Acta Math. Hungar., **155** (1) (2018), 130–140
https://doi.org/10.1007/s10474-018-0829-4
First published online May 8, 2018

COVERING COMPACT METRIC SPACES GREEDILY

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(Received October 23, 2017; accepted January 23, 2018)

Dedicated to the 70th birthday of Tibor Bisztriczky, Gábor Fejes Tóth, and Endre Makai

Abstract. A general greedy approach to construct coverings of compact metric spaces by metric balls is given and analyzed. The analysis is a continuous version of Chvátal's analysis of the greedy algorithm for the weighted set cover problem. The approach is demonstrated in an exemplary manner to construct efficient coverings of the n-dimensional sphere and n-dimensional Euclidean space to give short and transparent proofs of several best known bounds obtained from constructions in the literature on sphere coverings.

1. Introduction

Let X be a compact metric space having metric d. Given a scalar $r \in \mathbb{R}_{\geq 0}$ we define the *closed ball* of radius r around center $x \in X$ by

$$B(x,r) = \left\{ y \in X : d(x,y) \leq r \right\}.$$

The *covering number* of the space X and a positive number r is

$$\mathcal{N}(X,r) = \min \left\{ |Y| : Y \subseteq X,\ \bigcup_{y \in Y} B(y,r) = X \right\},$$

i.e. it is the smallest number of balls with radius r one needs to cover X. Determining the covering number is a fundamental problem in metric geometry (see for example the classical book by Rogers [16]) with many applications:

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† The second author is partially supported by the SFB/TRR 191 "Symplectic Structures in Geometry, Algebra and Dynamics", funded by the DFG.

Key words and phrases: geometric covering problem, set cover, greedy algorithm.

Mathematics Subject Classification: 52C17, 90C27.

compressive sensing [10], approximation theory and machine learning [5] — to name a few.

In this paper we are concerned with compact metric spaces which carry a probability measure ω; a Borel measure normalized by $\omega(X) = 1$. We will assume that this probability measure behaves homogeneously on balls and is non degenerate, i.e. it satisfies the following two conditions:

(a) $\omega(B(x,s)) = \omega(B(y,s))$ for all $x, y \in X$, and for all $s \geq 0$,

(b) $\omega(B(x,\varepsilon)) > 0$ for all $x \in X$, and for all $\varepsilon > 0$.

By (a) the measure of a ball does only depend on the radius s and not on the center x, so we simply denote $\omega(B(x,s))$ by ω_s throughout the paper.

THEOREM 1.1. *Let* (X, d) *be a compact metric space with probability measure* ω *satisfying conditions* (a) *and* (b). *Then for every* ε *with* $r/2 > \varepsilon > 0$ *the covering number satisfies*

$$\frac{1}{\omega_r} \leq \mathcal{N}(X,r) \leq \frac{1}{\omega_{r-\varepsilon}}\Big(\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1\Big).$$

The lower bound is obvious (using the σ-subadditivity of ω). We give a proof for the upper bound in Section 2. Our proof is based on a greedy approach to covering. We iteratively choose balls which cover the maximum measure of yet uncovered space.

This greedy algorithm has been analyzed in the finite setting of the *set cover* problem which is a fundamental problem in combinatorial optimization. The *set cover* problem is defined as follows. Given a collection $S_1, \dots, S_m$ of the ground set $\{1, \dots, n\}$ and given costs $c_1, \dots, c_m$ the task is find a set of indices $I \subseteq \{1, \dots, m\}$ such that $\bigcup_{i \in I} S_i = \{1, \dots, n\}$ and $\sum_{i \in I} c_i$ is as small as possible.

Computationally, the *set cover* problem is difficult; Dinur and Steurer [6] showed that for every $\varepsilon > 0$ it is NP-hard to find an approximation to the *set cover* problem within a factor of $(1 - \varepsilon) \ln n$.

On the other hand, Chvátal [4] (previously, Johnson [11], Stein [17] and Lovász [12] proved similar results for the case of uniform costs $c_1 = \dots = c_m = 1$) showed that the greedy algorithm gives an $(\ln n + 1)$-approximation for the *set cover* problem. More specifically, Chvátal showed that the natural linear programming relaxation of *set cover*

$$\text{minimize } \sum_{i=1}^{m} c_i x_i$$

$$\text{subject to } x_1, \dots, x_m \geq 0 \ \text{ and } \sum_{i : j \in S_i} x_i \geq 1 \text{ for all } j = 1, \dots, n$$

is at most a factor of $H_k = \sum_{n=1}^{k} \frac{1}{n} \leq \ln k + 1$, with $k = \max_i |S_i|$, away from an optimal solution of *set cover*. He proved this bound by exhibiting an appropriate feasible solution of the dual of the linear programming relaxation. The greedy algorithm is used to construct this feasible solution.

In Section 2 we transfer Chvátal's argument from the finite *set cover* setting to the setting of compact metric spaces. Function g appearing there features the feasible solution of the dual linear program. This will provide a proof of Theorem 1.1. In Section 3 we apply Theorem 1.1 to three concrete geometric settings and we retrieve some of the best known asymptotic results, unifying many results on sphere coverings.

We think that the NP-hardness of getting $(1-\varepsilon)\ln n$-approximations for the *set cover* problem is a natural barrier for getting better asymptotic results for geometric covering problems. This might serve as an explanation why progress for example on the sphere covering problem has been very slow since the initial work of Rogers [16].

We are not the first observing the strong relation between geometric covering problems and *set cover*[1]. In recent papers, Artstein-Avidan and Raz [1], Artstein-Avidan and Slomka [2] and especially Naszódi [13] used the results of Lovász [12] to unify old results and prove new results on geometric coverings. However, they apply the results from *set cover* directly after choosing a finite ε-net. Since we consider an infinite analogue of *set cover* we do not need to use an ε-net and by this we sometimes get slightly better constants and more importantly we think that the analysis becomes rather beautiful.

Using the relation between geometric covering problems and *set cover* has already turned out to be fruitful: Prosanov [15] found new upper bounds for the chromatic number of distance graphs on the unit sphere, Naszódi and Polyanskii [14] studied multi covers by this approach.

2. Proof of Theorem 1.1

We shall prove that the following greedy algorithm (Algorithm 1) will provide a covering of X with at most

$$\frac{1}{\omega_{r-\varepsilon}}\Big(\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_{\varepsilon}}\Big)+1\Big)$$

many balls of radius r.

[1] In fact, we realized this only after we, in an attempt to understand geometric covering problems from an optimization point of view, wrote down the main body of this paper.

Algorithm 1 Greedy algorithm.

$i \leftarrow 0$
$S_x^i = B(x, r-\varepsilon)$ for all $x \in X$
while $\bigcup_{j=1}^{i} B(y^j, r) \neq X$ **do**
  $i \leftarrow i+1$
  Choose $y \in X$ with $\omega(S_y^{i-1}) \geq \omega(S_x^{i-1})$ for all $x \in X$
  $y^i = y$
  $S_x^i = S_x^{i-1} \setminus S_y^{i-1}$ for all $x \in X$
end while

We split the proof into three lemmas where the following identity will become important:

$$S_x^{i-1} = B(x, r-\varepsilon) \setminus \bigcup_{j=1}^{i-1} B(y^j, r-\varepsilon). \tag{1}$$

The first lemma states that the step of the algorithm when we want to choose $y \in X$, with $\omega(S_y^{i-1}) \geq \omega(S_x^{i-1})$ for all $x \in X$, is indeed well-defined.

Lemma 2.1. *In every iteration i the supremum* $\sup\{\omega(S_x^{i-1}) : x \in X\}$ *is attained.*

Proof. We shall show that the function $f_i \colon X \to \mathbb{R}$, $f_i(x) = \omega(S_x^{i-1})$ is continuous for every iteration i. This implies that f_i attains its maximum since X is compact.

For $x, y \in X$ we have

$$|f_i(x) - f_i(y)| = \left|\omega(S_x^{i-1}) - \omega(S_y^{i-1})\right|$$
$$= \left|\omega(S_x^{i-1} \setminus S_y^{i-1}) + \omega(S_x^{i-1} \cap S_y^{i-1}) - (\omega(S_y^{i-1} \setminus S_x^{i-1}) + \omega(S_y^{i-1} \cap S_x^{i-1}))\right|$$
$$= \left|\omega(S_x^{i-1} \setminus S_y^{i-1}) - \omega(S_y^{i-1} \setminus S_x^{i-1})\right| \leq \max\left\{\omega(S_x^{i-1} \setminus S_y^{i-1}), \omega(S_y^{i-1} \setminus S_x^{i-1})\right\}.$$

Without loss of generality, the maximum is attained at $\omega(S_x^{i-1} \setminus S_y^{i-1})$. Then by (1) we see

$$S_x^{i-1} \setminus S_y^{i-1} \subseteq B(x, r-\varepsilon) \setminus B(y, r-\varepsilon).$$

By the triangle inequality

$$B(x, r-\varepsilon) \setminus B(y, r-\varepsilon) \subseteq B(y, r-\varepsilon+d(x,y)) \setminus B(y, r-\varepsilon).$$

Now consider the indicator function $\mathbb{1}_{B(y,r-\varepsilon+d(x,y)) \setminus B(y,r-\varepsilon)}$. When y tends to x, then we have a monotonously decreasing sequence of measurable functions tending to 0. By applying the theorem of monotone convergence we obtain that the integral

$$\int \mathbb{1}_{B(y,r-\varepsilon+d(x,y))\setminus B(y,r-\varepsilon)}(z)\,d\omega(z)$$

tends to 0 as well. Hence, $f_i(y)$ tends to $f_i(x)$. □

The second lemma states that the algorithm terminates after finitely many iterations.

LEMMA 2.2. *Algorithm* 1 *terminates after at most* ω_ε^{-1} *iterations and returns a covering.*

PROOF. Consider the i-th iteration of the algorithm and suppose there exists $z \in X$ with $z \notin \bigcup_{j=1}^{i-1} B(y^j, r)$. From the triangle inequality it follows that

$$B(z,\varepsilon) \cap B(y^j, r-\varepsilon) = \emptyset.$$

Together with (1) it implies that $B(z,\varepsilon) \subseteq S_z^{i-1}$. Choose $y \in X$ with $\omega(S_y^{i-1}) \geq \omega(S_x^{i-1})$ for every $x \in X$. Hence we have

$$\omega(S_y^{i-1}) \geq \omega(S_z^{i-1}) \geq \omega(B(z,\varepsilon)) = \omega_\varepsilon > 0,$$

where ω_ε is positive by assumption (b) and thus

$$1 = \omega(X) \geq \sum_{j=1}^{i} \omega(S_{y^j}^{j-1}) \geq i \cdot \omega_\varepsilon,$$

where the first inequality follows because the sets $S_{y^j}^{j-1}$, with $j = 1, \dots, i$, are pairwise disjoint. So after at most ω_ε^{-1} iterations, the algorithm terminates with a covering. □

The third lemma gives the desired upper bound for the covering number.

LEMMA 2.3. *Algorithm* 1 *terminates after at most*

$$\frac{1}{\omega_{r-\varepsilon}}\Big(\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1\Big)$$

iterations. In particular, this number gives an upper bound for the covering number $\mathcal{N}(X, r)$.

PROOF. Let $Y \subseteq X$ denote the covering produced by Algorithm 1 after $|Y|$ iterations. We shall prove

$$\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1 \geq |Y| \cdot \omega_{r-\varepsilon}. \tag{2}$$

For this we define the symmetric kernel $K\colon X \times X \to \mathbb{R}$ by

$$K(x,y) = \begin{cases} 1, & \text{if } y \in B(x, r-\varepsilon), \\ 0, & \text{otherwise.} \end{cases}$$

For every $x \in X$ the following equality

$$\int K(x,y)\, d\omega(y) = \omega_{r-\varepsilon}$$

holds because for every fixed $x \in X$ we have $K(x,y) = \mathbb{1}_{B(x,r-\varepsilon)}(y)$ for all $y \in X$. We will exhibit an integrable function $g\colon X \to \mathbb{R}$ satisfying

$$\int K(x,y) g(x)\, d\omega(x) \le \ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1 \tag{3}$$

for all $y \in X$ and satisfying

$$\int g(x)\, d\omega(x) = |Y|. \tag{4}$$

Combining (3) and (4), we get

$$\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1 \ge \int \int K(x,y) g(x)\, d\omega(x) d\omega(y)$$
$$= \int g(x) \int K(x,y)\, d\omega(y) d\omega(x) = \int g(x) \omega_{r-\varepsilon}\, d\omega(x) = |Y| \cdot \omega_{r-\varepsilon}$$

and we have proven (2).

Now we only have to exhibit the function g.

For brevity, we denote $\omega_y^{i-1} = \omega(S_y^{i-1})$. We define g as follows:

$$g(x) = \begin{cases} (\omega_{y^i}^{i-1})^{-1}, & \text{if } x \in S_{y^i}^{i-1}, \\ 0, & \text{otherwise,} \end{cases}$$

which is a valid definition since the sets $S_{y^i}^{i-1}$ are pairwise disjoint. Also observe that g is an integrable function on the compact set X.

From this definition of g we immediately get (4):

$$\int g(x)\, d\omega(x) = \sum_{i=1}^{|Y|} \omega_{y^i}^{i-1} (\omega_{y^i}^{i-1})^{-1} = |Y|,$$

To prove (3) we fix $y \in X$. We observe the equality

$$B(y, r-\varepsilon) \cap S_{y^i}^{i-1} = S_y^{i-1} \setminus S_y^i,$$

which describes which part of $B(y, r-\varepsilon)$ is cut away in iteration i. Then,

$$\int K(x,y)g(x)\,d\omega(x) = \sum_{i=1}^{|Y|} \int K(x,y)\mathbb{1}_{S_{y^i}^{i-1}}(x)(\omega_{y^i}^{i-1})^{-1}\,d\omega(x)$$

$$= \sum_{i=1}^{|Y|} \int \mathbb{1}_{S_y^{i-1}\setminus S_y^i}(x)(\omega_{y^i}^{i-1})^{-1}\,d\omega(x) = \sum_{i=1}^{|Y|}(\omega_y^{i-1} - \omega_y^i)(\omega_{y^i}^{i-1})^{-1}.$$

For $y \in X$ consider the last iteration b such that

$$\omega_{r-\varepsilon} = \omega(B(y, r-\varepsilon)) = \omega_y^0 \geq \omega_y^1 \geq \ldots \geq \omega_y^b \geq \omega(B(y,\varepsilon)) = \omega_\varepsilon \tag{5}$$

holds (here we used $r/2 > \varepsilon$). Note that $b < |Y|$. Note also that $\omega_y^{i-1} \leq \omega_{y^i}^{i-1}$ holds. We split the sum above into two parts:

$$\sum_{i=1}^{|Y|}(\omega_y^{i-1} - \omega_y^i)(\omega_{y^i}^{i-1})^{-1}$$

$$= \sum_{i=1}^{b}(\omega_y^{i-1} - \omega_y^i)(\omega_{y^i}^{i-1})^{-1} + \sum_{i=b+1}^{|Y|}(\omega_y^{i-1} - \omega_y^i)(\omega_{y^i}^{i-1})^{-1}$$

$$\leq \sum_{i=1}^{b}(\omega_y^{i-1} - \omega_y^i)(\omega_y^{i-1})^{-1} + (\omega_y^b - \omega_y^{b+1})(\omega_y^b)^{-1} + \sum_{i=b+2}^{|Y|}(\omega_y^{i-1} - \omega_y^i)\omega_\varepsilon^{-1}$$

$$\leq \left(\sum_{i=1}^{b}(\omega_y^{i-1} - \omega_y^i)(\omega_y^{i-1})^{-1} + \frac{\omega_y^b - \omega_\varepsilon}{\omega_y^b}\right) + \left(\frac{\omega_\varepsilon - \omega_y^{b+1}}{\omega_\varepsilon} + \frac{\omega_y^{b+1} - \omega_y^{|Y|}}{\omega_\varepsilon}\right).$$

The first sum is a lower Riemann sum of the function $x \mapsto \frac{1}{x}$ in the interval $[\omega_\varepsilon, \omega_{r-\varepsilon}]$ and thus we have $\ln(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon})$ as an upper bound. The second sum is clearly bounded above by 1. Hence, (3) holds. □

3. Applications of Theorem 1.1

3.1. Covering the n-dimensional sphere. As a first application of Theorem 1.1 we consider the problem of covering the n-dimensional sphere

$$X = S^n = \left\{x \in \mathbb{R}^{n+1} : x \cdot x = 1\right\},$$

equipped with spherical distance

$$d(x,y) = \arccos x \cdot y \in [0,\pi]$$

and with the rotationally invariant probability measure ω, by spherical caps / metric balls $B(x,r)$. Clearly, properties (a) and (b) are satisfied in this setting. Again we set $\omega_r = \omega(B(x,r))$.

We are especially interested in the covering number $\mathcal{N}(S^n, r)$ when $0 < r < \pi/2$ or equivalently in the covering density defined by $\omega_r \cdot \mathcal{N}(S^n, r)$. Theorem 1.1 says that the covering density is at most

$$\frac{\omega_r}{\omega_{r-\varepsilon}}\Big(\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1\Big). \tag{6}$$

This upper bound holds for every ε with $0 < \varepsilon < r/2$. By choosing ε depending on the dimension n and on the spherical distance r we can find an upper bound for the covering density which only depends on n.

For this we recall a useful estimate of fractions of the form ω_{tr}/ω_r due to Böröczky Jr. and Wintsche [3]:

$$\frac{\omega_{tr}}{\omega_r} \le t^n \quad \text{whenever } r < tr < \frac{\pi}{2}\,. \tag{7}$$

We set $\varepsilon = r/(\mu n + 1)$ with parameter $\mu > 1$ which we are going to adjust later. Furthermore, we set

$$t = \frac{r}{r-\varepsilon} = 1 + \frac{1}{\mu n} \quad \text{and} \quad t' = \frac{r-\varepsilon}{\varepsilon} = \mu n.$$

By using (6) and (7) we have the following upper bound for the covering density

$$\frac{\omega_r}{\omega_{r-\varepsilon}}\Big(\ln\Big(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\Big) + 1\Big) \le \Big(1 + \frac{1}{\mu n}\Big)^n (n \ln \mu n + 1)$$
$$\le e^{1/\mu}(n \ln \mu n + 1) \le \Big(1 + \frac{1}{\mu - 1}\Big)(n \ln \mu n + 1).$$

Thus we have proven:

Corollary 3.1. *The covering density of the n-dimensional sphere by spherical balls is at most*

$$\Big(1 + \frac{1}{\mu - 1}\Big)(n \ln \mu n + 1) \quad \textit{for all } \mu > 1.$$

In particular, for $\mu = \ln n$, the covering density is at most

$$n \ln n + n \ln \ln n + n + o(n).$$

In the asymptotic case the best known bound is $(1/2 + o(1))n \ln n$ due to Dumer [7] which comes from a randomized construction. Our corollary slightly improves the previously best known non-asymptotic bound $n \ln n + n \ln \ln n + 2n + o(n)$ by Böröczky Jr. and Wintsche [3] also coming from a randomized construction.

3.2. Covering *n*-dimensional Euclidean space. As a second application we consider coverings of n-dimensional Euclidean space $\mathbb{R}^n$ by congruent balls. We get a covering of $\mathbb{R}^n$ by applying Theorem 1.1 to the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ which is a compact metric space satisfying properties (a) and (b). Then we periodically extend the obtained covering of $\mathbb{T}^n$ to a covering of the entire $\mathbb{R}^n$ having the same covering density.

We repeat the choices and calculations as in the previous section (which are slightly simpler here because clearly $\omega_{tr}/\omega_r = t^n$ holds where here ω denotes the Lebesgues measure) and get:

COROLLARY 3.2. *The covering density of the n-dimensional Euclidean space by congruent balls is at most*

$$\Big(1 + \frac{1}{\mu - 1}\Big)(n \ln \mu n + 1) \quad \textit{for all } \mu > 1 \textit{ and all } n \in \mathbb{N}.$$

In particular, for $\mu = \ln n$, the covering density is at most

$$n \ln n + n \ln \ln n + n + o(n).$$

We remark that this bound coincides with the currently best known bound by G. Fejes Tóth [8] coming from a randomized construction. The best known asymptotic bound coming from a randomized construction is

$$(1/2 + o(1)) n \ln n$$

due to Dumer [7].

3.3. More general coverings. At last we want to demonstrate that the greedy approach to geometric covering problems is quite flexible. It is not restricted to finding coverings of compact metric spaces by balls but can be extended to finding coverings of compact metric spaces by finite unions of balls

$$\bigcup_{i=1}^{N} B(y_i, r),$$

where we choose the initial points $y_1, \ldots, y_N \in X$ arbitrarily.

We make this statement precise in the general setting of a compact metric space (X, d). Consider the group of continuous isometries of (X, d), these are all continuous bijective maps $\tau \colon X \to X$ which preserve the distance between every two points $x, y \in X$. We assume that the group acts transitively on X and that $\omega(\tau A) = \omega(A)$ holds for all continuous isometries τ and all measurable sets A. Then by the theorem of Arzelà–Ascoli (see for example [9, Ch. 4.6]) the group of continuous isometries is relatively compact in the compact space of continuous maps mapping X to itself equipped with the supremum norm. We need this compactness for Lemma 2.1. So we can transfer the analysis of the greedy algorithm given in Section 2 to this setting.

With small modifications this extension can for example be applied to prove the following theorem due to Naszódi [13, Theorem 1.3]:

THEOREM 3.3. *Let $K \subseteq \mathbb{R}^n$ be a bounded measurable set. Then there is a covering of $\mathbb{R}^n$ by translated copies of K of density at most*

$$\inf\left\{ \frac{\omega(K)}{\omega(K_{-\delta})}\left(\ln\left(\frac{\omega(K_{-\delta/2})}{\omega(B(0,\delta/2))}\right)+1\right) : \delta > 0, K_{-\delta} \neq \emptyset \right\},$$

where $K_{-\delta} = \{x \in K : B(x,\delta) \subseteq K\}$ is the δ-inner parallel body of K.

Here, we only sketch the proof, though filling in the details is easy. As in Section 3.2 we can work on the torus $\mathbb{T}^n$. We approximate the body K and its inner parallel bodies by a finite union of balls for which

$$\bigcup_{i=1}^{N} B(y_i,\delta) \subseteq K \quad \text{and} \quad K_{-\delta} \subseteq \bigcup_{i=1}^{N} B(y_i,\delta/2) \subseteq K_{-\delta/2}$$

holds. In the end going back from the torus $\mathbb{T}^n$ to $\mathbb{R}^n$ we get a covering of $\mathbb{R}^n$ by translated copies of K with density at most

$$\inf\left\{ \frac{\omega(K)}{\omega(K_{-\delta/2})}\left(\ln\left(\frac{\omega(K_{-\delta/2})}{\omega(B(0,\delta/2))}\right)+1\right) : \delta > 0, K_{-\delta} \neq \emptyset \right\},$$

improving the result of Naszódi slightly.

Another alternative of proving this bound is to verify that the proof of Theorem 1.1 also holds if we consider translates of $K_{-\delta/2} \subseteq \mathbb{R}^n$ instead of balls $B(x, r-\varepsilon)$. This further requires that $K_{-\delta}$ is nonempty and to consider translates of Minkowski sums $x + K_{-\delta/2} + B(0,\zeta)$ instead of $B(x, r-\varepsilon+\zeta)$ in the parts of the proofs of Lemmas 2.1 and 2.2 where we apply the triangle inequality.

Acknowledgements. We thank Markus Schweighofer, Cordian Riener, and the anonymous referee for helpful remarks.

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