are looking for the positive integer solutions of Eq. (1.1) which satisfy $d \nmid x_i$ for some i. If the solutions $(x_1, \ldots, x_r, y)$ satisfy $d \mid x_i$, $i = 1, \ldots, r$, we call them trivial. For $r = 2$, $k_i = 3$, Zhang and Cai [18] have proved that when $r = 2$, $k_i = 3$, for even number d, Eq. (1.1)

has infinitely many nontrivial positive integer solutions. For $r = 2$, $k_i = 3$, Katayama [10] showed that Eq. (1.1) also has infinitely many nontrivial positive integer solutions when the integer d is divisible by a prime p ($\equiv \pm 1 \pmod 8$).

In the following, we study Eq. (1.1) with $r \geq 2$, $k_i \geq 2$, $i = 2, \dots, r$, and $d \geq 2$ as [1]. Noting that the k_i are different, we replace the condition $x_i + (k_i - 1)d < x_{i+1}$ for $i = 1, \dots, r-1$ with the blocks of disjoint arithmetic progressions. Using the theory of Pellian equation, we prove

THEOREM 1.4. *For $r \geq 2$, $k_1 = 2$, $k_i \geq 2$, $i = 2, \dots, r$, and $d \geq 2$, Eq.* (1.1) *has infinitely many nontrivial positive integer solutions.*

THEOREM 1.5. *For $r \geq 2$, $k_1 = k_2 = 3$, $k_i \geq 3$, $i = 3, \dots, r$, and $d \geq 2$, Eq.* (1.1) *has infinitely many nontrivial positive integer solutions.*

THEOREM 1.6. *For $r = 3$, $k_i = 4$, $i = 1, 2, 3$, and even number $d \geq 2$, Eq.* (1.1) *has infinitely many nontrivial positive integer solutions, i.e., the Diophantine equation*

$$x(x+d)(x+2d)(x+3d)y(y+d)(y+2d)(y+3d)z(z+d)(z+2d)(z+3d) = w^2 \tag{1.2}$$

has infinitely many nontrivial positive integer solutions.

For an even number $d \geq 2$, the Diophantine equation

$$x(x+d)(x+2d)(x+3d)y(y+d)(y+2d)(y+3d) = z^2$$

has integer solutions

$$(x, y, z) = \left(\frac{d}{2}, 5d, 105d^4\right), \ \left(\frac{3d}{2}, 5d, 315d^4\right).$$

Since each number $r \geq 5$ is of the form $3s + 2$, $3s$, $3s + 4$, we have

COROLLARY 1.7. *For $r \geq 5$, $k_i = 4$, $i = 1, \dots, r$, and even number $d \geq 2$, Eq.* (1.1) *has infinitely many nontrivial positive integer solutions, i.e., the Diophantine equation*

$$\prod_{i=1}^{r} x_i(x_i + d)(x_i + 2d)(x_i + 3d) = y^2$$

has infinitely many nontrivial positive integer solutions for even number $d \geq 2$ and $r \geq 5$.

For $d = 3$, we have

THEOREM 1.8. *The Diophantine equation*

$$x(x+3)(x+6)(x+9)y(y+3)(y+6)(y+9)z(z+3)(z+6)(z+9)=w^2 \tag{1.3}$$

has infinitely many nontrivial positive integer solutions.

The Diophantine equation

$$x(x+3)(x+6)(x+9)y(y+3)(y+6)(y+9)=z^2$$

has integer solutions

$$(x,y,z)=(2,24,23760),\quad (4,36,98280),$$
$$(7,36,196560),\quad (99,5040,272307918960).$$

Since each number $r\geq 5$ is of the form $3s+2$, $3s$, $3s+4$, we have

COROLLARY 1.9. *For $r\geq 5$, $k_i=4$, $i=1,\dots,r$, and $d=3$, Eq.* (1.1) *has infinitely many nontrivial positive integer solutions*, *i.e.*, *the Diophantine equation*

$$\prod_{i=1}^{r} x_i(x_i+3)(x_i+6)(x_i+9)=y^2$$

has infinitely many nontrivial positive integer solutions for $r\geq 5$.

2. Proofs

PROOF OF THEOREM 1.1. 1) For $r=2$, $k_1=3$, $k_2=4$, and $d=1$, Eq. (1.1) equals

$$x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3)=y^2.$$

Let $x_2=u$ and

$$x_1=\frac{u(u+3)}{2},$$

then we have

$$\frac{u^2+3u+4}{2}\frac{[u(u+1)(u+2)(u+3)]^2}{4}=y^2.$$

Considering

$$\frac{u^2+3u+4}{2}=v^2,$$

it is equivalent to the Pellian equation $U^2 - 2V^2 = -7$, where $U = 2u + 3$, $V = 2v$.

All positive integer solutions of $U^2 - 2V^2 = -7$ are given by

$$U_n + V_n\sqrt{2} = (1 + 2\sqrt{2})(3 + 2\sqrt{2})^n, \quad n \geq 0.$$

Thus

$$\begin{cases} U_n = 6U_{n-1} - U_{n-2}, & U_0 = 1,\ U_1 = 11; \\ V_n = 6V_{n-1} - V_{n-2}, & V_0 = 2,\ V_1 = 8. \end{cases}$$

From

$$x = \frac{U-3}{2}, \quad v = \frac{V}{2},$$

we have

$$\begin{cases} u_n = 6u_{n-1} - u_{n-2} + 6, & u_0 = 0,\ u_1 = 4; \\ v_n = 6v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = 4. \end{cases}$$

Then

$$y_n = \frac{u_n(u_n+1)(u_n+2)(u_n+3)v_n}{2} \in \mathbb{Z}^+, \quad n \geq 1.$$

So the Diophantine equation

$$x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3) = y^2$$

has infinitely many positive integer solutions

$$\Big(\frac{u_n(u_n+3)}{2}, u_n, y_n\Big), \quad n \geq 1,$$

such that $x_1(x_1+1)(x_1+2)$ and $x_2(x_2+1)(x_2+2)(x_2+3)$ are disjoint.

2) For $r = 3$, $k_1 = 3$, $k_2 = k_3 = 4$, and $d = 1$, Eq. (1.1) reduces to

$$x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3)x_3(x_3+1)(x_3+2)(x_3+3) = y^2.$$

Let $x_3 = u$ and

$$x_2 = 2u + 3, \quad x_1 = \frac{u(2u+5)}{3},$$

then we have

$$\frac{2u^2 + 5u + 6}{3} \, \frac{[2u(u+1)(u+2)(u+3)(2u+3)(2u+5)]^2}{9} = y^2.$$

Considering

$$\frac{2u^2+5u+6}{3}=v^2,$$

it is equivalent to the Pellian equation $U^2-6V^2=-23$, where $U=4u+5$, $V=2v$.

All positive integer solutions of $U^2-6V^2=-23$ are given by

$$U_n+V_n\sqrt{6}=(1+2\sqrt{6})(5+2\sqrt{6})^n,\quad n\geq 0.$$

Thus

$$\begin{cases} U_n=10U_{n-1}-U_{n-2}, & U_0=1,\ U_1=29;\\ V_n=10V_{n-1}-V_{n-2}, & V_0=2,\ V_1=12.\end{cases}$$

From

$$u=\frac{U-5}{4},\quad v=\frac{V}{2},$$

we have

$$\begin{cases} u_n=10u_{n-1}-u_{n-2}+10, & u_0=0,\ u_1=6;\\ v_n=10v_{n-1}-v_{n-2}, & v_0=0,\ v_1=6.\end{cases}$$

Then

$$y_n=\frac{2u_n(u_n+1)(u_n+2)(u_n+3)(2u_n+3)(2u_n+5)v_n}{3}\in\mathbb{Z}^+,\quad n\geq 1.$$

Therefore, the Diophantine equation

$$x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3)x_3(x_3+1)(x_3+2)(x_3+3)=y^2$$

has infinitely many positive integer solutions

$$\Big(\frac{u_n(2u_n+5)}{3},2u_n+3,u_n,y_n\Big),\quad n\geq 1,$$

such that $x_1(x_1+1)(x_1+2)$, $x_2(x_2+1)(x_2+2)(x_2+3)$ and $x_3(x_3+1)(x_3+2)(x_3+3)$ are disjoint.

3) For $r=3$, $k_1=3$, $k_2=4$, $k_3=5$, and $d=1$, Eq. (1.1) leads to

$$x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3)$$
$$\times x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)=y^2.$$

Let $x_3=2$ and $x_2=u$, $x_1=u(u+3)$, then we have

$$5(u^2+3u+1)[12u(u+1)(u+2)(u+3)]^2=y^2.$$

Considering $u^2+3u+1=5v^2$, it is equivalent to the Pellian equation $U^2-5V^2=5$, where $U=2u+3$, $V=2v$.

All positive integer solutions of $U^2-5V^2=5$ are given by

$$U_n+V_n\sqrt{5}=(5+2\sqrt{5})(9+4\sqrt{5})^n, \quad n\geq 0.$$

Thus

$$\begin{cases} U_n=18U_{n-1}-U_{n-2}, & U_0=5,\ U_1=85; \\ V_n=18V_{n-1}-V_{n-2}, & V_0=2,\ V_1=38. \end{cases}$$

From

$$u=\frac{U-3}{2}, \quad v=\frac{V}{2},$$

we have

$$\begin{cases} u_n=18u_{n-1}-u_{n-2}+24, & u_0=1,\ u_1=41; \\ v_n=18v_{n-1}-v_{n-2}, & v_0=1,\ v_1=19. \end{cases}$$

Then

$$y_n=60u_n(u_n+1)(u_n+2)(u_n+3)v_n\in\mathbb{Z}^+, \quad n\geq 0.$$

So the Diophantine equation

$$x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3)$$
$$\times\, x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)=y^2$$

has infinitely many positive integer solutions $(u_n(u_n+3),u_n,2,y_n)$, $n\geq 1$, such that $x_1(x_1+1)(x_1+2)$, $x_2(x_2+1)(x_2+2)(x_2+3)$ and $x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4)$ are disjoint. $\square$

PROOF OF THEOREM 1.3. For $d=1$, $r\geq 3$, $k_1=3$, $k_2=5$, $k_i\geq 5$, $i=3,\dots,r$, let

$$x_1=u, \quad \prod_{i=3}^{r}x_i(x_i+1)\cdots(x_i+k_i-1)=A.$$

Choose $x_i\in\mathbb{Z}^+$, $k_i\geq 5$, $i=3,\dots,r$ such that $x_i(x_i+1)\cdots(x_i+k_i-1)$ are disjoint, $2A$ is not a perfect square, and the Pellian equation $U^2-2AV^2=1$ has a positive integer solution (U',V'). By the transformation $x_2=2x_1=2u$, Eq. (1.1) leads to

$$8u^2(u+1)^2(u+2)^2(2u+1)(2u+3)A=y^2.$$

Let

$$2(2u+1)(2u+3) = Av^2,$$

then $U^2 - 2AV^2 = 1$, where $U = 2u+2$, $V = \frac{v}{2}$.

If (U', V') is a fundamental solution of the Pellian equation $U^2 - 2AV^2 = 1$, then all positive integer solutions of it are given by

$$U_n + V_n\sqrt{2A} = \left(U' + V'\sqrt{2A}\right)^n, \quad n \geq 0.$$

Thus

$$\begin{cases} U_n = 2U'U_{n-1} - U_{n-2}, & U_0 = 1,\ U_1 = U'; \\ V_n = 2U'V_{n-1} - V_{n-2}, & V_0 = 0,\ V_1 = V'. \end{cases}$$

From

$$u = \frac{U-2}{2}, \quad v = 2V,$$

we have

$$\begin{cases} u_n = 2U'u_{n-1} - u_{n-2} + 2(U'-1), & u_0 = -\frac{1}{2},\ u_1 = \frac{U'-1}{2}; \\ v_n = 2U'v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = 2V'. \end{cases}$$

From $U'^2 - 2AV'^2 = 1$, we get U' is an odd number. By the recurrence relation of u_n, we have $u_{2n+1} \in \mathbb{Z}^+$, $v_{2n+1} \in \mathbb{Z}^+$, and

$$y_{2n+1} = 2Au_{2n+1}(u_{2n+1}+1)(u_{2n+1}+2)v_{2n+1} \in \mathbb{Z}^+, \quad n \geq 0.$$

Then for $d = 1$, $r \geq 3$, $k_1 = 3$, $k_2 = 5$, $k_i \geq 5$, $i = 3, \dots, r$, Eq. (1.1) has infinitely many positive integer solutions

$$(u_{2n+1}, 2u_{2n+1}, x_3, \dots, x_r, y_{2n+1}),$$

where $n \geq 0$, such that $x_i(x_i+1)\cdots(x_i+k_i-1), i = 1, \dots, r$ are disjoint. □

Example 2.1. When $r = 3$, $k_1 = 3$, $k_2 = 5$, $k_3 = 5$, let $x_3 = 1$, then $A = 120$. Noting that $(31, 2)$ is the fundamental solution of the Pellian equation $U^2 - 240V^2 = 1$, then (u, v) satisfying

$$\begin{cases} u_n = 62u_{n-1} - u_{n-2} + 60, & u_0 = -\frac{1}{2},\ u_1 = 15; \\ v_n = 62v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = 6. \end{cases}$$

Then $u_{2n+1} \in \mathbb{Z}^+$, $v_{2n+1} \in \mathbb{Z}^+$, and

$$y_{2n+1} = 240u_{2n+1}(u_{2n+1}+1)(u_{2n+1}+2)v_{2n+1} \in \mathbb{Z}^+, \quad n \geq 0.$$

The Diophantine equation

$$\begin{aligned} x_1(x_1+1)(x_1+2)x_2(x_2+1)(x_2+2)(x_2+3)(x_2+4) \\ \times\, x_3(x_3+1)(x_3+2)(x_3+3)(x_3+4) = y^2 \end{aligned}$$

has infinitely many positive integer solutions $(u_{2n+1}, 2u_{2n+1}, 1, y_{2n+1})$, $n \geq 0$.

PROOF OF THEOREM 1.4. Let

$$\prod_{i=2}^{r} x_i(x_i+d)\cdots(x_i+(k_i-1)d) = A.$$

Choose $x_i \in \mathbb{Z}^+$, $k_i \geq 2$, $i = 2, \ldots, r$, such that $x_i(x_i+d)\cdots(x_i+(k_i-1)d)$ are disjoint, $d \nmid x_2$, and A is not a perfect square, then Eq. (1.1) reduces to $x_1(x_1+d)A = y^2$. Let $x_1 = u$ and consider $u(u+d) = Av^2$, then we get a Pellian equation $U^2 - AV^2 = 1$, where

$$U = \frac{2u+d}{d}, \quad V = \frac{2v}{d}.$$

If (U', V') is a fundamental solution of $U^2 - AV^2 = 1$, then all positive integer solutions of it are given by

$$U_n + V_n\sqrt{A} = \left(U' + V'\sqrt{A}\right)^n, \quad n \geq 0.$$

Thus

$$\begin{cases} U_n = 2U'U_{n-1} - U_{n-2}, & U_0 = 1,\ U_1 = U'; \\ V_n = 2U'V_{n-1} - V_{n-2}, & V_0 = 0,\ V_1 = V'. \end{cases}$$

From

$$u = \frac{(U-1)d}{2}, \quad v = \frac{Vd}{2},$$

we have

$$\begin{cases} u_n = 2U'u_{n-1} - u_{n-2} + (U'-1)d, & u_0 = 0,\ u_1 = \frac{(U'-1)d}{2}; \\ v_n = 2U'v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = \frac{V'd}{2}. \end{cases}$$

By the recurrence relation of u_n, we can get $u_{2n} \in \mathbb{Z}^+$, $v_{2n} \in \mathbb{Z}^+$, and $y_{2n} = Av_{2n} \in \mathbb{Z}^+$, $n \geq 1$. Then for $r \geq 2$, $k_1 = 2$, $k_i \geq 2$, $i = 2, \ldots, r$, and $d \geq 2$, Eq. (1.1) has infinitely many nontrivial positive integer solutions $(u_{2n}, x_2, \ldots, x_r, y_{2n})$, where $n \geq 1$, such that $x_i(x_i+d)\cdots(x_i+(k_i-1)d)$, $i = 1, \ldots, r$ are disjoint. $\square$

Example 2.2. When $r=2$, $k_1=k_2=2$, $d=2$, let $x_1=1$, $x_2=u$. Then Eq. (1.1) leads to $3u(u+2)=y^2$. Let $u(u+2)=3v^2$, we get $(u+1)^2-3v^2=1$. Noting that $(2,1)$ is the fundamental solution of the Pellian equation $U^2-3V^2=1$, then (u,v) satisfy

$$\begin{cases} u_n = 4u_{n-1} - u_{n-2} + 2, & u_0 = 0,\ u_1 = 1; \\ v_n = 4v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = 3. \end{cases}$$

So $y_n = 3v_n \in \mathbb{Z}^+$, $n \ge 1$. The Diophantine equation

$$x_1(x_1+2)x_2(x_2+2) = y^2$$

has infinitely many nontrivial positive integer solutions $(1, u_n, y_n)$, $n \ge 2$.

Proof of Theorem 1.5. Let

$$\prod_{i=3}^{r} x_i(x_i+d)\cdots(x_i+(k_i-1)d) = A.$$

Choose $x_i \in \mathbb{Z}^+$, $k_i \ge 3$, $i = 3, \dots, r$ such that $x_i(x_i+d)\cdots(x_i+(k_i-1)d)$ are disjoint, $d \nmid x_3$, $2A$ is not a perfect square and the Pellian equation $U^2 - 2AV^2 = 1$ has a fundamental solution (U', V') with even number V'. Let $x_1 = u$ and $x_2 = 2u$, then Eq. (1.1) reduces to

$$4u^2(u+d)^2(u+2d)(2u+d)A = y^2.$$

Considering $(u+2d)(2u+d) = Av^2$, then we get a Pellian equation $U^2 - 2AV^2 = 1$, where $U = \frac{4u+5d}{3d}$, $V = \frac{2v}{3d}$.

If (U', V') is a fundamental solution of $U^2 - 2AV^2 = 1$, then all positive integer solutions of it are given by

$$U_n + V_n\sqrt{2A} = \left(U' + V'\sqrt{2A}\right)^n, \quad n \ge 0.$$

Thus

$$\begin{cases} U_n = 2U'U_{n-1} - U_{n-2}, & U_0 = 1,\ U_1 = U'; \\ V_n = 2U'V_{n-1} - V_{n-2}, & V_0 = 0,\ V_1 = V'. \end{cases}$$

From

$$u = \frac{(3U-5)d}{4}, \quad v = \frac{3Vd}{2},$$

we have

$$\begin{cases} u_n = 2U'u_{n-1} - u_{n-2} + \frac{5(U'-1)d}{2}, & u_0 = -\frac{d}{2},\ u_1 = \frac{(3U'-5)d}{4}; \\ v_n = 2U'v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = \frac{3V'd}{2}. \end{cases}$$

From $U'^2 - 2AV'^2 = 1$, we get U' is an even number. By the recurrence relation of u_n and V' is an even number, we have $u_{4n+2} \in \mathbb{Z}^+$, $v_{4n+2} \in \mathbb{Z}^+$, and

$$y_{4n+2} = 2Au_{4n+2}(u_{4n+2} + d)v_{4n+2} \in \mathbb{Z}^+, \quad n \geq 0.$$

Then for $r \geq 2$, $k_1 = k_2 = 3$, $k_i \geq 3$, $i = 3, \dots, r$, and $d \geq 2$, Eq. (1.1) has infinitely many nontrivial positive integer solutions

$$(u_{2n}, 2u_{2n}, x_3, \dots, x_r, y_{2n}),$$

where $n \geq 1$, such that $x_i(x_i + d) \cdots (x_i + (k_i - 1)d), i = 1, \dots, r$ are disjoint. □

Example 2.3. When $r = 3$, $k_1 = k_2 = k_3 = 3$, $d = 2$, let $x_1 = 1$, $x_2 = u$, $x_3 = 2u$, then Eq. (1.1) reduces to

$$120u^2(u + 2)^2(u + 4)(u + 1) = y^2.$$

Let $(u + 4)(u + 1) = 30v^2$, we have

$$\Big(\frac{2u + 5}{3}\Big)^2 - 30\Big(\frac{2v}{3}\Big)^2 = 1.$$

Noting that $(11, 2)$ is the fundamental solution of the Pellian equation $U^2 - 30V^2 = 1$, then (u, v) satisfying

$$\begin{cases} u_n = 22u_{n-1} - u_{n-2} + 50, & u_0 = -1,\ u_1 = 14; \\ v_n = 22v_{n-1} - v_{n-2}, & v_0 = 0,\ v_1 = 6, \end{cases}$$

and

$$y_n = 30u_n(u_n + 2)v_n \in \mathbb{Z}^+, \quad n \geq 1.$$

The Diophantine equation

$$x_1(x_1 + 2)(x_1 + 4)x_2(x_2 + 2)(x_2 + 4)x_3(x_3 + 2)(x_3 + 4) = y^2$$

has infinitely many nontrivial positive integer solutions $(1, u_n, 2u_n, y_n)$, $n \geq 1$.

Proof of Theorem 1.6. For an even number $d \geq 2$, let $d = 2b$ and $z = 2x + 3d = 2x + 6b$. From Eq. (1.2) we have

(2.1)
$$16(x + 4b)^2(x + 6b)^2x(x + 2b)(x + 3b)(x + 5b)y(y + 2b)(y + 4b)(y + 6b) = w^2.$$

Using the same method as Bauer and Bennett [1], if we let

$$x(x + 5b) = \frac{3}{4}y(y + 6b)\,, \tag{2.2}$$

then

$$(x+2b)(x+3b)=\frac{3}{4}(y+2b)(y+4b),$$

and Eq. (2.1) has positive integer solutions.

Eq. (2.2) is equivalent to the Pellian equation $X^2-3Y^2=-2b^2$, where $X=2x+5b$, $Y=y+3b$. An infinity of positive integer solutions are given by

$$X_n+Y_n\sqrt{3}=\left(b+b\sqrt{3}\right)\left(2+\sqrt{3}\right)^n, \quad n\geq 0.$$

Thus

$$\begin{cases} X_n=4X_{n-1}-X_{n-2}, & X_0=b,\ X_1=5b,\ X_2=19b;\\ Y_n=4Y_{n-1}-Y_{n-2}, & Y_0=b,\ Y_1=3b,\ Y_2=11b.\end{cases}$$

From

$$x=\frac{X-5b}{2}, \quad y=Y-3b,$$

we have

$$\begin{cases} x_n=4x_{n-1}-x_{n-2}+5b, & x_0=-2b,\ x_1=0,\ x_2=7b;\\ y_n=4y_{n-1}-y_{n-2}+6b, & y_0=-2b,\ y_1=0,\ y_2=8b.\end{cases}$$

It is easy to prove that $d=2b\nmid x_{4n+2}$, $n\geq 0$. Then

$$z_{4n+2}=2x_{4n+2}+6b\in\mathbb{Z}^+, \quad n\geq 0,$$

and

$$\begin{aligned} w_{4n+2}&=3(x_{4n+2}+4b)(x_{4n+2}+6b)\\ &\times y_{4n+2}(y_{4n+2}+2b)(y_{4n+2}+4b)(y_{4n+2}+6b)\in\mathbb{Z}^+, \quad n\geq 0.\end{aligned}$$

Therefore, Eq. (1.2) has infinitely many nontrivial positive integer solutions

$$(x_{4n+2},y_{4n+2},2x_{4n+2}+6b,w_{4n+2}),$$

where $n\geq 0$. □

PROOF OF THEOREM 1.8. Let $z=3x+9$, from Eq. (1.3) we have

$$81(x+3)^2(x+6)^2x(x+4)(x+5)(x+9)y(y+3)(y+6)(y+9)=w^2. \tag{2.3}$$

If we let

$$x(x+9)=\frac{10}{9}y(y+9), \tag{2.4}$$

then

$$(x+4)(x+5)=\frac{10}{9}(y+3)(y+6),$$

and Eq. (2.3) has positive integer solutions.

Eq. (2.4) is equivalent to the Pellian equation $X^2-10Y^2=-81$, where $X=6x+27$, $Y=2y+9$. An infinity of positive integer solutions are given by

$$X_n+Y_n\sqrt{10}=\left(3+3\sqrt{10}\right)\left(19+6\sqrt{10}\right)^n,\quad n\geq 0.$$

Thus

$$\begin{cases} X_n=38X_{n-1}-X_{n-2}, & X_0=3,\ X_1=237,\ X_2=9003;\\ Y_n=38Y_{n-1}-Y_{n-2}, & Y_0=3,\ Y_1=75,\ Y_2=2847.\end{cases}$$

From

$$x=\frac{X-27}{6},\quad y=\frac{Y-9}{2},$$

we have

$$\begin{cases} x_n=38x_{n-1}-x_{n-2}+162, & x_0=-4,\ x_1=35,\ x_2=1496;\\ y_n=38y_{n-1}-y_{n-2}+162, & y_0=-3,\ y_1=33,\ y_2=1419.\end{cases}$$

It is easy to prove that $3\nmid x_n$, $n\geq 1$. Then

$$3\nmid z_n=3x_n+9,\quad n\geq 1,$$

and

$$w_{4n+2}=10(x_n+3)(x_n+6)y_n(y_n+3)(y_n+6)(y_n+9)\in\mathbb{Z}^+,\quad n\geq 1.$$

Therefore, Eq. (1.3) has infinitely many nontrivial positive integer solutions

$$(x_n,y_n,3x_n+9,w_n),$$

where $n\geq 1$. $\square$

3. Some related questions

By searching on computer, we find that the Diophantine equation

$$x(x+1)(x+2)y(y+1)(y+2)(y+3)(y+4)=z^2$$

has some positive integer solutions, such as

$$(x,y,z)=(8,2,720),\ (14,6,10080),\ (64,9,205920),\ (168,14,2227680),$$

$$(169, 16, 3023280),\ (1680, 6, 11985120),\ (1804, 41, 928536840),$$
$$(2209, 64, 3674505120),\ (2540, 123, 22372396200),$$

and the Diophantine equation

$$x(x+1)(x+2)(x+3)y(y+1)(y+2)(y+3)(y+4) = z^2$$

has a nontrivial positive integer solution

$$(x, y, z) = (120, 242, 13726888560)$$

in the range $1 < x, y < 1000$. But it seems difficult to give a positive answer to the following question.

QUESTION 3.1. *Do the Diophantine equations*

$$x(x+1)(x+2)y(y+1)(y+2)(y+3)(y+4) = z^2$$

and

$$x(x+1)(x+2)(x+3)y(y+1)(y+2)(y+3)(y+4) = z^2$$

have infinitely many positive integer solutions?

When $d > 1$, we have

QUESTION 3.2. *For an even number* $d \geq 2,\ r = 4,\ k_i = 4,\ i = 1, 2, 3, 4,$ *does Eq.* (1.1) *have infinitely many nontrivial positive integer solutions*, *i.e.*, *does the Diophantine equation*

$$\prod_{i=1}^{4} x_i(x_i + d)(x_i + 2d)(x_i + 3d) = y^2$$

have infinitely many nontrivial positive integer solutions?

QUESTION 3.3. *For odd number* $d \geq 5,\ r = 3,\ k_i = 4,\ i = 1, 2, 3,$ *does Eq.* (1.1) *have infinitely many nontrivial positive integer solutions*, *i.e.*, *does the Diophantine equation*

$$\prod_{i=1}^{3} x_i(x_i + d)(x_i + 2d)(x_i + 3d) = y^2$$

have infinitely many nontrivial positive integer solutions?

Bennett and Van Luijk [3] constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square. Similarly, we can ask

QUESTION 3.4. *For $d \geq 2$, $r \geq 5$, $k_i = 5$, $i = 1, \ldots, r$, does Eq.* (1.1) *have infinitely many nontrivial positive integer solutions*, *i.e.*, *does the Diophantine equation*

$$\prod_{i=1}^{r} x_i(x_i + d)(x_i + 2d)(x_i + 3d)(x_i + 4d) = y^2$$

have infinitely many nontrivial positive integer solutions?

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