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A NILPOTENCY CRITERION FOR FINITE GROUPS

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Abstract. Let G be a finite group. We give a criterion of nilpotency of G based on the existence of elements of certain order in each section of G.

1. Introduction

The problem of detecting structural properties of finite groups by looking at element orders has been considered in many recent papers (see e.g. [1] and [3–6]). In the current note, we identify a new property detecting nilpotency of a finite group G that uses the function

$$\varphi(G) = \big|\{a \in G \mid o(a) = \exp(G)\}\big|$$

introduced and studied in [9]. The proof that we present is founded on the structure of minimal non-nilpotent groups (also called *Schmidt groups*) given by [8].

It is well known that a finite nilpotent group G contains elements of order $\exp(G)$. Moreover, all sections of G have this property. Under the above notation, this can be written alternatively as

$$\varphi(S) \neq 0 \quad \text{for any section } S \text{ of } G. \tag{1}$$

Our main theorem shows that the converse is also true, that is we have the following nilpotency criterion.

Theorem 1. *Let G be a finite group. Then G is nilpotent if and only if $\varphi(S) \neq 0$ for any section S of G.*

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Note that (1) implies

$$\varphi(S) \neq 0 \quad \text{for any subgroup } S \text{ of } G \tag{2}$$

and in particular

$$\varphi(G) \neq 0. \tag{3}$$

We observe that the condition (3) is not sufficient to guarantee the nilpotency of G, as shows the elementary example $G = \mathbb{Z}_6 \times S_3$; we can even construct a non-solvable group G for which $\varphi(G) \neq 0$, namely $G = \mathbb{Z}_n \times H$, where H is a simple group of exponent n. A similar thing can be said about the condition (2).

Example. Let G be a nontrivial semidirect product of a normal subgroup isomorphic to

$$E(5^3) = \left\langle x, y \mid x^5 = y^5 = [x,y]^5 = 1,\ [x,y] \in Z(E(5^3)) \right\rangle$$

by a subgroup $\langle a \rangle$ of order 3 such that a commutes with $[x,y]$. Then G is a non-CLT group of order 375, more precisely it does not have subgroups of order 75. We infer that its subgroups are: G, all subgroups contained in the unique Sylow 5-subgroup, all Sylow 3-subgroups, and all cyclic subgroups of order 15. Clearly, G satisfies the condition (2), but it is not nilpotent.

Finally, we note that our criterion can be used to prove the non-nilpotency of a finite group by looking to its sections. In [9] we have determined several classes of groups G satisfying $\varphi(G) = 0$, such as dihedral groups D_{2n} with n odd, non-abelian P-groups of order $p^{n-1}q$ ($p > 2$, q primes, $q \mid p-1$), symmetric groups S_n with $n \geq 3$, and alternating groups A_n with $n \geq 4$. These examples together with Theorem 1 lead to the following corollary.

Corollary 2. *If a finite group G contains a section isomorphic to one of the above groups, then it is not nilpotent.*

2. Proof of Theorem 1

We will prove that a finite group all of whose sections S satisfy $\varphi(S) \neq 0$ is nilpotent. Assume that G is a counterexample of minimal order. Then G is a Schmidt group since all its proper subgroups satisfy the hypothesis. By [8] (see also [2,7]) it follows that G is a solvable group of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p-subgroup P and a cyclic Sylow q-subgroup Q, and hence G is a semidirect product of P by Q. Moreover, we have:

– if $Q = \langle y \rangle$ then $y^q \in Z(G)$;

– $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P$, $P' = (G')' = \Phi(P)$;

– $|P/P'| = p^r$, where r is the order of p modulo q;

– if P is abelian, then P is an elementary abelian p-group of order p^r and P is a minimal normal subgroup of G;

– if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We infer that $S = G/Z(G)$ is also a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian p-group P_1 of order p^r by a cyclic group Q_1 of order q (note that S_3 and A_4 are examples of such groups). Clearly, we have $\exp(S) = pq$. On the other hand, it is easy to see that

$$L(S) = L(P_1) \cup \{Q_1^x \mid x \in S\} \cup \{S\}.$$

Thus, the section S does not have cyclic subgroups of order pq and consequently $\varphi(S) = 0$, a contradiction. This completes the proof. □

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