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ON THE EXISTENCE OF PERIODIC MOTIONS OF THE EXCITED INVERTED PENDULUM BY ELEMENTARY METHODS

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Abstract. Using purely elementary methods, necessary and sufficient conditions are given for the existence of $2T$-periodic and $4T$-periodic solutions around the upper equilibrium of the mathematical pendulum when the suspension point is vibrating with period $2T$. The equation of the motion is of the form

$$\ddot{\theta} - \frac{1}{l}(g + a(t))\theta = 0,$$

where l, g are constants and

$$a(t) := \begin{cases} A & \text{if } 2kT \le t < (2k+1)T, \\ -A & \text{if } (2k+1)T \le t < (2k+2)T, \end{cases} \qquad (k = 0, 1, \dots);$$

A, T are positive constants. The exact stability zones for the upper equilibrium are presented.

1. Introduction

It was a surprising discovery at the beginning of the last century (see [1,15]) that the upper (unstable) equilibrium of the mathematical pendulum can be stabilized by vibrating the point of suspension vertically with sufficiently high frequency. Many papers (see, e.g., [3,5,9,16–18,21,22] and the references therein) have been devoted to the description of this phenomenon (see also [1,6,20]). Investigating the small oscillations around the upper equilibrium V. I. Arnold [1] and, later on, M. Levi and W. Weckesser [18] estimated the stability zones on parameter planes. Levi and Weckesser

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supposed that the acceleration of the oscillation of the suspension point is essentially larger than the acceleration of gravity and therefore negligated the gravity. In the gravity free case the hyperbolic and elliptic frequencies (see $\omega_h,\ \omega_e$ later) coincide, and it is easy to handle trajectories. In this paper we consider the model containing the gravity, which cannot be treated by Levi's and Weckesser's method. It is well known [1] that the boundary curves of stability zones correspond to the equations of motions having $2T$-periodic or $4T$-periodic solutions, where $2T$ is the period of the vibration of the suspension point. We give necessary and sufficient conditions for the parameters in the equation of motions so that the equation has periodic solutions of $2T$ or $4T$. These conditions define the exact stability zones on the parameter plane. The conditions and their proofs are based upon purely elementary methods; we do not use even Floquet's theory [1,6,20].

In Section 2 we set up the model describing the small oscillations of the excited pendulum around the upper equilibrium. The model is a non-autonomous second order linear differential equation with a $2T$-periodic step function coefficient. We reduce this equation to an equivalent discrete dynamical system on the plane. In Section 3 we construct periodic solutions of period $2T$ or $4T$ to this equivalent system. In Section 4 we formulate the Oscillation Theorem and present the stability zones on the parameter plane introduced by Arnold [1].

2. The model

It is well-known [1,6,20] that motions of the mathematical pendulum are described by the second order differential equation

$$\ddot{\psi} + \frac{g}{l} \sin \psi = 0 \quad (-\infty < \psi < \infty), \tag{2.1}$$

where the state variable ψ denotes the angle between the rod of the pendulum and the direction downward measured counter-clockwise; g and l are the gravity acceleration and the length of the rod respectively. The lower equilibrium position $\psi \equiv 0 \pmod{2\pi}$ is stable and the upper one $\psi \equiv \pi \pmod{2\pi}$ is unstable. We want to stabilize the upper equilibrium position, so introducing the new angle variable $\theta = \psi - \pi$ and linearizing equation (2.1) we obtain the linear second order differential equation

$$\ddot{\theta} - \frac{g}{l} \theta = 0,$$

which describes the small oscillations of the pendulum around the upper equilibrium position $\theta = 0 \pmod{2\pi}$.

Suppose that the suspension point is vibrating vertically with the acceleration

$$a(t) := \begin{cases} A & \text{if } 2kT \le t < (2k+1)T, \\ -A & \text{if } (2k+1)T \le t < (2k+2)T, \end{cases} \qquad (k = 0, 1, \dots); \tag{2.2}$$

A, T are positive constants ($A > g$), so that the motion of the suspension point is $2T$-periodic. Since the suspending rod is rigid, the acceleration of the vibration is continuously added to the gravity, and the equation of motion of the pendulum is

$$\ddot{\theta} - \frac{1}{l}(g + a(t))\theta = 0. \tag{2.3}$$

Every motion of (2.3) has two phases during every period, a hyperbolic and an elliptic one, that are described by the equations

$$\ddot{\theta} - \omega_h^2 \theta = 0 \quad (2kT \le t < (2k+1)T) \tag{2.4}$$

and

$$\ddot{\theta} + \omega_e^2 \theta = 0 \quad ((2k+1)T \le t < (2k+2)T), \tag{2.5}$$

where

$$\omega_h := \sqrt{\frac{A+g}{l}}, \quad \omega_e := \sqrt{\frac{A-g}{l}}, \quad k \in \mathbb{Z}_0^+ := \{0, 1, 2, \dots\}$$

denotes the hyperbolic and the elliptic frequency of the pendulum, respectively.

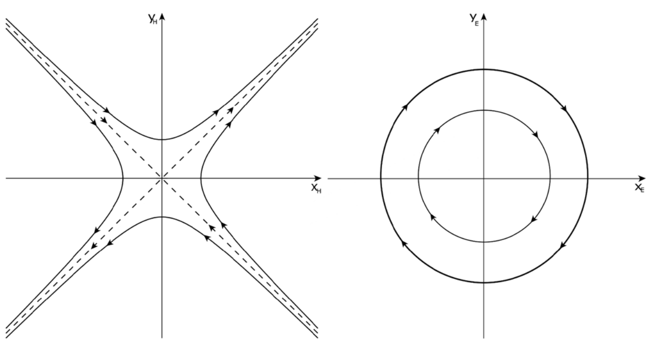

Fig. 1: Hyperbolic and elliptic rotation

We introduce two different phase planes for the two different phases of the motions. Starting with the hyperbolic case, we introduce the new phase variables

$$x_h = \theta, \quad y_h = \frac{\dot\theta}{\omega_h}\,,$$

in which (2.4) has the following symmetric form:

$$\dot x_h = \omega_h y_h, \quad \dot y_h = \omega_h x_h. \tag{2.6}$$

Using polar coordinates r_h, φ_h and the transformation rules

$$x_h = r_h \cos\varphi_h, \quad y_h = r_h \sin\varphi_h \quad (r_h > 0,\ -\infty < \varphi_h < \infty),$$

(2.6) can be rewritten into the system

$$\dot r_h = r_h\omega_h \sin 2\varphi_h\,, \quad \dot\varphi_h = \omega_h \cos 2\varphi_h\,. \tag{2.7}$$

The derivative of $H_h(x,y) := x_h^2 - y_h^2$ with respect to system (2.6) equals identically zero, i.e., H_h is a first integral of (2.6), so the trajectories of the system are hyperbolae; (2.7) describes "hyperbolic rotations" (see Fig. 1). We will need the solution of the second equation in (2.7). This equation is separable, so we can write

$$\int_0^t \frac{\dot\varphi_h(s)\ \mathrm{d}s}{\cos 2\varphi_h(s)} = \omega_h t, \quad 0 \le t \le T,$$

and so

$$\int_{\varphi_0}^{\varphi_h(t)} \frac{\mathrm{d}\varphi}{\cos 2\varphi} = \omega_h t, \quad \varphi_0 := \varphi_h(0) \neq -\frac{\pi}{4}\,. \tag{2.8}$$

Let $G(\varphi) := \int \mathrm{d}\varphi/\cos 2\varphi$. Then

$$G(\varphi) = -\frac{1}{2}\ln\Big|\tan\Big(\frac{\pi}{4} - \varphi\Big)\Big|,$$

whence

$$G(\varphi) := \begin{cases} -\frac{1}{2}\ln\tan(\frac{\pi}{4} - \varphi) & \text{if } -\pi/4 < \varphi < \pi/4, \\ -\frac{1}{2}\ln\tan(\varphi - \frac{\pi}{4}) & \text{if } -3\pi/4 < \varphi < -\pi/4. \end{cases} \tag{2.9}$$

From (2.8) we obtain

$$\varphi_h(t) = G^{-1}(\omega_h t + G(\varphi_0)).$$

Especially,

$$\varphi_h(T-0) = G^{-1}(\omega_h T + G(\varphi_0)), \tag{2.10}$$

where $\varphi_h(T-0)$ denotes the left-hand side limit of φ at T. Now, we can give the solution of the second equation of (2.7):

$$\varphi_h(t;\varphi_0) := \begin{cases} \frac{\pi}{4} - \arctan\big(e^{-2\omega_h t}\tan(\frac{\pi}{4}-\varphi_0)\big) & \text{if } -\pi/4 < \varphi_0 < \pi/4, \\ \frac{\pi}{4} + \arctan\big(e^{-2\omega_h t}\tan(\varphi_0 - \frac{\pi}{4})\big) & \text{if } -3\pi/4 < \varphi_0 < -\pi/4. \end{cases} \tag{2.11}$$

Let us repeat the same procedure for the second phase of the period with the new phase variables $x_e = \theta$, $y_e = \dot\theta/\omega_e$. Then we get the systems

$$\dot x_e = \omega_e y_e, \quad \dot y_e = -\omega_e x_e, \tag{2.12}$$

$$\dot r_e = 0, \quad \dot\varphi_e = -\omega_e. \tag{2.13}$$

Now $H_e(x,y) := x_e^2 + y_e^2$ is a first integral, and the trajectories of (2.12) are circles around the origin; (2.13) describes uniform "elliptic (ordinary) rotations".

Equation (2.3) has a piecewise continuous coefficient, so we have to modify the standard definition of a solution of a continuous second order differential equation. A function $\theta : \mathbb{R}_+ \to \mathbb{R}$ is a solution of (2.3) if it is continuously differentiable on $\mathbb{R}_+$, it is twice differentiable on the set

$$S := \mathbb{R}_+ \setminus \{kT\}_{k\in\mathbb{Z}_0^+},$$

and it satisfies equation (2.3) on the set S. Any solution θ consists of solutions $x_h\colon [2kT,(2k+1)T) \to \mathbb{R}$ and $x_e\colon [(2k+1)T,(2k+2)T) \to \mathbb{R}$ of (2.6) and (2.12), respectively ($k \in \mathbb{Z}_0^+$). To guarantee the continuity of the derivative $\dot\theta$ on $\mathbb{R}$ we have to require the "connecting conditions"

$$\begin{cases} x_e((2k+1)T) = \lim\limits_{t\to(2k+1)T-0} x_h(t), \\ x_h((2k+2)T) = \lim\limits_{t\to(2k+2)T-0} x_e(t); \\ \omega_e y_e((2k+1)T) = \lim\limits_{t\to(2k+1)T-0} \omega_h y_h(t), \\ \omega_h y_h((2k+2)T) = \lim\limits_{t\to(2k+2)T-0} \omega_e y_e(t). \end{cases} \tag{2.14}$$

Geometrically this means that at the ends of the hyperbolic and elliptic phases jumps happen in the dynamics: there acts on the phase point (x,y) a linear transformation (a contraction or a dilatation)

$$(x,y) \mapsto (x,dy) =: (x,\hat y) \quad (0 < d = \text{const.},\ d \neq 1)$$

in the direction of y-axis. Namely, $d = \omega_h/\omega_e$ at $t = (2k+1)T$, and $d = \omega_e/\omega_h$ at $t = (2k+2)T$.

The steps of dynamics of the system can be described as follows. The phase point starts from (x_0, y_0) and moves along a hyperbola during the interval $[0, T)$. At the moment $t = T$ a dilation of measure $\omega_h/\omega_e > 1$ happens parallel with $y-$axis. Then the phase point turns clockwise around the origin by $\omega_e T$. Finally, a contraction of measure $\omega_e/\omega_h < 1$ happens. These four steps are repeated ad infinitum.

Let us consider this system in polar coordinates. Denote by (r_R, φ_R), and $(r_C, \varphi_C) = (\rho(r, \varphi; d), \phi(\varphi; d))$ the image of the point (r, φ) at the rotation of a clockwise angle α and the contraction-dilatation, respectively. Then, obviously, $r_R(r, \varphi) = r$, $\varphi_R(r, \varphi) = \varphi - \alpha$; furthermore,

$$\rho(r, \varphi; d) = \sqrt{x^2 + d^2 y^2} = r\sqrt{1 + (d^2 - 1)\sin^2\varphi} = f(\varphi; d) r,$$

$$f(\varphi, d) := \sqrt{1 + (d^2 - 1)\sin^2\varphi}, \quad (d > 0,\ -\infty < \varphi < \infty).$$

It is easy to see that $\tan\phi(\varphi; d) = dy/x = d\tan\varphi$ $(x \neq 0$, i.e., $\varphi \not\equiv \pi/2 \pmod{\pi})$, so

$$\phi(\varphi; d) := \begin{cases} \arctan(d\tan\varphi) + \left[\frac{\varphi + \frac{\pi}{2}}{\pi}\right] \cdot \pi & \text{if } \varphi \neq (2k+1)\frac{\pi}{2}, \\ \varphi & \text{if } \varphi = (2k+1)\frac{\pi}{2}, \end{cases} \quad (k \in \mathbb{Z}),$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

The detailed description of properties of functions f and ϕ can be found in [11]. During our calculations we will use from these properties that f is even and ϕ is odd, furthermore $\phi(\cdot + k\pi; d) = \phi(\cdot; d) + k\pi$ $(k \in \mathbb{Z})$; $\phi(\phi(\varphi; d); 1/d) = \varphi$ $(\varphi \in \mathbb{R})$.

3. The construction of periodic solutions

Let us start a trajectory $t \mapsto (r(t), \varphi(t))$ from r_0, φ_0 at $t_0 = 0$. With $D := \omega_h/\omega_e > 1$, for the first five notable points of the trajectory we introduce the notations

$$\text{(3.1)} \qquad \begin{cases} r_0 := r(0), \quad \varphi_0 :\equiv \varphi(0) \pmod{2\pi},\ -2\pi < \varphi_0 \leq 0; \\ r_1 := r(T-0), \quad \varphi_1 := \varphi(T-0); \\ r_2 := r(T) = f(\varphi_1; D) r_1, \quad \varphi_2 := \varphi(T) = \phi(\varphi_1; D); \\ r_3 := r(2T-0) (= r_2), \quad \varphi_3 := \varphi(2T-0); \\ r_4 := r(2T) = f(\varphi_3; 1/D) r_3, \quad \varphi_4 := \varphi(2T) = \phi(\varphi_3; 1/D). \end{cases}$$

Since systems (2.4) and (2.5) are linear, it is obvious that if $t \mapsto (x(t), y(t))$ is a solution of a system then $t \mapsto (-x(t), -y(t))$ is also a solution. So, it is sufficient to consider the half plane of the right-hand side, namely, when $-\pi/2 \leqq \varphi_0 < \pi/2$.

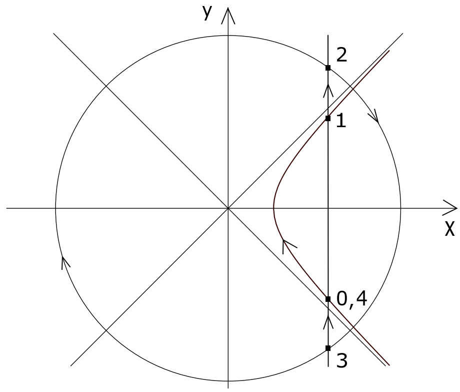

Fig. 2: Steps of the dynamics. 1

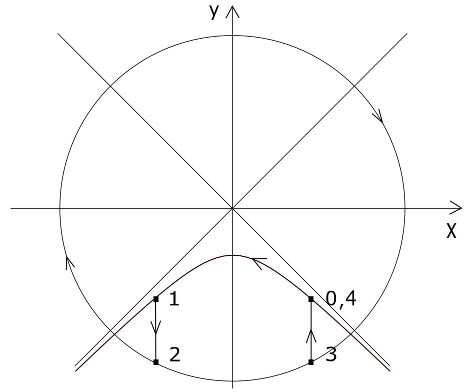

Fig. 3: Steps of the dynamics. 2

LEMMA 3.1. *Let $\varphi_0 \in [-\pi/2, \pi/2)$. Then $t \mapsto (r(t), \varphi(t))$ is a trajectory of a $2T$-periodic solution of (2.3) if and only if either*

(a) $-\pi/4 < \varphi_0 < 0$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 \\ \varphi_3 = -\varphi_2 - 2k\pi, \end{cases} \tag{3.2}$$

or

(b) $-\pi/2 < \varphi_0 < -\pi/4$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 - \pi \\ \varphi_3 = -\varphi_2 - \pi - 2(k+1)\pi. \end{cases} \tag{3.3}$$

PROOF. *Necessity.* Let θ be a $2T$-periodic solution of (2.3) such that $-\pi/2 < \varphi_0 < \pi/2$. From equations (2.7) we obtain that every hyperbola satisfies some differential equation

$$\frac{dr}{d\varphi} = r \tan 2\varphi \quad \Big(-\frac{\pi}{4} + m\frac{\pi}{2} < \varphi < \frac{\pi}{4} + m\frac{\pi}{2},\ m \in \{-1, 0, 1\}\Big). \tag{3.4}$$

(3.4) is separable, so integrating it we have

$$\frac{r}{r_0} = \sqrt{\frac{|\cos 2\varphi_0|}{|\cos 2\varphi|}} \quad \Big(-\frac{\pi}{4} + m\frac{\pi}{2} < \varphi_0, \varphi < \frac{\pi}{4} + m\frac{\pi}{2},\ m \in \{-1, 0, 1\}\Big). \tag{3.5}$$

Using (3.1) and the features of the function f we can write

$$r_3 = f(\varphi_0; D)r_0, \quad r_2 = f(\varphi_1; D)r_1.$$

Since the solution is $2T$-periodic and $r_3 = r_2$ we have

$$\frac{r_1}{r_0} = \sqrt{\frac{|\cos 2\varphi_0|}{|\cos 2\varphi_1|}} = \frac{f(\varphi_0; D)}{f(\varphi_1; D)} = \sqrt{\frac{1 + (D^2-1)\sin^2\varphi_0}{1 + (D^2-1)\sin^2\varphi_1}}. \tag{3.6}$$

By the use of the function

$$h(\varphi) := \frac{|\cos 2\varphi|}{1 + (D^2-1)\sin^2\varphi} \tag{3.7}$$

(3.6) can be expressed by $h(\varphi_0) = h(\varphi_1)$.

An elementary calculation shows that h is strictly increasing on the closed interval $[\pi/4 + m\pi/2, \pi/2 + m\pi/2]$, and strictly decreasing on $[m\pi/2, \pi/4 + m\pi/2]$ ($m \in \mathbb{Z}$).

If $\varphi_0 \in [0, \pi/4]$ or $\varphi_0 \in [\pi/4, \pi/2]$, then φ_1 must be found in the same interval. Since h is strictly monotonous in these intervals, $h(\varphi_0) = h(\varphi_1)$ cannot be satisfied. So, a $2T$-periodic solution cannot start from such a φ_0.

Function h is even and periodic of period π, so if $\varphi_0 \in (-\pi/4, 0)$ or $\varphi_0 \in (-\pi/2, -\pi/4)$ then there exists exactly one $\varphi_1 \in (0, \pi/4)$ or $\varphi_1 \in (-3\pi/4, -\pi/2)$ for which $h(\varphi_0) = h(\varphi_1)$; i.e., (3.2) or (3.3) are satisfied, respectively.

Sufficiency. Case (a). Let us suppose that the phase-point moves on the phase plane such that (3.2) is satisfied. Using the notations (3.1) and the properties of the function f and ϕ we have

$$\varphi_4 = \phi(-\varphi_2 - 2k\pi; 1/D) = \phi(-\varphi_2; 1/D) - 2k\pi$$
$$= \phi(-\phi(\varphi_1; D); 1/D) - 2k\pi = \phi(-\phi(-\varphi_0; D); 1/D) - 2k\pi = \varphi_0 - 2k\pi.$$

On the other hand

$$r_4 = f(\varphi_3; 1/D)r_3 = f(-\varphi_2 - 2k\pi; 1/D)r_2 = f(-\phi(\varphi_1; D); 1/D)f(\varphi_1; D)r_1$$
$$= f(\phi(-\varphi_0; D); 1/D)f(-\varphi_0; D)r_0 = f(-\phi(-\varphi_0; D); 1/D)f(\varphi_0; D)r_0 = r_0.$$

Case (b) can be treated by a similar computation. □

LEMMA 3.2. *Let $\varphi_0 \in [-\pi/2, \pi/2)$. Then $t \mapsto (r(t), \varphi(t))$ is the trajectory of such a $4T$-periodic solution of* (2.3) *which is not $2T$-periodic if and only if either*

(a) $-\pi/4 < \varphi_0 < 0$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 \\ \varphi_3 = -\varphi_2 - \pi - 2k\pi, \end{cases} \tag{3.8}$$

or

(b) $-\pi/2 < \varphi_0 < -\pi/4$ *and there is a non-negative integer* k *such that*

$$\begin{cases} \varphi_1 = -\varphi_0 - \pi \\ \varphi_3 = -\varphi_2 - 2\pi - 2k\pi. \end{cases} \tag{3.9}$$

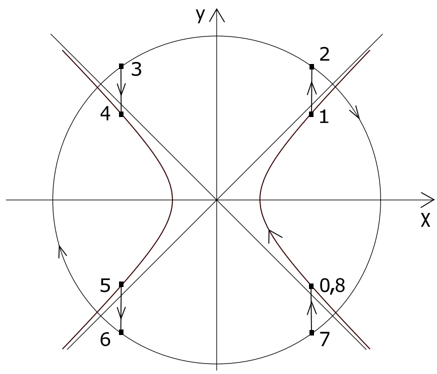

Fig. 4: 4T-periodic solution. 1

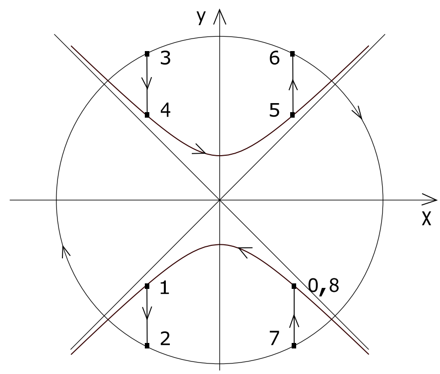

Fig. 5: 4T-periodic solution. 2

Proof. (2.3) is linear, so a solution $t \mapsto (r(t), \varphi(t))$ is $4T$-periodic but not $2T$-periodic if and only if $r(2T) = r(0)$, $\varphi(2T) \equiv \varphi(0) - \pi \pmod{2\pi}$. Therefore the necessity can be proved in the same way as in Lemma 3.1.

Sufficiency. Case (a). If (3.8) is satisfied, then, using notations (3.1) we can write

$$\begin{aligned} \varphi_4 &= \phi(-\varphi_2 - \pi - 2k\pi; 1/D) \\ &= \phi(-\varphi_2; 1/D) - (2k+1)\pi = \phi(-\phi(\varphi_1; D); 1/D) - (2k+1)\pi \\ &= \phi(-\phi(-\varphi_0; D); 1/D) - (2k+1)\pi = \varphi_0 - (2k+1)\pi. \end{aligned}$$

Therefore, $\varphi_4 \equiv \varphi_0 - \pi \pmod{2\pi}$, what we want to prove. Furthermore, we have

$$\begin{aligned} r_4 &= f(\varphi_3; 1/D) r_3 = f(-\varphi_2 - \pi - 2k\pi; 1/D) r_2 \\ &= f(-\phi(\varphi_1; D); 1/D) f(\varphi_1; D) r_1 = f(\phi(\varphi_0; D); 1/D) f(-\varphi_0; D) r_0 \\ &= f(\phi(\varphi_0; D); 1/D) f(\varphi_0; D) r_0 = r_0. \end{aligned}$$

Case (b). It is the same as in the proof of Lemma 3.1. □

We can give two theorems which yield necessary and sufficient conditions for the existence of $2T$-periodic and $4T$-periodic solutions of (2.3).

THEOREM 3.3. *There is a solution of* (2.3) *of period* $2T$ *if and only if there are positive constants* A *and* T *in* (2.2) *and a non-negative integer* k *such that either*

$$2\arctan\Big(D\,\frac{e^{\omega_h T}-1}{e^{\omega_h T}+1}\Big)+2k\pi=\omega_e T, \tag{3.10}$$

or

$$2\arctan\Big(D\,\frac{e^{\omega_h T}+1}{e^{\omega_h T}-1}\Big)+(2k+1)\pi=\omega_e T. \tag{3.11}$$

THEOREM 3.4. *There exists a* $4T$*-periodic solution of* (2.3) *which is not* $2T$*-periodic if and only if there are positive constants* A *and* T *in* (2.2) *and a non-negative integer* k *such that either*

$$2\arctan\Big(D\,\frac{e^{\omega_h T}-1}{e^{\omega_h T}+1}\Big)+(2k+1)\pi=\omega_e T, \tag{3.12}$$

or

$$2\arctan\Big(D\,\frac{e^{\omega_h T}+1}{e^{\omega_h T}-1}\Big)+2k\pi=\omega_e T. \tag{3.13}$$

PROOF OF THEOREM 3.3. *Necessity*. We suppose that θ is a $2T$-periodic solution of equation (2.3), furthermore, Case (a) of Lemma 3.1 is satisfied. Using notations (3.1) and the second equation of (2.13) we obtain

$$\varphi_3-\varphi_2=-\omega_e T. \tag{3.14}$$

We eliminate φ_2 and φ_3 in (3.14) in terms of φ_0. We can do it using the formulae

$$\varphi_2=\phi(\varphi_1;D)=\phi(-\varphi_0;D),\quad \varphi_3=\phi(\varphi_0;D)-2k\pi.$$

The first one is trivial. For the second one, take into consideration that $-\pi/4<\varphi_0<0$ and $\varphi_1=-\varphi_0$ imply

$$0<\varphi_2=\phi(\varphi_1;D)<\frac{\pi}{2}\,.$$

Therefore (3.2) yields

$$-2k\pi-\frac{\pi}{2}<\varphi_3=-\varphi_2-2k\pi<-2k\pi.$$

On the other hand, by the periodicity, $\varphi_0\equiv\varphi_4=\phi(\varphi_3;1/D)\pmod{2\pi}$, thus $\varphi_3=\phi(\varphi_4;D)\equiv\phi(\varphi_0;D)\pmod{2\pi}$. From these we get $\varphi_3=\phi(\varphi_0;D)-2k\pi$.

Now, (3.14) can be rewritten as

$$2\phi(\varphi_0; D) - 2k\pi = -\omega_e T,$$

therefore

$$2\arctan(D\tan\varphi_0) - 2k\pi = -\omega_e T. \tag{3.15}$$

Using (2.10) and (2.11) we obtain

$$e^{-2\omega_h T} = \frac{\tan(\frac{\pi}{4} - \varphi_1)}{\tan(\frac{\pi}{4} - \varphi_0)} = \frac{\tan(\frac{\pi}{4} + \varphi_0)}{\tan(\frac{\pi}{4} - \varphi_0)} = \Big(\frac{1 + \tan\varphi_0}{1 - \tan\varphi_0}\Big)^2,$$

whence we have

$$\tan\varphi_0 = \frac{1 - e^{\omega_h T}}{1 + e^{\omega_h T}}\,. \tag{3.16}$$

Substituting (3.16) into (3.15) we get (3.10).

Now, let us suppose that Case (b) of Lemma 3.1 is satisfied. Similarly we obtain

$$2\arctan(D\tan(-\varphi_0)) - \pi + (k+1)2\pi = \omega_e T, \tag{3.17}$$

and

$$\tan\varphi_0 = \frac{1 + e^{\omega_h T}}{1 - e^{\omega_h T}} \tag{3.18}$$

which yield (3.11).

Sufficiency. Suppose that (3.10) is satisfied. We show that the solution with

$$\varphi_0 := \arctan\frac{1 - e^{\omega_h T}}{1 + e^{\omega_h T}} \tag{3.19}$$

is $2T$-periodic. Obviously,

$$e^{\omega_h T} = \frac{1 - \tan\varphi_0}{1 + \tan\varphi_0}\,. \tag{3.20}$$

By (2.11), this means that

$$\tan\Big(\frac{\pi}{4} - \varphi_1\Big) = \frac{1 + \tan\varphi_0}{1 - \tan\varphi_0}\,, \tag{3.21}$$

and thus $\varphi_1 = -\varphi_0$. We show that the second equality in (3.2) is also satisfied. In fact, from (2.7) and (3.10) we obtain

$$2\arctan\Big(D\,\frac{e^{\omega_h T}-1}{e^{\omega_h T}+1}\Big)+2k\pi=-(\varphi_3-\varphi_2).$$

But in view of (3.16) we can write

$$\varphi_2=\phi(\varphi_1;D)=-\phi(\varphi_0;D)=-\arctan(D\tan\varphi_0)=-\arctan\Big(D\frac{1-e^{\omega_h T}}{1+e^{\omega_h T}}\Big),$$

therefore $2\varphi_2+2k\pi=-\varphi_3+\varphi_2$, i.e., $\varphi_3=-\varphi_2-2k\pi$.

So we have proved that (3.2) is satisfied. Lemma 3.1 guarantees that the solution is $2T$-periodic.

If (3.11) is satisfied, then we define

$$\varphi_0:=-\arctan\frac{e^{\omega_h T}+1}{e^{\omega_h T}-1}\in\Big(-\frac{\pi}{2},-\frac{\pi}{4}\Big). \tag{3.22}$$

Repeating step by step the previous reasoning we get that (3.3) is satisfied, and the solution with this φ_0 is $2T$-periodic. □

Proof of Theorem 3.4. *Necessity*. Suppose that we have a $4T$-periodic solution such that (3.8) is satisfied. We will obtain (3.12) if we express φ_2 and φ_3 in terms of φ_0 in the equality $\varphi_3-\varphi_2=-\omega_e T$. Since $-\pi/4<\varphi_0<0$ and $\varphi_2=\phi(\varphi_1;D)=\phi(-\varphi_0;D)$ we have $0<\varphi_2<\pi/2$. Together (3.8) this implies

$$\frac{-3\pi}{2}-2k\pi<\varphi_3=-\varphi_2-\pi-2k\pi<-\pi-2k\pi.$$

The solution is $4T$-periodic, consequently $\varphi_3=\phi(\varphi_4;D)\equiv\phi(\varphi_0-\pi;D)$ (mod 2π), therefore

$$\varphi_3=\phi(\varphi_0-\pi;D)-2k\pi=\phi(\varphi_0;D)-\pi-2k\pi.$$

Now the equality $\varphi_3-\varphi_2=-\omega_e T$ has the form

$$\begin{aligned}\omega_e T&=-\big(\phi(\varphi_0;D)-\pi-2k\pi\big)-\phi(-\varphi_0;D)\\&=2\arctan(D\tan(-\varphi_0))+(2k+1)\pi.\end{aligned}$$

Taking into account (3.16) we obtain (3.12).

If (3.9) is satisfied for a $4T$-periodic solution, then the use of (3.18) instead of (3.16) and a similar calculation yield (3.13).

Sufficiency. Similarly to the proof of Theorem 3.3 one can show that if (3.12) is satisfied and φ_0 is defined by (3.19), then (3.8) is true and the solution is $4T$-periodic by Lemma 3.2. Analogously, if (3.13) is satisfied and φ_0 is defined by (3.22), then (3.9) is true and the solution is $4T$-periodic. □

4. Remarks

4.1. Oscillation theorem. (2.3) is a special Hill's equation [13]. As is known [12,19], one of main results about Hill's equations is the Oscillation Theorem. From Theorems 3.3 and 3.4 we can deduce an oscillation theorem for (2.3), which is analogous to Theorem 3.7 in [8].

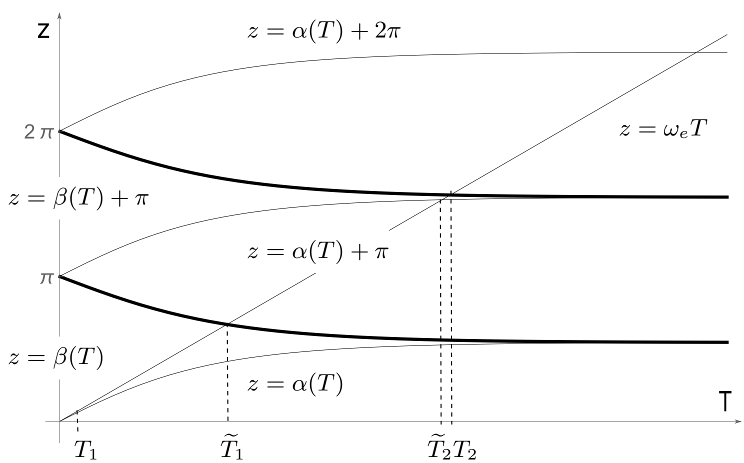

Fig. 6: Conditions (3.10)–(3.13)

Introduce the notations

$$\alpha(T) := 2\arctan\Big(D\,\frac{e^{\omega_h T}-1}{e^{\omega_h T}+1}\Big), \quad \beta(T) := 2\arctan\Big(D\,\frac{e^{\omega_h T}+1}{e^{\omega_h T}-1}\Big).$$

α is concave and β is convex from below, consequently, every equation of (3.10)-(3.13) has exactly one solution provided that the non-negative k, A (i.e., ω_h and ω_e) are fixed (see Fig. 6).

COROLLARY 4.1. *For every $A > g$ there exist sequences $\{T_n\}_{n=1}^{\infty}$, $\{\widetilde{T}_n\}_{n=1}^{\infty}$ such that*

$$0 < T_1 < \widetilde{T}_1 < \widetilde{T}_2 < T_2 < T_3 < \cdots \widetilde{T}_n < \widetilde{T}_{n+1} < T_{n+1} < T_{n+2} \cdots ,$$

and equation (2.3) *with $T = T_n$* (*respectively, with $T = \widetilde{T}_n$*) *has $2T$-periodic* (*respectively, $4T$-periodic*) *solutions.*

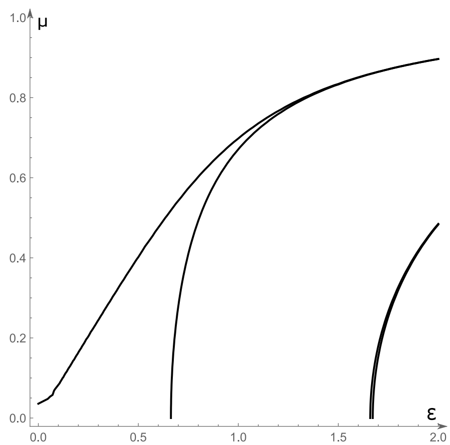

Fig. 7: The exact stability zones

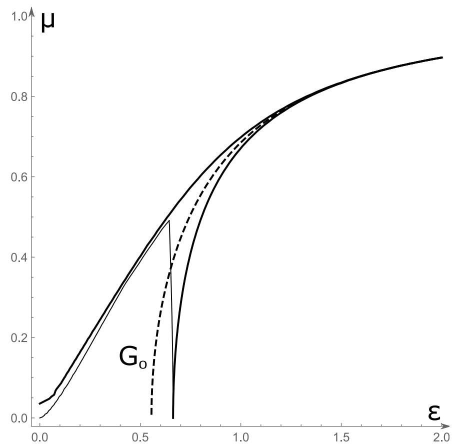

Fig. 8: Curve G_0 (dashed); the first earlier approximating, respectively exact stability zone (thin, respectively thick boundary curve)

4.2. Stability zones. Equation (2.3) contains two independent parameters T, A. Accordingly, there is a bijection between the set of points (T, A) ($T > 0$, $A > g$) on the T-A plane and all the possible equations of form (2.3). An equation (2.3) is called *strongly stable* (*strongly unstable*) [1] if it is stable (unstable) in the sense of Lyapunov and, in addition, all the equations close enough are also stable (unstable) in Lyapunov's sense. The set on the T-A plane consisting of all the points corresponding to the strongly stable (strongly unstable) equations is called the *stability region* (*instability region*) of (2.3).

Arnold [1] suggested using the new parameters

$$\varepsilon = \frac{1}{\sqrt{8l}} T\sqrt{A} \quad (\varepsilon > 0), \quad \mu = \frac{\sqrt{g}}{\sqrt{A}} \quad (0 < \mu < 1).$$

In [7], using a totally different method, we approximated the stability region on the ε-μ plane and proved that it has infinitely many components, which are tangential to the ε-axis, they are located "along" the curves

$$G_m : \quad 2\sqrt{2}\varepsilon\sqrt{1-\mu^2} = (2m+1)\frac{\pi}{2} \quad (m = 0, 1, \ldots)$$

in the sense that the heights and the width of the components (stability zones, stability tongues) tend to zero, as $m \to \infty$.

By Floquet Theory [1] the stability region and the instability region are separated by curves whose points correspond to the equations of form (2.3) having $2T$- or $4T$-periodic solutions. Therefore if we draw the solution sets of equations (3.10)-(3.13) on the ε-μ plane, then we get the exact boundary curves of the stability zones (Fig. 7). Fig. 8 shows curve G_0 and the earlier first approximating and exact stability zones.

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