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ON THE STRUCTURE OF A MUTUALLY PERMUTABLE PRODUCT OF FINITE GROUPS

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Abstract. Let a finite group $G = AB$ be the product of the mutually permutable subgroups A and B. We investigate the structure of G given by conditions on conjugacy class sizes of elements in $A \cup B$. Some recent results are extended.

1. Introduction

All groups considered in this paper are finite. For an element x in G, by x^G we denote the conjugacy class containing x and by $|x^G|$ the length of x^G, that is, the number of elements in x^G. By $\pi(G)$ we denote the set of prime divisors of $|G|$ and by G_p we denote a Sylow p-subgroup of G, where $p \in \pi(G)$. If G is p-soluble we denote by $l_p(G)$ the p-length of G. We call an element x of G p-regular element (p-singular element) if $o(x)$ is prime to p (p divides $o(x)$).

The product $G = AB$ of the subgroups A and B of a group G is said to be a mutually permutable product of A and B if $UB = BU$ for all subgroups U of A and $AV = VA$ for all subgroups V of B. The relation between properties of A and B and the properties of G and the relation between conditions on conjugacy classes of a group and its structure have been extensively investigated. Surveys of these can be found at [1] and [3]. Recently conditions on

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the conjugacy classes of A and B have been used to investigate the structure of a mutually permutable product $G = AB$ ([2], [6]).

In this paper we prove an extension of [9, Theorem B].

THEOREM 1. *Let p be a prime in $\pi(G)$. Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B of G. Suppose that for every x of prime power order in $A \cup B$, $|x^G|$ is not divisible by p^{p-1}. Then G is p-soluble and the p-length of G is at most one.*

We also give examples to show that some of the results in [8] and [10] can not be extended in the same way to mutually permutable products.

2. Preliminaries

The proofs of our results rely heavily on results from the book [1]. For the convenience of the reader we state the results we use from this book.

LEMMA 2.1 [1, Lemma 4.1.10]. *Assume that the group G is a mutually permutable product of the subgroups A and B and that N is a normal subgroup of G. Then G/N is a mutually permutable product of AN/N and BN/N.*

LEMMA 2.2 [1, Theorem 4.3.11]. *Assume that the non-trivial group G is a mutually permutable product of the subgroups A and B. Then $A_G B_G$ is non-trivial.*

LEMMA 2.3 [1, Lemma 4.1.21]. *Assume that the group G is a mutually permutable product of the subgroups A and B. If U is a normal subgroup of G, then $(U \cap A)(U \cap B)$ is a normal subgroup of G.*

LEMMA 2.4 [1, Theorem 4.1.15]. *Assume that the group G is a mutually permutable product of the subgroups A and B. If A and B are p-soluble, then G is p-soluble.*

LEMMA 2.5 [1, Theorem 4.3.3]. *Assume that the group G is a mutually permutable product of the subgroups A and B.*

(1) *If N is a minimal normal subgroup of G, then $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$.*

(2) *If N is a minimal normal subgroup of G contained in A and $B \cap N = 1$, then $N \le C_G(A)$ or $N \le C_G(B)$. If furthermore N is not cyclic, then $N \le C_G(B)$.*

LEMMA 2.6 [9, Theorem B]. *Let G be a p-soluble group, where p is a fixed prime integer. Suppose that for every element x of prime power order in G, $|x^G|$ is not divisible by p^{p-1}. Then the p-length of G is at most one.*

The following facts are well known and we will use them without further reference.

(1) If N is a subnormal subgroup of G then $|x^N|\,|\,|x^G|$ for any $x \in G$;

(2) $|(xN)^{G/N}|\,|\,|x^G|$ if $x \in G$ and N is normal in G.

(3) Let N be a normal subgroup of G and p a prime. If xN is a p-element of G/N, then there is a p-element x_1 of G such that $xN = x_1N$.

3. Proof of Theorem 1

We know that G is p-soluble by Lemma 2.4.

Assume the the theorem is false and G is a counterexample with minimal order. Then $O_{p'}(G) = 1$. If N is a minimal normal subgroup of G then N is an elementary abelian p-group. By Lemma 2.1, we know that $G/N = AN/N \cdot BN/N$ satisfies the condition of the theorem and $l_p(G/N) \leq 1$. The minimality of G gives $N = \mathrm{Soc}(G) = O_p(G) = F(G) = C_G(N)$ and $\Phi(G) = 1$ by standard arguments. We can write $G = NH$ and $H \cap N = 1$ and $l_p(H) \leq 1$. By Lemma 2.2, we know that $A_G B_G \neq 1$. Hence we can assume that $A_G \neq 1$ and $N \leq A$. If $N \not\leq B$, then $N \cap B = 1$ by Lemma 2.5(1). If N is cyclic, then H is cyclic of order dividing $p - 1$, giving $l_p(G) = 1$, a contradiction. Hence N is not cyclic and so $N \leq C_G(B)$ by Lemma 2.5(2). Then $B \leq C_G(N) = N \leq A$. So $G = AB = A$. By Lemma 2.6, $l_p(G) \leq 1$, a contradiction. Hence $N \leq A \cap B$. Note also that since $O_{p'}(A) \leq C_A(N) \leq C_G(N) = N$, we have $O_{p'}(A) = 1$.

Next we set $L = O^{p'}(G)$ and $K = (A \cap L)(B \cap L)$. Then, by Lemma 2.3, K is a normal subgroup of G. Since $K \leq L$, $A \cap K \leq A \cap L$. On the other hand, $A \cap L \leq K$, so $A \cap L \leq A \cap K$. Hence $A \cap K = A \cap L$. Now we have

$$A/(A \cap K) = A/(A \cap L) \cong AL/L,$$

which is a p'-group. This implies that AK/K is a p'-group. Similarly BK/K is a p'-group. Hence G/K is a p'-group. Hence $L \leq K$ and so $K = L$. So L satisfies the hypotheses of the theorem. If $L < G$, then $l_p(L) \leq 1$ by the minimal choice of G. Then $l_p(G) \leq 1$, a contradiction. Hence $L = G$.

We now have $O^{p'}(G/N) = G/N$ and so $O^{p'}(H) = H$. Since the p-length of H is at most 1, if P is a Sylow p-subgroup of H then $PO_{p'}(H)$ is normal in H and $H/(PO_{p'}(H))$ is a p'-group. Since $O^{p'}(G) = G$ we have $H = PO_{p'}(H)$ is p-nilpotent.

We now prove that for any $x \in H$, $C_G(x) = C_N(x)C_H(x)$. To do this we only need to prove that $C_G(x) \leq C_N(x)C_H(x)$. Pick $g \in C_G(x)$. Then $g = nh$, where $n \in N$ and $h \in H$. If $x^g = x^{nh} = x$ then $x^n = x^{h^{-1}}$ and so

$$x^{-1}x^n = x^{-1}x^{h^{-1}}.$$

Since $x^{-1}x^n \in N$ and $x^{-1}x^{h^{-1}} \in H$ we have $x^{-1}x^n = x^{-1}x^{h^{-1}} \in N \cap H = 1$. Hence $n \in C_N(x)$ and $h \in C_H(x)$. So $C_G(x) \le C_N(x)C_H(x)$.

Since $G_p = (A \cap H)_p(B \cap H)_p \neq 1$, we can assume that $(A \cap H)_p \neq 1$. Since

$$(B \cap H)_{p'}(A \cap H)_p$$
$$= (A \cap H)_{p'}(A \cap H)_p(B \cap H)_{p'} \cap (A \cap H)_{p'}(B \cap H)_p(B \cap H)_{p'},$$

$(B\cap H)_{p'}(A\cap H)_p$ is a group. Since H is p-nilpotent, so are $(A\cap H)_{p'}(A\cap H)_p$ and $(B\cap H)_{p'}(A\cap H)_p$. Similarly, $(B\cap H)_{p'}(A\cap H)_p$ and $(B\cap H)_{p'}(B\cap H)_p$ are also p-nilpotent groups.

We now consider the actions of $(A\cap H)_p$ on $(A\cap H)_{p'}$ and $(B\cap H)_{p'}$. Assume that $(A \cap H)_p$ does not act trivially on both $(A \cap H)_{p'}$ and $(B \cap H)_{p'}$. Hence we can assume that $(A \cap H)_p$ acts non-trivially on $(A \cap H)_{p'}$. Hence there exists a p-element x in $(A \cap H)_p$ such that x acts non-trivially on $(A \cap H)_{p'}$. Let Q be an $\langle x\rangle$-invariant subgroup of $(A \cap H)_{p'}$ such that $\langle x\rangle$ acts non-trivially on Q and $\langle x\rangle$ acts trivially on any proper $\langle x\rangle$-invariant subgroup of Q. Since $C_{A\cap H}(N) = 1$, $Q\langle x\rangle$ acts faithfully on N. Now applying [9, Lemma 3.2], there exists an element $y \in Q\langle x\rangle$ of prime power order such that $|N : C_N(y)| \ge p^{p-1}$. We know that $C_G(x) = C_N(x)C_H(x)$ for any $x \in H$, hence

$$|y^G| = [G : C_G(y)] = \frac{|G|}{|C_G(y)|} = \frac{|H|}{|C_H(y)|} \cdot \frac{|N|}{|C_N(y)|}\,.$$

So $|y^G|$ is divisible by p^{p-1}. Since $y \in A$, this is a contradiction.

Now suppose that $(A\cap H)_p$ acts trivially on both $(A\cap H)_{p'}$ and $(B\cap H)_{p'}$. If $(B\cap H)_p = 1$, then $(A\cap H)_p$ is a normal p-group of $H \cong G/C_G(N) = G/N$. By [5, Theorem B 3.12], this is impossible. Hence assume that $(B \cap H)_p \neq 1$. Now consider the actions of $(B\cap H)_p$ on $(A\cap H)_{p'}$ and $(B\cap H)_{p'}$. If $(B\cap H)_p$ acts trivially on both $(A \cap H)_{p'}$ and $(B \cap H)_{p'}$, then $H_p = (A \cap H)_p(B \cap H)_p$ is normal in H. By [5, Theorem B. 3.12], this is impossible. Hence $(B \cap H)_p$ does not act trivially on $(A\cap H)_{p'}(B\cap H)_{p'} = H_{p'}$. Repeating the arguments in the above, we get a contradiction. □

4. Remarks

It is not always possible to extend results in the manner above. As an example, [8, Theorem B] tells us: if p is a prime and P a Sylow p-subgroup of a group G then $|x^G|$ is not divisible by p for every p-singular element x of G if and only if P is an abelian TI-subgroup and $N_G(P)/C_G(P)$ acts Frobeniusly on P if $N_G(P) > C_G(P)$.

If we replace the condition on p-singular elements of G by the same condition on p-singular elements of $A \cup B$, the same conclusion would imply that the p-singular elements x of G would satisfy $|x^G|$ not divisible by p. It is easy to find counterexamples even for totally permutable products. If $G = A \times B$ with $A \cong B \cong S_3$, then $|x^G| = 2$ for all 3-singular elements of $A \cup B$, but the 3-singular element xu with $x \in A_3$ and $y \in B_2$ has $|(xy)^G| = 6$.

Another example is given by the following well known result [10, Theorem 5.4]. Suppose that G is a finite group and p is a prime. Then p does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal abelian Sylow p-subgroup ([10, Theorem 5.4]). If we replace the condition on characters of G by the same condition on $\mathrm{Irr}(A) \cup \mathrm{Irr}(B)$ then any non-abelian product (including mutually permutable and totally permutable) of abelian subgroups A and B will give a counterexample.

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References

[1] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of Finite Groups*, Walter de Gruyter (Berlin, New-York, 2010).

[2] A. Ballester-Bolinches, J. Cossey and Y.-M. Li, Mutually permutable products and conjugacy classes, *Monatsh. Math.*, **170** (2013), 305–310.

[3] A. R. Camina and R. D. Camina, The influence of conjugacy class sizes on the structure of finite groups: a survey, *Asian-Eur. J. Math.*, **4** (2011), 559–588.

[4] J. Cossey and Y. Wang, Remarks on the length of conjugacy classes of finite groups, *Comm. Algebra*, **27** (1999), 4347–4353.

[5] K. Doerk and T. O. Hawkes, *Finite Soluble Groups*, De Gruyter (Berlin, 1992).

[6] M. J. Felipe, A. Martínez-Pastor and V. M. Ortíz-Sotomayor, Square-free class sizes in mutually permutable products, *J. Algebra*, **491** (2017), 190–206.

[7] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press (New York, 1976).

[8] G. Qian and Y. Wang, On class size of p-singular elements in finite groups, *Comm. Algebra*, **37** (2009), 1172–1181.

[9] G. Qian and Y. Wang, On conjugacy class sizes and character degrees of finite groups, *J. Algebra Appl.*, **13** (2014), 1–8.

[10] G. O. Michler, A finite simple group of Lie type has p-blocks with different defects, $p \neq 2$, *J. Algebra*, **104** (1986), 220–230.