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THE φ-BRUNN–MINKOWSKI INEQUALITY

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Abstract. For strictly increasing concave functions φ whose inverse functions are log-concave, the φ-Brunn–Minkowski inequality for planar convex bodies is established. It is shown that for convex bodies in $\mathbb{R}^n$ the φ-Brunn–Minkowski is equivalent to the φ-Minkowski mixed volume inequalities.

1. Introduction

The Brunn–Minkowski theorem states that for convex bodies K, L in $\mathbb{R}^n$ and for $\lambda \in [0,1]$,

$$|(1-\lambda)K+\lambda L|^{1/n} \geq (1-\lambda)|K|^{1/n}+\lambda|L|^{1/n}. \tag{1.1}$$

Equality for some $\lambda \in (0,1)$ holds if and only if K and L either lie in parallel hyperplanes or are homothetic.

Over the decades, the Brunn–Minkowski inequality (1.1) and its extensions and ramifications have been playing the part of the foundationstone in convex geometric analysis, with applications to extremal, uniqueness and other problems. Excellent references for the Brunn–Minkowski theory are [4,5,8,18].

In the 1960s, Firey [3] introduced for $p \geq 1$ the so-called Minkowski–Firey L_p sum of convex bodies that contain the origin in their interiors. Let h_K and h_L be support functions (see the next section for definitions) of convex bodies K and L that contain the origin in their interiors. If $\lambda \in [0,1]$, then the Minkowski–Firey L_p-sum, $(1-\lambda)\cdot K +_p \lambda \cdot L$, is defined by

$$(1-\lambda)\cdot K +_p \lambda\cdot L = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x\cdot u \leq ((1-\lambda)h_K(u)^p+\lambda h_L(u)^p)^{1/p}\right\}. \tag{1.2}$$

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Firey also proved the L_p-Brunn–Minkowski inequality for $p > 1$

$$|(1-\lambda)\cdot K +_p \lambda \cdot L|^{p/n} \geq (1-\lambda)|K|^{p/n} + \lambda |L|^{p/n}, \tag{1.3}$$

with equality for $\lambda \in (0,1)$ if and only if K and L are dilates.

In the mid 1990s, Lutwak and his colleagues brought the Minkowski–Firey L_p theory to a great height of development (see e.g. [11–13]). Among many others, it has been noticed that the Minkowski–Firey L_p-sum makes sense for all $p > 0$. The case where $p = 0$ is the limit case, which is known as the log-Minkowski sum, $(1-\lambda)\cdot K +_0 \lambda \cdot L$, of K and L that contain the origin in their interiors, defined by

$$(1-\lambda)\cdot K +_0 \lambda \cdot L = \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda} \right\}. \tag{1.4}$$

In [2], Böröczky et al. conjectured and proved the planar case of the following log-Brunn–Minkowski inequality: If K and L are o-symmetric convex bodies in $\mathbb{R}^n$, then for all $\lambda \in [0,1]$,

$$|(1-\lambda)\cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda} |L|^{\lambda}. \tag{1.5}$$

By Jensen's inequality, it is easily seen that the log-Brunn–Minkowski inequality (1.5) is stronger than the L_p Brunn–Minkowski inequality (1.3) for $p > 0$. Böröczky et al. also showed that the (conjectured) log-Brunn–Minkowski inequality is equivalent to the following log-Minkowski mixed volume inequality: If K and L are o-symmetric convex bodies in $\mathbb{R}^n$, then

$$\int_{S^{n-1}} \log \frac{h_L}{h_K} \, d\overline{V}_K \geq \frac{1}{n} \log \frac{|L|}{|K|}. \tag{1.6}$$

Here S^{n-1} denotes the standard unit sphere in $\mathbb{R}^n$ and $\overline{V}_K$ is the cone-volume probability measure of K; see the next section for detailed definitions. Note that one can easily find counterexamples of the log-Brunn–Minkowski inequality (equivalently, the log-Minkowski mixed volume inequality) for not o-symmetric convex bodies. Inspired by this feature, recently Xi and Leng [19] studied a "dilation" version of the log-Brunn–Minkowski inequality for general convex bodies. For more recent progress on the conjectured log-Brunn–Minkowski inequality, one can refer to [10,17] and the references therein.

Grounded on the convexity of the function $\varphi(t) = t^p$ for $p \geq 1$, Lutwak, Yang, and Zhang [14,15] instituted the so-called Orlicz–Brunn–Minkowski theory. The Orlicz–Brunn–Minkowski theory is a natural extension of the L_p Brunn–Minkowski theory for $p \geq 1$. Since it was set up, the theory has attracted an increasing research interest, see e.g. [6,7,9,20,21].

Along the lines of extending the Minkowski–Firey L_p sum not only for $p \geq 1$ but also for $0 < p < 1$, in this paper we shall introduce the φ-Minkowski sum of convex bodies that contain the origin in their interiors, and then study the φ-Brunn–Minkowski inequality for general functions φ. Following but going beyond the pattern taken in the Orlicz–Brunn–Minkowski theory, we shall replace the power function t^p for all $p > 0$ by general φ (not necessarily convex). Then, we shall study the so-called φ-Brunn–Minkowski inequality. Indeed, such a maneuver provides a unified treat for the Orlicz–Brunn–Minkowski inequality, the L_p Brunn–Minkowski inequality for $0 < p < 1$, as well as the log-Brunn–Minkowski inequality.

To facilitate all of the mentioned cases, we first define the class of general functions φ as follows.

Let Φ be the set of strictly increasing functions $\varphi\colon (0,\infty) \to I \subseteq \mathbb{R}$ which are continuously differentiable on $(0,\infty)$ with positive derivative, and satisfy that $\lim_{t\to\infty} \varphi(t) = \infty$ and that $\log \circ \varphi^{-1}$ is concave. Observe that whenever $\varphi \in \Phi$ is convex, the composite function $\log \circ \varphi^{-1}$ is log-concave. The collection of convex functions from Φ shall be denoted by $\mathcal{C}$.

There are many fundamental examples of the functions $\varphi \in \Phi$. Convex examples of functions in Φ include the power function $\varphi(t) = t^p$ with $p \geq 1$; the logistic function $\varphi(t) = t + 2\log(1 + e^{-t})$; the Laplace function $\varphi(t) = e^{-t}$, and so on. Non-convex examples of Φ include $\varphi(t) = t^p$ with $0 < p < 1$, the log function $\varphi = \log$, and $\varphi(t) = \frac{1}{q}\log(1 + t)$ with $q \in (0,1)$.

Let $\lambda \in [0,1]$ and $\varphi \in \Phi$. For $u \in S^{n-1}$, we define a function $h_\lambda(u)$ as

$$h_\lambda(u) = \inf\Big\{\tau > 0 : (1-\lambda)\varphi\Big(\frac{h_K(u)}{\tau}\Big) + \lambda\varphi\Big(\frac{h_L(u)}{\tau}\Big) \leq \varphi(1)\Big\}. \tag{1.7}$$

It then follows from the strict monotonicity of φ that

$$\varphi(1) = (1-\lambda)\varphi\Big(\frac{h_K}{h_\lambda}\Big) + \lambda\varphi\Big(\frac{h_L}{h_\lambda}\Big). \tag{1.8}$$

Throughout, we denote by $\mathcal{K}_o^n$ the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors.

We define the *φ-combination* $Q_{\varphi,\lambda} = (1-\lambda)\cdot K +_\varphi \lambda \cdot L$ of $K, L \in \mathcal{K}_o^n$ as

$$Q_{\varphi,\lambda} = \bigcap_{u\in S^{n-1}} \big\{x \in \mathbb{R}^n : x \cdot u \leq h_\lambda(u)\big\}. \tag{1.9}$$

Since the function h_λ defined by (1.7) is both positive and continuous on S^{n-1}, the φ-combination $Q_{\varphi,\lambda}$ must be an element of $\mathcal{K}_o^n$. Moreover, if $\varphi \in \mathcal{C}$, then $Q_{\varphi,\lambda}$ has h_λ as its support function. If $\varphi \in \Phi$ is non-convex, the support function of $Q_{\varphi,\lambda}$ may not precisely be h_λ, but $h_{Q_{\varphi,\lambda}} = h_\lambda$ a.e. with respect to the surface area measure $S_{Q_{\varphi,\lambda}}$.

We note that if $\varphi(t) = t^p$ with $p > 0$, then the φ-combination reduces to the Minkowski–Firey L_p-combination defined by (1.2); i.e.,

$$Q_{p,\lambda} = (1-\lambda)\cdot K +_p \lambda \cdot L.$$

Further, if $\varphi(t) = \log(t)$, then we retrieve the log-combination, of $K, L \in \mathcal{K}_o^n$, given by (1.4); i.e.,

$$Q_{0,\lambda} = (1-\lambda)\cdot K +_0 \lambda \cdot L.$$

One main result of this paper is the following φ-Brunn–Minkowski inequality for planar o-symmetric convex bodies.

THEOREM 1.1. *Let $\lambda \in [0,1]$ and let $\varphi \in \Phi$ be concave on $(0,\infty)$. If K, L are o-symmetric convex bodies in the plane and $Q_{\varphi,\lambda} = (1-\lambda)\cdot K +_\varphi \lambda \cdot L$, then*

$$(1-\lambda)\varphi\left(\frac{|K|^{1/2}}{|Q_{\varphi,\lambda}|^{1/2}}\right) + \lambda\varphi\left(\frac{|L|^{1/2}}{|Q_{\varphi,\lambda}|^{1/2}}\right) \le \varphi(1). \tag{1.10}$$

Equality for some $\lambda \in (0,1)$ holds if and only if K and L are dilates.

The above theorem follows by the planar log-Minkowski mixed volume inequality [2, Theorem 1.4] and the following equivalence of the φ-Minkowski mixed volume inequality and the φ-Brunn–Minkowski inequality. Even though they remain open for convex bodies in $\mathbb{R}^n$ for $n > 2$, the equivalence is of great significance. As will be shown, these two inequalities are tied to the log-Minkowski mixed volume inequality (equivalently, the log-Brunn–Minkowski inequality) in $\mathbb{R}^n$ for $n > 2$.

THEOREM 1.2. *Let $\lambda \in [0,1]$ and $\varphi \in \Phi$ be, in addition, convex (or concave) on $(0,\infty)$. If $K, L \in \mathcal{K}_o^n$ are o-symmetric convex bodies in $\mathbb{R}^n$, and $Q_{\varphi,\lambda} = (1-\lambda)\cdot K +_\varphi \lambda \cdot L$, then the φ-Brunn–Minkowski inequality*

$$(1-\lambda)\varphi\left(\frac{|K|^{1/n}}{|Q_{\varphi,\lambda}|^{1/n}}\right) + \lambda\varphi\left(\frac{|L|^{1/n}}{|Q_{\varphi,\lambda}|^{1/n}}\right) \le \varphi(1) \tag{1.11}$$

is equivalent to the φ-Minkowski mixed volume inequality

$$\int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) d\overline{V}_K \ge \varphi\left(\frac{|L|^{1/n}}{|K|^{1/n}}\right). \tag{1.12}$$

Note that if $\varphi \in \Phi$ is convex on $(0,\infty)$, then the assumption that K and L are o-symmetric, imposed on the φ-Brunn–Minkowski inequality (1.11) as well as the φ-Minkowski mixed volume inequality (1.12), is not necessary. In this instance, the corresponding φ-Brunn–Minkowski inequality on $\mathbb{R}^n$ is exactly the Orlicz–Brunn–Minkowski inequality. The reader can refer to [20] for a symmetrization argument about it. For the case where $\varphi \in \Phi$ is concave on $(0, \infty)$, the φ-Brunn–Minkowski inequality (as well as the φ-Minkowski mixed volume inequality) remains open if $n > 2$. However, it turns out that once the conjectured log-Brunn–Minkowski inequality (1.5) is proved, the φ-Brunn–Minkowski inequality (1.11) will hold true consequently.

We also note that it is the log-concavity of φ^{-1} that allows us to unify the log-Brunn–Minkowski inequality, the L_p-Brunn–Minkowski inequality for $0 < p < 1$, and the Orlicz Brunn–Minkowski inequality. Associated to the log-concavity of φ^{-1} is the following comparison result linking the φ-means to the log-means:

$$\varphi^{-1}\bigg(\int_{S^{n-1}} \varphi\Big(\frac{h_L}{h_K}\Big)\, d\overline{V}_K\bigg) \ge \exp\bigg(\int_{S^{n-1}} \log\Big(\frac{h_L}{h_K}\Big)\, d\overline{V}_K\bigg),$$

which in turn indicates that the (conjectured) log-Brunn–Minkowski inequality (1.5), or equivalently, the (conjectured) log-Minkowski mixed volume inequality (1.6) is the sharpest one among all of the φ-Brunn–Minkowski inequalities for $\varphi \in \Phi$.

2. Preliminaries

The setting for this paper is the n-dimensional Euclidean space, $\mathbb{R}^n$. We shall write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$. Let B_2^n and S^{n-1} denote the standard unit ball and the unit sphere in $\mathbb{R}^n$. The most fundamental functional for convex body in $\mathbb{R}^n$ is the volume (Lebesgue measure), denoted by $|\cdot|$. A *convex body* in $\mathbb{R}^n$ is understood as a compact, convex subset of $\mathbb{R}^n$ with nonempty interior.

Let K be a convex body in $\mathbb{R}^n$ and $\nu_K \colon \partial' K \to S^{n-1}$ the Gauss map, where $\partial' K$ is the set of boundary points of K that have only one unit normal vector. It is worth noting that $\partial K \backslash \partial' K$ has $\mathcal{H}^{n-1}$-measure equal to zero. For each Borel set $\omega \subseteq S^{n-1}$ the *inverse spherical image* $\nu_K^{-1}(\omega)$ is defined as a subset of $\partial' K$ such that the outer normal of $x \in \partial' K$ belongs to ω. For a convex body K in $\mathbb{R}^n$, the classical *surface area measure* of K is defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)),$$

for each Borel set $\omega \subseteq S^{n-1}$. That is to say, $S_K(\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all points on $\partial' K$.

The *support function* of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n \backslash \{0\}$.

For real $a,b \ge 0$ (not both zero), the *Minkowski linear combination* $aK+bL$ of convex bodies K, L can be defined either by

$$aK+bL = \{ax+by : x \in K,\ y \in L\},$$

or by

$$h_{aK+bL} = ah_K + bh_L. \tag{2.1}$$

More generally, if $K,L \in \mathcal{K}_o^n$, then for $p>1$ the *Minkowski–Firey L_p-combination* $a\cdot K +_p b\cdot L$ can be defined by

$$h^p_{a\cdot K+_p b\cdot L} = ah^p_K + bh^p_L. \tag{2.2}$$

Obviously, $a\cdot K +_p b\cdot L \in \mathcal{K}_o^n$.

Let $I \subseteq [0,\infty)$ be an interval in $\mathbb{R}$. The left derivative and right derivative of a function $f\colon I \to \mathbb{R}$ are denoted by f'_l and f'_r, respectively.

For a convex body $K \in \mathcal{K}_o^n$, the *cone-volume measure* V_K of K is defined as

$$dV_K = \frac{1}{n} h_K\, dS_K.$$

Observing that

$$|K| = \int_{S^{n-1}} dV_K(u),$$

we can define the *cone-volume probability measure* $\overline{V}_K$ of K by

$$\overline{V}_K = \frac{dV_K}{|K|}\,.$$

Let $I \subset \mathbb{R}$ be an interval containing the origin and suppose that $h_\lambda(u) = h(\lambda,u)\colon I \times S^{n-1} \to (0,\infty)$ is continuous. For fixed $\lambda \in I$, one can define the Wulff shape (or Alekdandrov body) associated with the function h_λ as

$$K_\lambda = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x\cdot u \le h(\lambda,u)\right\}.$$

It is well-known that

$$h_{K_\lambda} \le h_\lambda \quad \text{and} \quad h_{K_\lambda} = h_\lambda, \quad \text{a.e. with respect to } S_{K_\lambda},$$

for each $\lambda \in I$.

The following Aleksandrov lemma (see e.g., [1, p.103], [9, Lemma 1], or [18, P.345]) will be needed.

LEMMA 2.1. *Let $h(\lambda,u)\colon I\times S^{n-1}\to(0,\infty)$ be continuous, where I is an interval such that $[0,1]\subseteq I\subset\mathbb{R}$. Suppose that the convergence in*

$$h'_+(0,u)=\lim_{\lambda\to0^+}\frac{h(\lambda,u)-h(0,u)}{\lambda}$$

is uniform on S^{n-1}. If $\{K_\lambda\}_{\lambda\in I}$ is the family of Wulff shapes associated with h_λ, then

$$\lim_{\lambda\to0^+}\frac{|K_\lambda|-|K_0|}{\lambda}=\int_{S^{n-1}}h'_+(0,u)\,dS_{K_0}(u).$$

Suppose $K,L\in\mathcal{K}_o^n$. For $\varphi\in\Phi$, the *φ-mixed volume* $V_\varphi(K,L)$ can be defined as

$$V_\varphi(K,L)=\int_{S^{n-1}}\varphi\Big(\frac{h_L}{h_K}\Big)\,dV_K \tag{2.3}$$

We define the *normalized φ-mixed volume* $\overline{V}_\varphi(K,L)$ of $K,L\in\mathcal{K}_o^n$ as

$$\overline{V}_\varphi(K,L)=\varphi^{-1}\Big(\frac{V_\varphi(K,L)}{|K|}\Big)=\varphi^{-1}\Big(\int_{S^{n-1}}\varphi\Big(\frac{h_L}{h_K}\Big)\,d\overline{V}_K\Big). \tag{2.4}$$

In particular, if $\varphi(t)=t^p$ with $p>0$, the normalized φ-mixed volume $\overline{V}_\varphi(K,L)$ reduces to the *normalized L_p mixed volume* $\overline{V}_p(K,L)$ of $K,L\in\mathcal{K}_o^n$:

$$\overline{V}_p(K,L)=\Big(\int_{S^{n-1}}\Big(\frac{h_L}{h_K}\Big)^p\,d\overline{V}_K\Big)^{1/p}.$$

As $p\to0$, it leads to the *normalized* log-*mixed volume* $\overline{V}_0(K,L)$ of $K,L\in\mathcal{K}_o^n$:

$$\overline{V}_0(K,L)=\exp\Big(\int_{S^{n-1}}\log\frac{h_L}{h_K}\,d\overline{V}_K\Big). \tag{2.5}$$

3. Equivalence of the φ-Minkowski mixed volume and the φ-Brunn–Minkowski inequalities

In [2], Böröczky et al. proved the equivalence of the log-Brunn–Minkowski and the log-Minkowski mixed volume inequalities, as well as the equivalence of the L_p-Brunn–Minkowski and the L_p-Minkowski mixed volume inequalities for $p>0$. In this section, we shall establish the equivalence of the φ-Brunn–Minkowski and the φ-Minkowski mixed volume inequalities. This provides a unified treat for the log-case, the L_p-case, and the Orlicz case. The last one of them is a natural generalization of the L_p-case for $p>1$; see, e.g., [6,20] for the details of the Orlicz–Brunn–Minkowski inequality.

LEMMA 3.1. *Suppose $\varphi \in \Phi$ and $K, L \in \mathcal{K}_o^n$. If $\lambda \in (0,1)$, then h_λ converges to h_K uniformly on S^{n-1} as $\lambda \to 0^+$.*

PROOF. We first prove $\lim_{\lambda \to 0^+} h_\lambda = h_K$. In fact, we only need to show that for any sequence $\{\lambda_i\} \subset (0,1)$ converging to $\bar\lambda \in [0,1]$ as $i \to \infty$, it follows that $\lim_{i\to\infty} h_{\lambda_i} = h_{\bar\lambda}$ on S^{n-1}. Suppose $\lambda_i \in (0,1)$, we have

$$\varphi(1) = (1-\lambda_i)\varphi\Big(\frac{h_K}{h_{\lambda_i}}\Big) + \lambda_i \varphi\Big(\frac{h_L}{h_{\lambda_i}}\Big) < \varphi\Big(\frac{h_K + h_L}{h_{\lambda_i}}\Big).$$

This, together with the strict monotonicity of φ^{-1}, gives $h_{\lambda_i} < h_K + h_L$ and shows that h_{λ_i} is bounded. Thus, the sequence $\{h_{\lambda_i}\}$ has a convergent subsequence (denoted also by h_{λ_i}) converging to $h_{\bar\lambda'}$ for some $\bar\lambda' \in [0,1]$. By the continuity of φ, we see that $h_{\bar\lambda'} > 0$ and

$$\varphi(1) = \lim_{i\to\infty} \Big[(1-\lambda_i)\varphi\Big(\frac{h_K}{h_{\lambda_i}}\Big) + \lambda_i\varphi\Big(\frac{h_L}{h_{\lambda_i}}\Big)\Big] = (1-\bar\lambda)\varphi\Big(\frac{h_K}{h_{\bar\lambda'}}\Big) + \bar\lambda\varphi\Big(\frac{h_L}{h_{\bar\lambda'}}\Big).$$

That proves $h_{\bar\lambda'} = h_{\bar\lambda}$. In particular, if $\bar\lambda = 0$, then from $h_0 = h_K$ we achieve the desired convergence.

Next we show that the convergence is uniform. It is easily seen that over S^{n-1} there exists a $\bar c > 0$ such that $h_\lambda > \bar c$ for any $\lambda \in [0,1]$. To this end, we let $\bar c_1 B_2^n \subset K$ and $\bar c_2 B_2^n \subset L$ with $\bar c_1, \bar c_2 > 0$, and set $\bar c = \min\{\bar c_1, \bar c_2\}$. By (1.7) and the log-concavity of φ^{-1}, we obtain

$$\begin{aligned} &= \log \circ \varphi^{-1}\Big((1-\lambda)\varphi\Big(\frac{h_K}{h_\lambda}\Big) + \lambda\varphi\Big(\frac{h_L}{h_\lambda}\Big)\Big) \\ &\geq (1-\lambda)\log\frac{h_K}{h_\lambda} + \lambda \log \frac{h_L}{h_\lambda} = \log \frac{h_K^{1-\lambda} h_L^\lambda}{h_\lambda}. \end{aligned}$$

Thus, $h_\lambda \geq h_K^{1-\lambda} h_L^\lambda > \bar c$.

Let $0 < M < \infty$ be such that $L \subset MB_2^n$, $K \subset MB_2^n$, and $h_\lambda \leq M$. Define

$$\alpha = \sup_{u \in S^{n-1}} \frac{h_L(u)}{h_\lambda(u)} \leq \frac{M}{\bar c} < \infty; \quad \beta = \inf_{u\in S^{n-1}} \frac{h_L(u)}{h_\lambda(u)} \geq \frac{\bar c}{M} > 0.$$

If $h_K \leq h_\lambda$, then from the fact that $\varphi \in \Phi$ is strictly increasing, and (1.8) we have

$$\varphi(1) - \lambda\varphi(\alpha) \leq (1-\lambda)\varphi\Big(\frac{h_K}{h_\lambda}\Big) \leq \varphi\Big(\frac{h_K}{h_\lambda}\Big).$$

If λ is small enough, then $\varphi(1) - \lambda\varphi(\alpha) > 0$ and hence

$$h_\lambda \varphi^{-1}(\varphi(1) - \lambda\varphi(\alpha)) \leq h_K.$$

It follows that

$$0 \le h_\lambda - h_K \le h_\lambda(1 - \varphi^{-1}(\varphi(1) - \lambda\varphi(\alpha))) \le M_1(\lambda), \tag{3.1}$$

where $M_1(\lambda) = M(1 - \varphi^{-1}(\varphi(1) - \lambda\varphi(\alpha)))$.

If $h_K \ge h_\lambda$, then

$$\varphi(1) - \lambda\varphi(\beta) \ge (1-\lambda)\varphi(h_K/h_\lambda),$$

which implies

$$h_K \le h_\lambda \varphi^{-1}\Big(\frac{\varphi(1) - \lambda\varphi(\beta)}{1-\lambda}\Big).$$

Observing $\frac{\varphi(1)-\lambda\varphi(\beta)}{1-\lambda} > 0$ and $\varphi^{-1}\big(\frac{\varphi(1)-\lambda\varphi(\beta)}{1-\lambda}\big) > 1$, we see that

$$0 \le h_K - h_\lambda \le h_K\left(1 - \frac{1}{\varphi^{-1}\big(\frac{\varphi(1)-\lambda\varphi(\beta)}{1-\lambda}\big)}\right) \le M_2(\lambda), \tag{3.2}$$

where $M_2(\lambda) = M\Big(1 - \frac{1}{\varphi^{-1}(\frac{\varphi(1)-\lambda\varphi(\beta)}{1-\lambda})}\Big)$.

Combining (3.1) with (3.2) shows that

$$|h_\lambda(u) - h_K(u)| \le M(\lambda) = \max\big\{M_1(\lambda), M_2(\lambda)\big\}$$

holds for all $u \in S^{n-1}$. Since $M(\lambda) \to 0^+$ as $\lambda \to 0^+$, we confirm that the convergence $\lim_{\lambda\to 0^+} h_\lambda = h_K$ is uniform on S^{n-1}. □

LEMMA 3.2. *Suppose $\lambda \in (0,1)$ and $K, L \in \mathcal{K}_o^n$. If $\varphi \in \Phi$, then*

$$\lim_{\lambda\to 0^+} \frac{h_\lambda - h_K}{\lambda} = \frac{h_K}{\varphi'(1)}\Big[\varphi\Big(\frac{h_L}{h_K}\Big) - \varphi(1)\Big]. \tag{3.3}$$

If, in addition, φ is convex (or concave) on $(0,\infty)$ then the convergence in (3.3) is uniform on S^{n-1}.

PROOF. From Lemma 3.1, (1.8), and the continuity of φ, we have

$$\begin{aligned}\lim_{\lambda\to 0^+} \frac{h_\lambda - h_K}{\lambda} &= \lim_{\lambda\to 0^+} h_\lambda \lim_{\lambda\to 0^+} \frac{1 - h_K/h_\lambda}{\lambda}\\ &= h_K \lim_{\lambda\to 0^+} \frac{1 - h_K/h_\lambda}{\varphi(1) - \varphi(h_K/h_\lambda)} \lim_{\lambda\to 0^+} \frac{\varphi(1) - \varphi(h_K/h_\lambda)}{\lambda}\\ &= \frac{h_K}{\varphi'(1)} \lim_{\lambda\to 0^+} \big[\varphi(h_L/h_\lambda) - \varphi(h_K/h_\lambda)\big] = \frac{h_K}{\varphi'(1)}\big[\varphi(h_L/h_K) - \varphi(1)\big].\end{aligned}$$

Let $t_\lambda = \varphi(\frac{h_K}{h_\lambda})$. Then $\frac{h_K}{h_\lambda} \to 1$ as $\lambda \to 0^+$. Since $\varphi \in \Phi$ is convex (concave) on $(0, \infty)$, φ^{-1} is concave (convex) on $(0, \infty)$. This together with the facts that $t_0 = \varphi(1)$ and $\varphi^{-1}(t_\lambda) = 1$ gives

$$\frac{1}{\varphi'(1)} \underset{(\geq)}{\leq} \frac{1 - \varphi^{-1}(t_\lambda)}{t_0 - t_\lambda} \underset{(\geq)}{\leq} \frac{1}{\varphi'(\varphi^{-1}(t_\lambda))}.$$

Observing that

$$\frac{h_\lambda - h_K}{\lambda} = h_\lambda \frac{1 - \varphi^{-1}(t_\lambda)}{t_0 - t_\lambda} \Big[\varphi\Big(\frac{h_L}{h_\lambda}\Big) - \varphi\Big(\frac{h_K}{h_\lambda}\Big)\Big],$$

we have

$$\frac{h_\lambda\big[\varphi(\frac{h_L}{h_\lambda}) - \varphi(\frac{h_K}{h_\lambda})\big]}{\varphi'(1)} \underset{(\geq)}{\leq} \frac{h_\lambda - h_K}{\lambda} \underset{(\geq)}{\leq} \frac{h_\lambda\big[\varphi(\frac{h_L}{h_\lambda}) - \varphi(\frac{h_K}{h_\lambda})\big]}{\varphi'(\varphi^{-1}(t_\lambda))}. \tag{3.4}$$

By Lemma 3.1, we see that $h_\lambda \to h_K$ uniformly on S^{n-1} as $\lambda \to 0^+$, which implies that $\frac{h_L}{h_\lambda}$, $\frac{h_K}{h_\lambda}$ converge uniformly to $\frac{h_L}{h_K}$, 1, respectively, on S^{n-1}. Thus, $\frac{h_L}{h_\lambda}$, $\frac{h_K}{h_\lambda}$ are uniformly bounded on some compact interval $I \subset (0, \infty)$. From the fact that φ is uniformly continuous on any compact subset of $(0, \infty)$, we see that the left side of (3.4) converges uniformly to $\frac{h_K}{\varphi'(1)}\big[\varphi(\frac{h_L}{h_K}) - \varphi(1)\big]$.

In order to show that the right hand side of (3.4) also converges uniformly to $\frac{h_K}{\varphi'(1)}\big[\varphi(\frac{h_L}{h_K}) - \varphi(1)\big]$, we need to prove that $\varphi'(\frac{h_K}{h_\lambda})$ converges uniformly to $\varphi'(1)$ on S^{n-1} as $\lambda \to 0^+$. But this is a direct consequence of the continuity of φ' on an open interval $I \subset (0, \infty)$ such that $1 \in \operatorname{int} I$, and the fact that the convergence in $\lim_{\lambda \to 0^+} \frac{h_K}{h_\lambda} = 1$ is uniform on S^{n-1}. $\square$

THEOREM 3.3. *Let $\lambda \in [0, 1]$ and $\varphi \in \Phi$ be, in addition, convex (or concave) on $(0, \infty)$. If $K, L \in \mathcal{K}_o^n$ are o-symmetric convex bodies in $\mathbb{R}^n$, and $Q_\lambda = (1 - \lambda) \cdot K +_\varphi \lambda \cdot L$, then the φ-Brunn–Minkowski inequality*

$$(1 - \lambda)\varphi\bigg(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\bigg) + \lambda\varphi\bigg(\frac{|L|^{1/n}}{|Q_\lambda|^{1/n}}\bigg) \leq \varphi(1) \tag{3.5}$$

is equivalent to the φ-Minkowski mixed volume inequality

$$\overline{V}_\varphi(K, L)^n \geq |L|/|K|. \tag{3.6}$$

PROOF. For $\lambda \in [0, 1]$, let $Q_\lambda = (1 - \lambda) \cdot K +_\varphi \lambda \cdot L$.

Firstly, suppose that the φ-Minkowski mixed volume inequality (3.6) holds. From (1.8), (2.4), the fact that $h_\lambda = h_{Q_\lambda}$ a.e. with respect to the surface area measure S_{Q_λ}, and (3.6), we have

$$
\begin{aligned}
\varphi(1) &= \frac{1}{n|Q_\lambda|}\int_{S^{n-1}} \varphi(1) h_{Q_\lambda}\, dS_{Q_\lambda} \\
&= \frac{1}{n|Q_\lambda|}\int_{S^{n-1}} \Big[(1-\lambda)\varphi\Big(\frac{h_K}{h_\lambda}\Big) + \lambda\varphi\Big(\frac{h_L}{h_\lambda}\Big)\Big] h_{Q_\lambda}\, dS_{Q_\lambda} \\
&= (1-\lambda)\int_{S^{n-1}} \varphi\Big(\frac{h_K}{h_{Q_\lambda}}\Big)\, d\overline{V}_{Q_\lambda} + \lambda\int_{S^{n-1}} \varphi\Big(\frac{h_L}{h_{Q_\lambda}}\Big)\, d\overline{V}_{Q_\lambda} \\
&= (1-\lambda)\varphi\left(\overline{V}_\varphi(Q_\lambda, K)\right) + \lambda\varphi\left(\overline{V}_\varphi(Q_\lambda, L)\right) \\
&\ge (1-\lambda)\varphi\bigg(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\bigg) + \lambda\varphi\bigg(\frac{|L|^{1/n}}{|Q_\lambda|^{1/n}}\bigg).
\end{aligned}
\tag{3.7}
$$

Conversely, we shall show that if the φ-Brunn–Minkowski inequality (3.5) holds, then the φ-Minkowski mixed volume inequality (3.6) holds accordingly. To this end, we define a function $f\colon [0,1] \mapsto \mathbb{R}$ as

$$
f(\lambda) = (1-\lambda)\varphi\bigg(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\bigg) + \lambda\varphi\bigg(\frac{|L|^{1/n}}{|Q_\lambda|^{1/n}}\bigg) - \varphi(1).
$$

From (3.5) and the fact that $f(0) = 0$, we see that

$$
\begin{aligned}
0 &\ge \lim_{\lambda\to 0^+} \frac{f(\lambda) - f(0)}{\lambda} = \lim_{\lambda\to 0^+} \frac{(1-\lambda)\varphi\big(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\big) + \lambda\varphi\big(\frac{|L|^{1/n}}{|Q_\lambda|^{1/n}}\big) - \varphi(1)}{\lambda} \\
&= \varphi\bigg(\frac{|L|^{1/n}}{|K|^{1/n}}\bigg) - \varphi(1) + \lim_{\lambda\to 0^+} \frac{\varphi\big(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\big) - \varphi(1)}{\lambda}.
\end{aligned}
\tag{3.8}
$$

Since $\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}} \to 1$ as $\lambda \to 0^+$, we obtain that

$$
\begin{aligned}
&\lim_{\lambda\to 0^+} \frac{\varphi\big(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\big) - \varphi(1)}{\lambda} \\
&= \lim_{\lambda\to 0^+} \frac{\varphi\big(\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}}\big) - \varphi(1)}{\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}} - 1} \cdot \lim_{\lambda\to 0^+} \frac{\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}} - 1}{\lambda} = \varphi'(1) \lim_{\lambda\to 0^+} \frac{\frac{|K|^{1/n}}{|Q_\lambda|^{1/n}} - 1}{\lambda} \\
&= \frac{\varphi'(1)}{|K|^{1/n}} \lim_{\lambda\to 0^+} \frac{|K|^{1/n} - |Q_\lambda|^{1/n}}{\lambda} = -\frac{\varphi'(1)}{n|K|} \frac{d}{d\lambda}\bigg|_{\lambda=0} |Q_\lambda|.
\end{aligned}
\tag{3.9}
$$

Further, by Lemmas 2.1 and 3.2, we have

$$\frac{d}{d\lambda}\Big|_{\lambda=0}|Q_\lambda| = \int_{S^{n-1}} \frac{h_K}{\varphi'(1)}\Big[\varphi\Big(\frac{h_L}{h_K}\Big) - \varphi(1)\Big]\,dS_K. \tag{3.10}$$

It now follows from (3.8)–(3.10) and definition (2.3) that

$$\varphi\Big(\frac{|L|^{1/n}}{|K|^{1/n}}\Big) - \frac{V_\varphi(K,L)}{|K|} \le 0. \tag{3.11}$$

In view of definition (2.4), we see that (3.11) is exactly the φ-Minkowski mixed volume inequality (3.6). □

4. The φ-Brunn–Minkowski inequality for planar convex bodies

We shall show that once the log-Brunn–Minkowski inequality holds, then so does the φ-Brunn–Minkowski inequality. This assertion is based on the following fact: If $\varphi \in \Phi$ is such that φ^{-1} is strictly log-concave, then the log-Minkowski mixed volume inequality is sharper than the φ-Minkowski mixed volume inequality.

As mentioned in the Introduction, the authors in [2] showed that (see [16] for an alternate proof): If K, L are o-symmetric convex bodies in the plane, then

$$\int_{S^1} \log\frac{h_L}{h_K}\,d\overline{V}_K \ge \frac{1}{2}\log\frac{|L|}{|K|}\,, \tag{4.1}$$

with equality if and only if, either K and L are dilates or K and L are parallelograms with parallel sides.

Theorem 4.1. *Let $\varphi \in \Phi$ and $K, L \in \mathcal{K}_o^2$ be o-symmetric convex bodies in the plane. Then*

$$\int_{S^1} \varphi\Big(\frac{h_L}{h_K}\Big)\,d\overline{V}_K \ge \varphi\Big(\frac{|L|^{1/2}}{|K|^{1/2}}\Big), \tag{4.2}$$

with equality if and only if K and L are dilates.

Proof. We first claim that

$$\overline{V}_\varphi(K,L) \ge \overline{V}_0(K,L), \tag{4.3}$$

with equality if and only if K and L are dilates. In fact, from the log-concavity of φ^{-1}, we have

$$\int_{S^{n-1}} \log\Big(\frac{h_L}{h_K}\Big)\,d\overline{V}_K \le \log\circ\varphi^{-1}\Big(\int_{S^{n-1}} \varphi\Big(\frac{h_L}{h_K}\Big)\,d\overline{V}_K\Big). \tag{4.4}$$

Denote $h(u) = h_L(u)/h_K(u)$ and define a function $\theta\colon S^{n-1} \to \varphi((0,\infty))$ by $\theta(u) = \varphi(h(u))$. We obtain that $h(u) = \varphi^{-1}(\theta)$. Then (4.4) becomes

$$\int_{S^{n-1}} \log\circ\varphi^{-1}(\theta(u))\, d\overline{V}_K(u) \le \log\circ\varphi^{-1}\Big(\int_{S^{n-1}} \theta(u)\, d\overline{V}_K(u)\Big). \tag{4.5}$$

However, inequality (4.5) is equivalent to the concavity of the composite function $\log\circ\varphi^{-1}$. Moreover, if the concavity of $\log\circ\varphi^{-1}$ is strict, then the equality holds if and only if there exists a constant $c > 0$ such that $\varphi(h(u)) = c$, that is, $h(u) = \varphi^{-1}(c)$ for all $u \in S^{n-1}$. That proves that equality in (4.3) holds if and only if K and L are dilates.

Now the inequality (4.3) together with (4.1) gives the desired inequality (4.2). The equality follows from the equality conditions of (4.1) and (4.3). □

THEOREM 4.2. *Let $\lambda \in [0,1]$ and let $\varphi \in \Phi$ be concave on $(0,\infty)$. If $K, L \in \mathcal{K}_o^2$ are o-symmetric convex bodies in the plane and $Q_\lambda = (1-\lambda)\cdot K +_\varphi \lambda\cdot L$, then*

$$(1-\lambda)\varphi\Big(\frac{|K|^{1/2}}{|Q_\lambda|^{1/2}}\Big) + \lambda\varphi\Big(\frac{|L|^{1/2}}{|Q_\lambda|^{1/2}}\Big) \le \varphi(1). \tag{4.6}$$

When $\lambda \in (0,1)$, equality in the inequality holds if and only if K and L are dilates.

PROOF. Inequality (4.6) is an immediate consequence of Theorem 3.3 and Theorem 4.1.

From (3.7) and the equality conditions of Theorem 4.1, we see that the equality in inequality (4.6) holds if and only if K and L are dilates. □

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