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AN IMPROVED UPPER BOUND FOR THE SIZE OF A SUNFLOWER-FREE FAMILY

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Abstract. We combine here Tao's slice-rank bounding method and Gröbner basis techniques and apply it to the Erdős–Rado Sunflower Conjecture.

Let $0 \le k \le n$ be integers. We prove that if $\mathcal{F}$ is a k-uniform family of subsets of $[n]$ without a sunflower with 3 petals, then

$$|\mathcal{F}| \le 3 \binom{n}{\lfloor n/3 \rfloor}.$$

This result allows us to improve slightly a recent upper bound of Naslund and Sawin for the size of a sunflower-free family in $2^{[n]}$.

1. Introduction

Let $[n]$ stand for the set $\{1, 2, \dots, n\}$. We denote the family of all subsets of $[n]$ by $2^{[n]}$. Let X be a fixed subset of $[n]$. For an integer $0 \le k \le n$ we denote by $\binom{X}{k}$ the family of all k element subsets of X. This is the *complete* k-uniform family. We say that a family $\mathcal{F}$ is *k-uniform*, if $|F| = k$ for each $F \in \mathcal{F}$.

A family $\mathcal{F} = \{F_1, \dots, F_t\}$ of subsets of $[n]$ is a *sunflower* (or a Δ-*system*) with t petals if

$$F_i \cap F_j = \bigcap_{s=1}^{t} F_s$$

for each $1 \le i < j \le t$. Here the intersection of the members of a sunflower form its *kernel*.

Erdős and Rado conjectured the following famous statement in [6].

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Conjecture 1. *For each $t > 2$, there exists a constant $C(t)$ such that if $\mathcal{F}$ is a k-uniform set system with more than $C(t)^k$ members, then $\mathcal{F}$ contains a sunflower with t petals.*

Erdős offered 1000 dollars for the proof or disproof of this conjecture for $t = 3$ (see [5]).

Erdős and Rado gave also an upper bound for the size of a k-uniform family without a sunflower with t petals in [6].

Theorem 1.1 (Sunflower theorem). *If $\mathcal{F}$ is a k-uniform set system with more than*

$$k!\,(t-1)^k$$

members, then $\mathcal{F}$ contains a sunflower with t petals.

Define $F(n,t)$ to be the largest integer so that there exists a family $\mathcal{F}$ of subsets of $[n]$ which does not contain a sunflower with t petals and $|\mathcal{F}| = F(n,t)$.

Define β_t as

$$\beta_t := \limsup_{n\to\infty} F(n,t)^{1/n}.$$

Naslund and Sawin gave the following upper bound for the size of a sunflower-free family in [8]. Their proof based on Tao's slice-rank bounding method (see [9]).

Theorem 1.2. *Let $\mathcal{F}$ be a family of subsets of $[n]$ without a sunflower with 3 petals. Then*

$$|\mathcal{F}| \le 3(n+1) \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i}.$$

As a simple consequence of Theorem 1.2 Naslund and Sawin derived the following upper bound for β_3:

$$\beta_3 \le \frac{3}{2^{2/3}} = 1.88988\ldots.$$

Our main result is the following new upper bound for the size of a sunflower-free family. In the proof we mix Tao's slice-rank bounding method with Gröbner basis techniques. Our proof is a simple modification of the proof of [8, Theorem 1].

Theorem 1.3. *Let $\mathcal{F}$ be a k-uniform family of subsets of $[n]$ without a sunflower with 3 petals. Then*

$$|\mathcal{F}| \le 3\binom{n}{\lfloor n/3 \rfloor}.$$

Theorem 1.3 implies easily the following Corollary.

COROLLARY 1.4. *Let $\mathcal{F}$ be a sunflower-free family of subsets of $[n]$. Then*

$$|\mathcal{F}| \le 3(n+1)\binom{n}{\lfloor n/3 \rfloor}$$

In Section 2 we collected some useful preliminaries about the slice rank of functions and Gröbner bases. We present our proofs in Section 3.

2. Preliminaries

2.1. Slice rank. We define first the slice rank of functions. This definition appeared first in Tao's blog [9]. Let A be a fixed finite set, $m \ge 1$ be a fixed integer and $\mathbb{F}$ be a field. Recall that a function $F\colon A^m \to \mathbb{F}$ has *slice-rank* one, if it has the form

$$(x_1, \dots, x_m) \mapsto f(x_i)\, g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m),$$

for some $i = 1, \dots, m$ and some functions $f\colon A \to \mathbb{F}$, $g\colon A^{m-1} \to \mathbb{F}$. The slice rank $\mathrm{rank}(F)$ of a function $F\colon A^m \to \mathbb{F}$ is the least number of rank one functions needed to generate F as a linear combination. For instance, if $m = 2$, then we get back the usual definition of the rank of a function $F\colon A^2 \to \mathbb{F}$.

Let $\delta_\alpha(\mathbf{x})$ denote the Kronecker delta function. Tao proved the following result about the slice rank of diagonal hyper-matrices in [9, Lemma 1] (see also [2, Lemma 4.7]).

THEOREM 2.1. *Let $\mathbb{F}$ be a fixed field, A be a finite subset and denote $c_\alpha \in \mathbb{F}$ a coefficient for each $\alpha \in A$. Let $m \ge 2$ be a fixed integer. Consider the function*

$$F(\mathbf{x}_1, \dots, \mathbf{x}_m) := \sum_{\alpha \in A} c_\alpha \delta_\alpha(\mathbf{x}_1) \dots \delta_\alpha(\mathbf{x}_m) : A^m \to \mathbb{F}.$$

Then

$$\mathrm{rank}(F) = \big|\{\alpha \in A : c_\alpha \ne 0\}\big|.$$

2.2. Gröbner theory. Let $\mathbb{F}$ be a field. In the following $\mathbb{F}[x_1, \dots, x_n] = \mathbb{F}[\mathbf{x}]$ denotes the ring of polynomials in commuting variables $x_1, \dots, x_n$ over $\mathbb{F}$. For a subset $F \subseteq [n]$ we write $\mathbf{x}_F = \prod_{j \in F} x_j$. In particular, $\mathbf{x}_\emptyset = 1$. We denote by $\mathbf{v}_F \in \{0,1\}^n$ the characteristic vector of a set $F \subseteq [n]$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, define $V(\mathcal{F})$ as the subset $\{\mathbf{v}_F : F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$. A polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$ can be considered as a function from $V(\mathcal{F})$ to $\mathbb{F}$ in a natural way.

We can describe several interesting properties of finite set systems $\mathcal{F} \subseteq 2^{[n]}$ as statements about *polynomial functions on* $V(\mathcal{F})$. As for polynomial functions on $V(\mathcal{F})$, it is natural to consider the ideal $I(V(\mathcal{F}))$:

$$I(V(\mathcal{F})) := \left\{ f \in \mathbb{F}[\mathbf{x}] : \; f(\mathbf{v}) = 0 \text{ whenever } \mathbf{v} \in V(\mathcal{F}) \right\}.$$

Clearly the substitution gives an $\mathbb{F}$ algebra homomorphism from $\mathbb{F}[\mathbf{x}]$ to the $\mathbb{F}$ algebra of $\mathbb{F}$-valued functions on $V(\mathcal{F})$. It is easy to verify that this homomorphism is surjective, and the kernel is exactly $I(V(\mathcal{F}))$. Hence we can identify the algebra $\mathbb{F}[\mathbf{x}]/I(V(\mathcal{F}))$ and the algebra of $\mathbb{F}$ valued functions on $V(\mathcal{F})$. It follows that

$$\dim_{\mathbb{F}} \mathbb{F}[\mathbf{x}] \big/ I(V(\mathcal{F})) = |\mathcal{F}|.$$

Now we recall some basic facts about to Gröbner bases and standard monomials. For details we refer to [1], [3], [4].

A linear order $\prec$ on the monomials over variables $x_1, x_2, \dots, x_m$ is a *term order*, or *monomial order*, if 1 is the minimal element of $\prec$, and $\mathbf{uw} \prec \mathbf{vw}$ holds for any monomials $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ with $\mathbf{u} \prec \mathbf{v}$. Two important term orders are the lexicographic order $\prec_l$ and the deglex order $\prec_d$. We have

$$x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \prec_l x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$$

iff $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$. The definition of the deglex order is similar: we have $\mathbf{u} \prec_d \mathbf{v}$ iff either $\deg \mathbf{u} < \deg \mathbf{v}$, or $\deg \mathbf{u} = \deg \mathbf{v}$, and $\mathbf{u} \prec_l \mathbf{v}$.

The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial $f \in \mathbb{F}[\mathbf{x}]$ is the $\prec$-largest monomial which appears with nonzero coefficient in the canonical form of f as a linear combination of monomials. We denote by $\mathrm{lc}(f)$ the leading coefficient of f, where $f \in \mathbb{F}[\mathbf{x}]$ is a a nonzero polynomial.

Let I be an ideal of $\mathbb{F}[\mathbf{x}]$. We say that a finite subset $\mathcal{G} \subseteq I$ is a *Gröbner basis* of I if for every $f \in I$ there exists a $g \in \mathcal{G}$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. In other words, the leading monomials $\mathrm{lm}(g)$ for $g \in \mathcal{G}$ generate the semigroup ideal of monomials $\{\mathrm{lm}(f) : f \in I\}$. Consequently $\mathcal{G}$ is actually a basis of I, i.e. $\mathcal{G}$ generates I as an ideal of $\mathbb{F}[\mathbf{x}]$ (cf. [4, Corollary 2.5.6]). A well-known fact is (cf. [3, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal I of $\mathbb{F}[\mathbf{x}]$ has a Gröbner basis.

A monomial $\mathbf{w} \in \mathbb{F}[\mathbf{x}]$ is a *standard monomial for* I if it is not a leading monomial for any $f \in I$. We denote by $\mathrm{sm}(I)$ the set of standard monomials of I.

Let $\mathcal{F} \subseteq 2^{[n]}$ be a set family. Then the characteristic vectors in $V(\mathcal{F})$ are all 0,1-vectors, consequently the polynomials $x_i^2 - x_i$ all vanish on $V(\mathcal{F})$. It follows that the standard monomials of the ideal $I(\mathcal{F}) := I(V(\mathcal{F}))$ are square-free monomials.

Now we give a short introduction to the notion of reduction. Let $\prec$ be a fixed term order. Let $\mathcal{G}$ be a set of polynomials in $\mathbb{F}[x_1, \dots, x_n]$ and let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a fixed polynomial. We can reduce f by the set $\mathcal{G}$ with respect to $\prec$. This gives a new polynomial $h \in \mathbb{F}[x_1, \dots, x_n]$.

Here *reduction* means that we possibly repeatedly replace monomials in f by smaller ones (with respect to $\prec$) in the following way: if w is a monomial occurring in f and $\mathrm{lm}(g)$ divides w for some $g \in \mathcal{G}$ (i.e. $w = \mathrm{lm}(g)u$ for some monomial u), then we replace w in f with $u(\mathrm{lm}(g) - \frac{g}{lc(g)})$. It is easy to verify that the monomials in $u(\mathrm{lm}(g) - \frac{g}{lc(g)})$ are $\prec$-smaller than w.

It is a key fact that $\mathrm{sm}(I)$ gives a basis of the $\mathbb{F}$-vector-space $\mathbb{F}[\mathbf{x}]/I$ in the sense that every polynomial $g \in \mathbb{F}[\mathbf{x}]$ can be uniquely expressed as $h + f$ where $f \in I$ and h is a unique $\mathbb{F}$-linear combination of monomials from $\mathrm{sm}(I)$. Hence if $g \in \mathbb{F}[\mathbf{x}]$ is an arbitrary polynomial and $\mathcal{G}$ is a Gröbner basis of I, then we can reduce g with $\mathcal{G}$ into a linear combination of standard monomials for I. In particular, $f \in I$ if and only if f can be $\mathcal{G}$-reduced to 0.

Let $0 \le k \le n/2$, where k and n are integers. Denote by $\mathcal{M}_{k,n}$ the set of all monomials $\mathbf{x}_G$ such that $G = \{s_1 < s_2 < \dots < s_j\} \subset [n]$ for which $j \le k$ and $s_i \ge 2i$ holds for every i, $1 \le i \le j$. These monomials $\mathbf{x}_G$ are the *ballot monomials* of degree at most k. If n is clear from the context, then we write $\mathcal{M}_k$ instead of the more precise $\mathcal{M}_{k,n}$. It is known that

$$|\mathcal{M}_k| = \binom{n}{k}.$$

Let $\mathcal{M}_{k,n}$ denote the set of all sets $H = \{s_1 < s_2 < \dots < s_j\} \subset [n]$ for which $j \le k$ and $s_i \ge 2i$ holds for every i, $1 \le i \le j$.

In [7] we described completely the Gröbner bases and the standard monomials of the complete uniform families of all k element subsets of $[n]$.

Theorem 2.2. *Let $\prec$ an arbitrary term order such that $x_1 \prec \dots \prec x_n$. Let $0 \le k \le n$ and $j := min(k, n-k)$. Then*

$$\mathrm{sm}\left(V\binom{[n]}{k}\right) = \mathcal{M}_{j,n}.$$

Let $0 \le k \le n$ be arbitrary integers. Define the vector system

$$\mathcal{F}(n,k,3) := V\left(\binom{[n]}{k}\right) \times V\left(\binom{[n]}{k}\right) \times V\left(\binom{[n]}{k}\right) \subseteq \{0,1\}^{3n}.$$

It is easy to verify the following Corollary.

Corollary 2.3. *Let $\prec$ an arbitrary term order such that $x_1 \prec \dots \prec x_n$. Let $0 \le k \le n$ be arbitrary integers. Let $j := min(k, n-k)$. Then*

$$\mathrm{sm}(\mathcal{F}(n,k,3)) = \left\{ x_{M_1} \cdot y_{M_2} \cdot z_{M_3} : M_1, M_2, M_3 \in \mathcal{M}_{j,n} \right\} \subseteq \mathbb{F}[\mathbf{x},\mathbf{y},\mathbf{z}].$$

3. Proofs

We use in our proof essentially the same argument as the one appeared in [8, Theorem 1], but we extended their proof with a simple Gröbner basis technique.

Proof of Theorem 1.3. Let $\mathcal{F}$ be a k-uniform sunflower–free family of subsets of $[n]$. Let $H_1, H_2, H_3 \in \mathcal{F}$ be arbitrary subsets. Since $\mathcal{F}$ is sunflower-free, if

$$\mathbf{v}(H_1) + \mathbf{v}(H_2) + \mathbf{v}(H_3) \in \{0, 1, 3\}^n,$$

then $H_1 = H_2 = H_3$. Namely first suppose that $H_1 \neq H_2$, $H_1 \neq H_3$ and $H_2 \neq H_3$. Then the triple (H_1, H_2, H_3) is not a sunflower, hence there exist indices $1 \le i < j \le 3$ such that $(H_i \cap H_j) \setminus (H_1 \cap H_2 \cap H_3) \neq \emptyset$. Let $t \in (H_i \cap H_j) \setminus (H_1 \cap H_2 \cap H_3)$. Then $\mathbf{v}(H_1)_t + \mathbf{v}(H_2)_t + \mathbf{v}(H_3)_t = 2$.

Suppose that $H_1 \neq H_2$ but $H_2 = H_3$. Since $|H_1| = |H_2| = k$, we have $H_2 \setminus H_1 \neq \emptyset$. Let $t \in H_2 \setminus H_1$. Then it is easy to see that $\mathbf{v}(H_1)_t + \mathbf{v}(H_2)_t + \mathbf{v}(H_3)_t = 2$.

Consider the polynomial function

$$T : (V(\mathcal{F}))^3 \to \mathbb{R}$$

given by

$$T(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \prod_{i=1}^{n} (2 - (x_i + y_i + z_i))$$

for each $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{z} = (z_1, \ldots, z_n) \in V(\mathcal{F}) \subseteq V\binom{[n]}{k}$.

Let $\mathcal{G}$ denote a deglex Gröbner basis of the ideal $I := I(\mathcal{F}(n, k, 3))$. Let H denote the reduction of T via $\mathcal{G}$.

Then

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) = T(\mathbf{x}, \mathbf{y}, \mathbf{z}) \tag{1}$$

for each $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{z} = (z_1, \ldots, z_n) \in V(\mathcal{F}) \subseteq V\binom{[n]}{k}$, because we reduced T with a Gröbner basis of the ideal I.

On the other hand

$$T(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq 0 \text{ if and only if } \mathbf{x} = \mathbf{y} = \mathbf{z} \in V(\mathcal{F}),$$

hence by equation (1)

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq 0 \text{ if and only if } \mathbf{x} = \mathbf{y} = \mathbf{z} \in V(\mathcal{F}). \tag{2}$$

Let $j := \min(k, n-k)$.

Since $V(\mathcal{F}) \subseteq V\binom{[n]}{k}$, it follows from Corollary 2.3 that we can write $H(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as a linear combination of standard monomials

$$x_I y_K z_L,$$

where $x_I, y_K, z_L \in \mathcal{M}_{j,n}$ and $\deg(x_I y_K z_L) \leq n$. Here we used that $\mathcal{G}$ is a *deglex* Gröbner basis of the ideal I.

It follows from the pigeonhole principle that at least one of $|I|$, $|K|$ and $|L|$ is at most $\lfloor n/3 \rfloor$. First we can consider the contribution of the standard monomials to the sum for which $|I| \leq \lfloor \frac{n}{3} \rfloor$. We can regroup this contribution as

$$\sum_M x_M g_M(\mathbf{y}, \mathbf{z}),$$

where M ranges over those subsets $\{i_1, \dots, i_t\}$ of $[n]$ with $t \leq \lfloor n/3 \rfloor$ and $i_s \geq 2s$ for every $1 \leq s \leq t$. Here $g_M \colon (V(\mathcal{F}))^2 \to \mathbb{R}$ are some explicitly computable functions. The number of such sets M is at most $\binom{n}{\lfloor n/3 \rfloor}$, so this contribution has slice-rank at most $\binom{n}{\lfloor n/3 \rfloor}$.

The cases $|K| \leq \lfloor \frac{n}{3} \rfloor$ and $|L| \leq \lfloor \frac{n}{3} \rfloor$ can be treated the same way.

H and T are the same functions on $\mathcal{F}(n, k, 3)$, hence we get that

$$\mathrm{rank}(H) = \mathrm{rank}(T) \leq 3\binom{n}{\lfloor n/3 \rfloor}.$$

It follows from Theorem 2.1 and equation (2) that

$$\mathrm{rank}(H) = |\mathcal{F}|.$$

These together imply

$$|\mathcal{F}| \leq 3\binom{n}{\lfloor n/3 \rfloor}. \quad \square$$

Proof of Corollary 1.4. Let $\mathcal{F} \subseteq \{0,1\}^n$ be a fixed sunflower-free subset. Define the families

$$cF(s) := \mathcal{F} \cap \binom{[n]}{s}$$

for each $0 \leq s \leq n$. We can apply Theorem 1.3 for the family $\mathcal{F}(s)$ and we get

$$|\mathcal{F}(s)| \leq 3\binom{n}{\lfloor n/3 \rfloor}.$$

These together imply

$$|\mathcal{F}| = \sum_{s=0}^{n} |\mathcal{F}(s)| \leq 3(n+1)\binom{n}{\lfloor n/3 \rfloor}. \qquad \square$$

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