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NONLINEAR $*$-LIE-TYPE DERIVATIONS ON STANDARD OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. It is shown that each nonlinear $*$-Lie-type derivation δ on $\mathcal{A}$ is a linear $*$-derivation. Moreover, δ is an inner $*$-derivation as well.

1. Introduction

Let $\mathcal{A}$ be an associative $*$-algebra over the complex field $\mathbb{C}$. For any $A, B \in \mathcal{A}$, we denote a "new product" of A and B by $[A, B]_* = AB - BA^*$. Such kind of product based on Lie bracket naturally appears in relation with the so-called Jordan $*$-derivations and plays an important role in the problem of representability of quadratic functionals by sesqui-linear functionals on left-modules over $*$-algebras (see [16,17]). The product is workable for us to characterize ideals (see, [3,13]). Particular attention has been paid to understanding mappings which preserve the product $AB - BA^*$ between $*$-algebras (see [2,4,5,8,10]).

The question of to what extent the multiplicative structure of an algebra determines its additive structure has been considered by many researchers over the past decades. In particular, they have investigated under which conditions bijective mappings between algebras preserving the multiplicative structure necessarily preserve the additive structure as well. The most fundamental result in this direction is due to W. S. Martindale III [15] who proved that every bijective multiplicative mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Later, a number of authors considered the Jordan-type product or Lie-type product and proved that, on certain associative algebras or rings, bijective mappings which preserve any of those products are automatically additive.

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An additive mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called an *additive derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Furthermore, δ is said to be an *additive $*$-derivation* provided that δ is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$. Let $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping (without the additivity assumption). We say that δ is a *nonlinear $*$-Lie derivation* if

$$\delta([A,B]_*) = [\delta(A), B]_* + [A, \delta(B)]_*,$$

holds true for all $A, B \in \mathcal{A}$. Similarly, a mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called a *nonlinear $*$-Lie triple derivation* if it satisfies the condition

$$\delta([[A,B]_*, C]_*) = [[\delta(A), B]_*, C]_* + [[A, \delta(B)]_*, C]_* + [[A,B]_*, \delta(C)]_*$$

for all $A, B, C \in \mathcal{A}$.

Given the consideration of $*$-Lie derivations and $*$-Lie triple derivations, we can further develop them in one natural way. Suppose that $n \geq 2$ is a fixed positive integer. Let us see a sequence of polynomials with $*$

$$\begin{aligned}
p_1(x_1) &= x_1,\\
p_2(x_1,x_2) &= [x_1,x_2]_* = x_1x_2 - x_2x_1^*,\\
p_3(x_1,x_2,x_3) &= [p_2(x_1,x_2),x_3]_* = [[x_1,x_2]_*,x_3]_*,\\
p_4(x_1,x_2,x_3,x_4) &= [p_3(x_1,x_2,x_3),x_4]_* = [[[x_1,x_2]_*,x_3]_*,x_4]_*,\\
&\dots,\\
p_n(x_1,x_2,\dots,x_n) &= [p_{n-1}(x_1,x_2,\dots,x_{n-1}),x_n]_*.
\end{aligned}$$

Accordingly, a *nonlinear $*$-Lie n-derivation* is a mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the condition

$$\delta(p_n(x_1,x_2,\dots,x_n)) = \sum_{k=1}^{n} p_n(x_1,\dots,x_{k-1},\delta(x_k),x_{k+1},\dots,x_n)$$

for all $x_1, x_2, \dots, x_n \in \mathcal{A}$. This notion makes the best use of the definition of Lie n-derivations, see [1,6]. By the definition, it is clear that every $*$-Lie derivation is a $*$-Lie 2-derivation and every $*$-Lie triple derivation is a $*$-Lie 3-derivation. One can easily check that every nonlinear $*$-Lie derivation on any $*$-algebra is a nonlinear $*$-Lie triple derivation. But we do not know whether the converse statement is still valid. $*$-Lie 2-derivations, $*$-Lie 3-derivations and $*$-Lie n-derivations are collectively referred to as *$*$-Lie-type derivations*. $*$-Lie-type derivations in different backgrounds are extensively studied by several authors, see [9,11,12,18].

Yu and Zhang [18] proved that every nonlinear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. This result was extended to the case of nonlinear $*$-Lie triple derivations by Li et al. [12]. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. In [11], Li et al showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections, then $\delta\colon \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a nonlinear $*$-Lie derivation if and only if δ is an additive $*$-derivation. Jing [9] investigated nonlinear $*$-Lie derivations on standard operator algebras. Let $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. It was shown that every nonlinear $*$-Lie derivation δ on $\mathcal{A}$ is automatically linear. Moreover, δ is an inner $*$-derivation.

Motivated by the afore-mentioned works, we will concentrate on giving a description of nonlinear $*$-Lie-type derivations on standard operator algebras. The framework of this paper is as follows. We recall and collect some indispensable facts with respect to standard operator algebras in Section 2. Section 3 is to provide a detailed proof of our main result. The main theorem states that every nonlinear $*$-Lie n-derivation on a standard operator algebra is an additive $*$-derivation. Moreover, it is an inner $*$-derivation. Section 4 is devoted to certain potential topics in this vein for the future.

2. Notation and preliminaries

Before beginning detailed demonstration and stating our main result, we need to give some notation and preliminaries. Throughout the paper, all algebras and spaces are defined over the field $\mathbb{C}$ of complex numbers.

In this paper, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We denote the subalgebra of all bounded finite rank operators by $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. We call a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ a *standard operator algebra* if it contains $\mathcal{F}(\mathcal{H})$. It should be remarked that a standard operator algebra is not necessarily closed in the sense of weak operator topology. This is quite different from von Neumann algebras which are always weakly closed. We refer the reader to [9,14] about basic facts of standard operator algebras.

From ring theoretic perspective, standard operator algebras and factor von Neumann algebras are both prime, whereas von Neumann algebras are usually semiprime. Recall that an algebra $\mathcal{A}$ is *prime* if $A\mathcal{A}B = \{0\}$ implies either $A = 0$ or $B = 0$. An algebra is *semiprime* if $A\mathcal{A}A = \{0\}$ implies $A = 0$. Every standard operator algebra has the center $\mathbb{C}I$, which is also the center of an arbitrary factor von Neumann algebra. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a *projection* provided $P^* = P$ and $P^2 = P$. Any operator $A \in \mathcal{B}(\mathcal{H})$ can be expressed as $A = \Re A + i\Im A$, where i is the imaginary unit, $\Re A = \frac{A+A^*}{2}$ and $\Im A = \frac{A-A^*}{2i}$. Note that both $\Re A$ and $\Im A$ are self-adjoint.

LEMMA 2.1 [9, Lemma 2.1]. *Let $\mathcal{A}$ be a standard operator algebra containing the identity operator I in a complex Hilbert space which is closed* *under the adjoint operation. If $AB = BA^*$ holds true for all $B \in \mathcal{A}$, then $A \in \mathbb{R}I$.*

We now choose a projection $P_1 \in \mathcal{A}$ and let $P_2 = I - P_1$. Let us write $\mathcal{A}_{jk} = P_j \mathcal{A} P_k$ for all $j, k = 1, 2$. Then we have the Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Thus an arbitrary operator $A \in \mathcal{A}$ can be written as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{jk} \in \mathcal{A}_{jk}$ and $A_{jk}^* \in \mathcal{A}_{kj}$.

LEMMA 2.2 [9, Proposition 2.7]. *Let $\mathcal{A}$ be a standard operator algebra with identity I. For any $A \in \mathcal{A}$,*
(1) $[iP_1, A]_* = 0$ *implies that* $A_{11} = A_{12} = A_{21} = 0$,
(2) $[iP_2, A]_* = 0$ *implies that* $A_{12} = A_{21} = A_{22} = 0$,
(3) $[i(P_2 - P_1), A]_* = 0$ *implies that* $A_{11} = A_{22} = 0$.

THEOREM 2.3 [16]. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional Hilbert space $\mathcal{H}$. Then every additive derivation $D : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is of the form $DA = AS - SA$ for some $S \in \mathcal{B}(\mathcal{H})$.*

LEMMA 2.4 [7, Problem 230]. *Let $\mathcal{A}$ be a Banach algebra with identity I. For any $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, if $[A, B] = AB - BA = \lambda I$, then $\lambda = 0$.*

LEMMA 2.5. *Let $\mathcal{A}$ be a standard operator algebra with identity I. For any $A \in \mathcal{A}$ and for any positive integer $n \geq 2$, we have*

$$p_n\Big(A, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = \frac{1}{2}(A - A^*).$$

PROOF. A recursive calculation gives that

$$\begin{aligned} p_n\Big(A, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) &= p_{n-1}\Big((\frac{1}{2}(A - A^*), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ = p_{n-2}\Big(\frac{1}{2}(A - A^*), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) &= p_{n-3}\Big(\frac{1}{2}(A - A^*), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ &= \dots = \frac{1}{2}(A - A^*). \quad \square \end{aligned}$$

3. Nonlinear $*$-Lie-type derivations on standard operator algebras

The key task of this section is to prove our main theorem.

THEOREM 3.1. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. If $\delta\colon \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a nonlinear $*$-Lie-type derivation, then δ is a linear $*$-derivation. Moreover, there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\delta(A) = AT - TA$ for all $A \in \mathcal{A}$, i.e., δ is inner.*

PROOF. Let $\delta\colon \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be a nonlinear $*$-Lie-n derivation. Since $*$-Lie 2-derivations are also $*$-Lie 3-derivations, it may and will be supposed that $n \geq 3$ in the proof of this theorem, which will be laid out nicely in several claims.

CLAIM 1. $\delta(0) = 0$.

Namely,

$$\delta(0) = \delta(p_n(0, 0, \dots, 0)) = \sum_{k=1}^{n} p_n(0, \dots, \overset{k}{\delta(0)}, \dots, 0) = 0.$$

CLAIM 2. *For any* $\lambda \in \mathbb{R}$, $\delta(\lambda I) \in \mathbb{R}I$.

For any $\lambda \in \mathbb{R}$ and $A \in \mathcal{A}$, by Lemma 2.5, we know that

$$p_n\Big(\lambda I, A, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = p_{n-1}\Big(0I, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = 0.$$

In light of Claim 1 and Lemma 2.5, we have

$$\begin{aligned}
0 &= \delta\Big(p_n\Big(\lambda I, A, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(\delta(\lambda I), A, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(\lambda I, \delta(A), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&= p_{n-1}\Big(\delta(\lambda I)A - A\delta(\lambda I)^*, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&= \frac{1}{2}\Big(\delta(\lambda I)A - A\delta(\lambda I)^*\Big) - \frac{1}{2}\Big(\delta(\lambda I)A - A\delta(\lambda I)^*\Big)^* \\
&= \frac{1}{2}\delta(\lambda I)(A + A^*) - \frac{1}{2}(A + A^*)\delta(\lambda I)^*
\end{aligned}$$

for all $A \in \mathcal{A}$. That is,

$$\delta(\lambda I)(A + A^*) = (A + A^*)\delta(\lambda I)^*.$$

holds true for all $A \in \mathcal{A}$. Thus we can say that

$$\delta(\lambda I)B = B\delta(\lambda I)^*$$

holds true for all $B = B^* \in \mathcal{A}$. Since for each $B \in \mathcal{A}$, $B = \Re B + i\Im B$ with $\Re B = \frac{B+B^*}{2}$ and $\Im B = \frac{B-B^*}{2i}$, it follows that

$$\delta(\lambda I)B = B\delta(\lambda I)^*$$

for all $B \in \mathcal{A}$. By Lemma 2.1 we assert that $\delta(\lambda I) \in \mathbb{R}I$.

CLAIM 3. *For any $A \in \mathcal{A}$ with $A = A^*$, we have $\delta(A) = \delta(A^*) = \delta(A)^*$.*

For any $A = A^* \in \mathcal{A}$, we set $x_1 = A$, $x_2 = I$, $x_k = \frac{1}{2}I$ $(3 \le k \le n)$ and get

$$p_n\Big(A, I, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = 0.$$

Using Lemma 2.5 and Claim 2, we arrive at

$$\begin{aligned} 0 &= \delta\Big(p_n\Big(A, I, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ &= p_n\Big(\delta(A), I, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(A, \delta(I), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ &= p_{n-1}\Big(\delta(A) - \delta(A)^*, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = \delta(A) - \delta(A)^*. \end{aligned}$$

Thus we obtain $\delta(A) = \delta(A^*) = \delta(A)^*$.

CLAIM 4. *For any $\lambda \in \mathbb{C}$, $\delta(\lambda I) \in \mathbb{C}I$.*

For any $A = A^*$, $B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we put $x_1 = A$, $x_2 = I$, $x_3 = B$, $x_k = \frac{1}{2}I$ $(4 \le k \le n)$ and get

$$p_n\Big(A, \lambda I, B, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = 0.$$

By invoking Lemma 2.5 and Claim 3, we obtain

$$\begin{aligned} 0 &= \delta\Big(p_n\Big(A, \lambda I, B, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ &= p_n\Big(\delta(A), \lambda I, B, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(A, \delta(\lambda I), B, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ &= p_{n-2}\Big([[A, \delta(\lambda I)]_*, B]_*, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ &= \frac{1}{2}[[A, \delta(\lambda I)]_*, B]_* - \frac{1}{2}([[A, \delta(\lambda I)])_*, B]_*)^*. \end{aligned}$$

Thus we have

$$[[A, \delta(\lambda I)]_*, B]_* = ([[A, \delta(\lambda I)])_*, B]_*)^*.$$

A direct calculation gives

$$(A\delta(\lambda I) - \delta(\lambda I)A)(B + B^*) = (B + B^*)(A\delta(\lambda I) - \delta(\lambda I)A)^*.$$

We therefore conclude that

$$(A\delta(\lambda I) - \delta(\lambda I)A)D = D(A\delta(\lambda I) - \delta(\lambda I)A)^*$$

for all $D = D^* = B + B^* \in \mathcal{A}$. Since for arbitrary $D \in \mathcal{A}$, $D = \Re D + i\Im D$ with $\Re D = \frac{D+D^*}{2}$ and $\Im D = \frac{D-D^*}{2i}$, we get

$$(A\delta(\lambda I) - \delta(\lambda I)A)D = D(A\delta(\lambda I) - \delta(\lambda I)A)^*$$

for all $D \in \mathcal{A}$. Applying Lemma 2.1 yields that

$$A\delta(\lambda I) - \delta(\lambda I)A \in \mathbb{R}I$$

Taking into account Lemma 2.4, we obtain $[A, \delta(\lambda I)] = 0$. So $\delta(\lambda I) \in \mathbb{C}I$.

CLAIM 5. *For any $A \in \mathcal{A}$, we have $\delta(\frac{1}{2}I) = \delta(\frac{1}{2}iI) = 0$ and $\delta(iA) = i\delta(A)$, where i is the imaginary unit.*

By Claim 2 and Claim 4, we can write

$$\delta\Big(\frac{1}{2}I\Big) = \alpha I, \delta\Big(-\frac{1}{2}I\Big) = \beta I, \quad \delta\Big(\frac{1}{2}iI\Big) = (\gamma_1 + \gamma i)I, \tag{3.1}$$
$$\delta\Big(-\frac{1}{2}iI\Big) = (\omega_1 + \omega i)I,$$

where $\alpha, \beta, \gamma_1, \gamma, \omega_1, \omega \in \mathbb{R}$. Since $p_n(-\frac{1}{2}iI, \frac{1}{2}iI, \frac{1}{2}I, \ldots, \frac{1}{2}I) = 0$, we get

$$\begin{aligned} 0 &= \delta\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}iI, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ &= p_n\Big(\delta\Big(-\frac{1}{2}iI\Big), \frac{1}{2}iI, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(-\frac{1}{2}iI, \delta\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ &= p_{n-1}\Big(\frac{1}{2}i\Big(\delta\Big(-\frac{1}{2}iI\Big) - \delta\Big(-\frac{1}{2}iI\Big)^*\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ &\quad + p_{n-1}\Big(-i\delta\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = -\frac{1}{2}i\Big(\delta\Big(\frac{1}{2}iI\Big) + \delta\Big(\frac{1}{2}iI\Big)^*\Big). \end{aligned}$$

This implies that $\gamma_1 = 0$. Similarly, using the equality

$$p_n\Big(\frac{1}{2}iI, -\frac{1}{2}iI, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = 0,$$

one can show that $\omega_1 = 0$.

Now (3.1) becomes

$$\delta\Big(\frac{1}{2}I\Big) = \alpha I, \delta\Big(-\frac{1}{2}I\Big) = \beta I, \delta\Big(\frac{1}{2}iI\Big) = \gamma iI, \delta\Big(-\frac{1}{2}iI\Big) = \omega iI. \tag{3.2}$$

Let us next write $x_1 = -\frac{1}{2}iI$, $x_k = \frac{1}{2}I$ $(2 \le k \le n)$. In view of Lemma 2.5, we obtain

$$p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = -\frac{1}{2}iI.$$

We therefore have

$$\delta\Big(-\frac{1}{2}iI\Big)=\delta\Big(p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\dots,\frac{1}{2}I\Big)\Big) \tag{3.3}$$

$$=p_n\Big(\delta\Big(-\frac{1}{2}iI\Big),\frac{1}{2}I,\dots,\frac{1}{2}I\Big)+\sum_{k=2}^{n}p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\dots,\overset{k}{\delta\Big(\frac{1}{2}I\Big)},\dots,\frac{1}{2}I\Big).$$

We now calculate each term in (3.3) step by step.

Step 1. By Lemma 2.5 and (3.2), it is not difficult to see that

$$p_n\Big(\delta\Big(-\frac{1}{2}iI\Big),\frac{1}{2}I,\dots,\frac{1}{2}I\Big)=\delta\Big(-\frac{1}{2}iI\Big). \tag{3.4}$$

Step 2. Let us calculate the second term $p_n(-\frac{1}{2}iI,\delta(\frac{1}{2}I),\frac{1}{2}I,\dots,\frac{1}{2}I)$ (wherein $k=2$) in equality (3.3). By Lemma 2.5 and (3.2), we have

$$\begin{aligned}&p_n\Big(-\frac{1}{2}iI,\delta\Big(\frac{1}{2}I\Big),\frac{1}{2}I,\dots,\frac{1}{2}I\Big)\\&=p_{n-1}\Big(-i\delta\Big(\frac{1}{2}I\Big),\frac{1}{2}I,\dots,\frac{1}{2}I\Big)=-i\delta\Big(\frac{1}{2}I\Big).\end{aligned} \tag{3.5}$$

Step 3. We next calculate the k-th term $p_n(-\frac{1}{2}iI,\frac{1}{2}I,\dots,\overset{k}{\delta(\frac{1}{2}I)},\dots,\frac{1}{2}I)$ (wherein $3\le k\le n-1$).

By invoking Lemma 2.5 and (3.2), we get

$$\begin{aligned}&p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\dots,\overset{k}{\delta\Big(\frac{1}{2}I\Big)},\dots,\frac{1}{2}I\Big)\\&=p_{n-k+2}\Big(-\frac{1}{2}iI,\delta\Big(\frac{1}{2}I\Big),\dots,\frac{1}{2}I\Big)\\&=p_{n-k+1}\Big(-i\delta\Big(\frac{1}{2}I\Big),\frac{1}{2}I,\dots,\frac{1}{2}I\Big)=-i\delta\Big(\frac{1}{2}I\Big).\end{aligned} \tag{3.6}$$

Step 4. Let us check the last term $p_n(-\frac{1}{2}iI,\frac{1}{2}I,\dots,\delta(\frac{1}{2}I))$ in (3.3) (wherein $k=n$). In view of Lemma 2.5 and equality (3.2), we obtain

$$p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\dots,\delta\Big(\frac{1}{2}I\Big)\Big)=p_2\Big(-\frac{1}{2}iI,\delta\Big(\frac{1}{2}I\Big)\Big)=-i\delta\Big(\frac{1}{2}I\Big). \tag{3.7}$$

Taking equalities (3.4)–(3.7) into equality (3.3) gives

$$-(n-2)i\delta\Big(\frac{1}{2}I\Big)=0.$$

This shows that

$$\delta\Big(\frac{1}{2}I\Big) = 0. \tag{3.8}$$

Let us next put $x_1 = \frac{1}{2}iI$, $x_2 = -\frac{1}{2}I$, $x_k = \frac{1}{2}I$ $(3 \le k \le n)$. It follows from Lemma 2.5 that

$$p_n\Big(\frac{1}{2}iI, -\frac{1}{2}I, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = p_{n-1}\Big(-\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = -\frac{1}{2}iI.$$

Taking into account Lemma 2.5 and (3.8) again, we arrive at

$$\begin{aligned}
\delta\Big(-\frac{1}{2}iI\Big) &= \delta\Big(p_n\Big(\frac{1}{2}iI, -\frac{1}{2}I, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big)\\
&= p_n\Big(\delta\Big(\frac{1}{2}iI\Big), -\frac{1}{2}I, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(\frac{1}{2}iI, \delta\Big(-\frac{1}{2}I\Big), \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\\
&= p_{n-1}\Big(-\delta\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_{n-1}\Big(i\delta\Big(-\frac{1}{2}I\Big), \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\\
&= -\delta\Big(\frac{1}{2}iI\Big) + i\delta\Big(-\frac{1}{2}I\Big).
\end{aligned}$$

Taking (3.2) into the above equality, we get

$$\omega = -\gamma + \beta. \tag{3.9}$$

Using Lemma 2.5 again, we assert that

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \frac{1}{2}iI\Big) = p_2\Big(\frac{1}{2}iI, \frac{1}{2}iI\Big) = -\frac{1}{2}I.$$

Considering Lemma 2.5 together with equalities (3.2) and (3.8), we obtain

$$\begin{aligned}
\beta I = \delta\Big(-\frac{1}{2}I\Big) &= \delta\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \frac{1}{2}iI\Big)\Big)\\
&= p_n\Big(\delta\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \dots, \frac{1}{2}I, \frac{1}{2}iI\Big) + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta\Big(\frac{1}{2}iI\Big)\Big)\\
&= p_2\Big(\delta\Big(\frac{1}{2}iI\Big), \frac{1}{2}iI\Big) + p_2\Big(\frac{1}{2}iI, \delta\Big(\frac{1}{2}iI\Big)\Big) = 2i\delta\Big(\frac{1}{2}iI\Big) = -2\gamma I.
\end{aligned} \tag{3.10}$$

At last, we set $x_1 = -\frac{1}{2}iI$, $x_k = \frac{1}{2}I$ $(2 \le k \le n-1)$, $x_n = \frac{1}{2}iI$. By Lemma 2.5, we know that

$$p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \frac{1}{2}iI\Big) = p_2\Big(-\frac{1}{2}iI, \frac{1}{2}iI\Big) = \frac{1}{2}I.$$

By invoking Lemma 2.5, equalities (3.2) and (3.8), we conclude

$$0 = \delta\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \frac{1}{2}iI\Big)\Big) \tag{3.11}$$
$$= p_n\Big(\delta\Big(-\frac{1}{2}iI\Big), \frac{1}{2}I, \dots, \frac{1}{2}I, \frac{1}{2}iI\Big) + p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta\Big(\frac{1}{2}iI\Big)\Big)$$
$$= p_2\Big(\delta\Big(-\frac{1}{2}iI\Big), \frac{1}{2}iI\Big) + p_2\Big(-\frac{1}{2}iI, \delta\Big(\frac{1}{2}iI\Big)\Big) = i(\omega - \gamma)I.$$

Combining (3.9), (3.10) with (3.11) gives $\beta = \omega = \gamma = 0$. Now we see that $\alpha = \beta = \omega = \gamma = 0$. Thus $\delta(\frac{1}{2}I) = \delta(\frac{1}{2}iI) = 0$.

For some $A \in \mathcal{A}$, we have

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, A\Big) = p_2\Big(\frac{1}{2}iI, A\Big) = iA.$$

We therefore get

$$\delta(iA) = \delta\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, A\Big)\Big)$$
$$= p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(A)\Big) = p_2\Big(\frac{1}{2}iI, \delta(A)\Big) = i\delta(A).$$

In order to continue our discussions, we need the Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Then for any operator $A \in \mathcal{A}$, we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$ for some $A_{jk} \in \mathcal{A}_{jk}$ $(j, k = 1, 2)$.

CLAIM 6. *For any $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, we have $\delta(B_{12}+C_{21}) = \delta(B_{12}) + \delta(C_{21})$.*

It is sufficient to show that

$$M = \delta(B_{12} + C_{21}) - \delta(B_{12}) - \delta(C_{21}) = 0.$$

In view of Lemma 2.5, we obtain

$$p_n\Big(i(P_2 - P_1), \frac{1}{2}I, \dots, \frac{1}{2}I, B_{12}\Big) = p_2(i(P_2 - P_1), B_{12}) = 0$$

and

$$p_n\Big(i(P_2 - P_1), \frac{1}{2}I, \dots, \frac{1}{2}I, C_{21}\Big) = p_2(i(P_2 - P_1), C_{21}) = 0.$$

Thus we get

$$0 = \delta\Big(p_n\Big(i(P_2 - P_1), \frac{1}{2}I, \dots, \frac{1}{2}I, B_{12} + C_{21}\Big)\Big)$$

$$= p_n\Big(\delta\Big(i(P_2-P_1)\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I, B_{12}+C_{21}\Big)$$
$$+ p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, \delta\Big(B_{12}+C_{21}\Big)\Big).$$

At the same time, we also get

$$0 = \delta\Big(p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, B_{12}\Big)\Big)$$
$$+ \delta\Big(p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, C_{21}\Big)\Big) = p_n\Big(\delta\Big(i(P_2-P_1)\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I, B_{12}\Big)$$
$$+ p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, \delta(B_{12})\Big) + p_n\Big(\delta\Big(i(P_2-P_1)\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I, C_{21}\Big)$$
$$+ p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, \delta(C_{21})\Big).$$

Comparing the above two identities gives

$$p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, \delta(B_{12}+C_{21}) - \delta(B_{12}) - \delta(C_{21})\Big) = 0,$$

that is,

$$p_n\Big(i(P_2-P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I, M\Big) = 0.$$

Using Lemma 2.5 again, we arrive at

$$p_2(i(P_2-P_1), M) = [i(P_2-P_1), M]_* = 0.$$

Applying Lemma 2.2 yields that $M_{11} = M_{22} = 0$. Notice that

$$p_n\Big(B_{12}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = 0$$

and

$$p_n\Big(C_{21}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) = C_{21} - C_{21}^*.$$

Let us now calculate $\delta(C_{21} - C_{21}^*)$ by two different approaches. On the one hand,

$$\delta(C_{21} - C_{21}^*) = \delta\Big(p_n\Big(B_{12}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) + \delta\Big(p_n\Big(C_{21}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big)$$
$$= p_n\Big(\delta(B_{12}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(B_{12}, \delta\Big(P_1\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)$$
$$+ p_n\Big(\delta(C_{21}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(C_{21}, \delta\Big(P_1\Big), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big).$$

On the other hand,

$$\delta(C_{21}-C_{21}^*)=\delta\Big(p_n\Big(B_{12}+C_{21},P_1,\frac{1}{2}I,\dots,\frac{1}{2}I\Big)\Big)$$
$$=p_n\Big(\delta\Big(B_{12}+C_{21}\Big),P_1,\frac{1}{2}I,\dots,\frac{1}{2}I\Big)+p_n\Big(B_{12}+C_{21},\delta\Big(P_1\Big),\frac{1}{2}I,\dots,\frac{1}{2}I\Big).$$

The above two equalities imply that

$$p_n\Big(M,P_1,\frac{1}{2}I,\dots,\frac{1}{2}I\Big)=0.$$

By Lemma 2.5 and the fact $M_{11}=M_{22}=0$ it follows that $M_{21}=M_{21}^*=0$. Thus we get $M_{21}=0$.

Notice the facts

$$p_n\Big(B_{12},P_2,\frac{1}{2}I,\dots,\frac{1}{2}I\Big)=B_{12}-B_{12}^*$$

and

$$p_n\Big(C_{21},P_2,\frac{1}{2}I,\dots,\frac{1}{2}I\Big)=0.$$

Adopting similar methods as above, one can show that $M_{12}=0$. Then we get $\delta(B_{12}+C_{21})=\delta(B_{12})+\delta(C_{21})$.

Claim 7. *For any $A_{11}\in\mathcal{A}_{11}$, $B_{12}\in\mathcal{A}_{12}$, $C_{21}\in\mathcal{A}_{21}$ and $D_{22}\in\mathcal{A}_{22}$, we have*

(1) $\delta(A_{11}+B_{12}+C_{21})=\delta(A_{11})+\delta(B_{12})+\delta(C_{21})$.
(2) $\delta(B_{12}+C_{21}+D_{22})=\delta(B_{12})+\delta(C_{21})+\delta(D_{22})$.

To simplify the computational process, we write

$$M=\delta(A_{11}+B_{12}+C_{21})-\delta(A_{11})-\delta(B_{12})-\delta(C_{21}).$$

We shall show that $M=0$.

In light of the relation

$$iB_{12}+iC_{21}=p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)$$
$$=p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}\Big)+p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}\Big)$$
$$+p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,C_{21}\Big),$$

we adopt two different ways to calculate $\delta(iB_{12}+iC_{21})$.

We first have

$$\begin{aligned}\delta(iB_{12}+iC_{21}) &= \delta\Big(p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)\Big)\\ &= p_n\Big(\delta(iP_2),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)\\ &\quad + p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11}+B_{12}+C_{21})\Big).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\delta(iB_{12}+iC_{21}) &= \delta\Big(p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}\Big)\Big)\\ &+ \delta\Big(p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}\Big)\Big)+\delta\Big(p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,C_{21}\Big)\Big)\\ &= p_n\Big(\delta(iP_2),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}\Big)+p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11})\Big)\\ &+ p_n\Big(\delta(iP_2),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}\Big)+p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(B_{12})\Big)\\ &+ p_n\Big(\delta(iP_2),\frac{1}{2}I,\dots,\frac{1}{2}I,C_{21}\Big)+p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(C_{21})\Big).\end{aligned}$$

Using Claim 6, we therefore get

$$\begin{aligned}&p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11}+B_{12}+C_{21})\Big)\\ &= p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11})\Big)+p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(B_{12})\Big)\\ &\quad + p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(C_{21})\Big).\end{aligned}$$

That is

$$p_n\Big(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,M\Big)=[iP_2,M]_*=0.$$

In view of Lemma 2.2, we assert that $M_{12}=M_{21}=M_{22}=0$.

Let us now show $M_{11}=0$. By Lemma 2.5, we obtain three equalities

$$p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}\Big)=-2iA_{11},$$

$$p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}\Big)=0,$$

$$p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,C_{21}\Big)=0.$$

Adopting analogous arguments as above, we get

$$\begin{aligned}\delta(-2iA_{11}) &= \delta\Big(p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)\Big)\\ &= p_n\Big(\delta\Big(i(P_2-P_1)\Big),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)\\ &\quad+p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11}+B_{12}+C_{21})\Big),\end{aligned}$$

and

$$\begin{aligned}&\delta(-2iA_{11}) = \delta\Big(p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}\Big)\Big)\\ &+\delta\Big(p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}\Big)\Big)+\delta\Big(p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,C_{21}\Big)\Big)\\ &= p_n\Big(\delta(i(P_2-P_1)),\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}\Big)+p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11})\Big)\\ &+p_n\Big(\delta(i(P_2-P_1)),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}\Big)+p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(B_{12})\Big)\\ &+p_n\Big(\delta(i(P_2-P_1)),\frac{1}{2}I,\dots,\frac{1}{2}I,C_{21}\Big)+p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(C_{21})\Big).\end{aligned}$$

Comparing the two equalities gives

$$0=p_n\Big(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,M\Big)=[i(P_2-P_1),M]_*.$$

Applying Lemma 2.2 yields that $M_{11}=0$. Thus we obtain

$$\delta(A_{11}+B_{12}+C_{21})=\delta(A_{11})+\delta(B_{12})+\delta(C_{21}).$$

Using the same methods to calculate $p_n(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}+C_{21}+D_{22}))$ and $p_n(i(P_2-P_1),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{12}+C_{21}+D_{22})$, we can show that

$$\delta(B_{12}+C_{21}+D_{22})=\delta(B_{12})+\delta(C_{21})+\delta(D_{22}).$$

CLAIM 8. *For any $A_{11}\in\mathcal{A}_{11}$, $B_{12}\in\mathcal{A}_{12}$, $C_{21}\in\mathcal{A}_{21}$ and $D_{22}\in\mathcal{A}_{22}$, we have*

$$\delta(A_{11}+B_{12}+C_{21}+D_{22})=\delta(A_{11})+\delta(B_{12})+\delta(C_{21})+\delta(D_{22}).$$

To simplify the computational process, we set

$$M=\delta(A_{11}+B_{12}+C_{21}+D_{22})-\delta(A_{11})-\delta(B_{12})-\delta(C_{21})-\delta(D_{22}).$$

Notice the fact $p_n(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,D_{22})=0$. Applying (1) in Claim 7 yields that

$$\begin{aligned}
&p_n\Big(\delta(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}+D_{22}\Big)\\
&+p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11}+B_{12}+C_{21}+D_{22})\Big)\\
&=\delta\Big(p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}+D_{22}\Big)\Big)\\
&=\delta\Big(p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)\Big)+\delta\Big(p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,D_{22}\Big)\Big)\\
&=p_n\Big(\delta(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}\Big)\\
&+p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11}+B_{12}+C_{21})\Big)\\
&+p_n\Big(\delta(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,D_{22}\Big)+p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(D_{22})\Big)\\
&=p_n\Big(\delta(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}+D_{22}\Big)\\
&+p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(A_{11})+\delta(B_{12})+\delta(C_{21})+\delta(D_{22})\Big).
\end{aligned}$$

Thus we have

$$0=p_n\Big(iP_1,\frac{1}{2}I,\dots,\frac{1}{2}I,M\Big)=[iP_1,M]_*.$$

By Lemma 2.2 it follows that $M_{11}=M_{12}=M_{21}=0$.

Similarly,we consider this polynomial $p_n(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11}+B_{12}+C_{21}+D_{22})$. By using the fact $p_n(iP_2,\frac{1}{2}I,\dots,\frac{1}{2}I,A_{11})=0$ and (2) in Claim 7, we can get $M_{22}=0$.

Claim 9. *For any $A_{jk},B_{jk}\in\mathcal{A}_{jk}(j,k=1,2)$, we have*

$$\delta(A_{jk}+B_{jk})=\delta(A_{jk})+\delta(B_{jk}).$$

Case 1. $j\neq k$. Notice that

$$\begin{aligned}
p_n\Big(iP_j+iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,P_k+B_{jk}\Big)&=p_2\Big(iP_j+\frac{1}{2}i(A_{jk}+A_{jk}^*),P_k+B_{jk}\Big)\\
&=\frac{1}{2}iA_{jk}+iB_{jk}+\frac{1}{2}iA_{jk}^*B_{jk}+\frac{1}{2}iA_{jk}^*+\frac{1}{2}iB_{jk}A_{jk}^*.
\end{aligned}$$

By Claim 8 we arrrive at

$$\begin{aligned}
&\delta\Big(p_n\Big(iP_j+iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,P_k+B_{jk}\Big)\Big)\\
&=\delta\Big(\frac{1}{2}iA_{jk}+iB_{jk}\Big)+\delta\Big(\frac{1}{2}iA^*_{jk}B_{jk}\Big)+\delta\Big(\frac{1}{2}iA^*_{jk}\Big)+\delta\Big(\frac{1}{2}iB_{jk}A^*_{jk}\Big).
\end{aligned}$$

On the other hand, by Claim 8, we also have

$$\begin{aligned}
&\delta\Big(p_n\Big(iP_j+iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,P_k+B_{jk}\Big)\Big)\\
&=p_n\Big(\delta\Big(iP_j+iA_{jk}\Big),\frac{1}{2}I,\dots,\frac{1}{2}I,P_k+B_{jk}\Big)\\
&\quad+p_n\Big(iP_j+iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,\delta\Big(P_k+B_{jk}\Big)\Big)\\
&=p_n\Big(\delta(iP_j),\frac{1}{2}I,\dots,\frac{1}{2}I,P_k\Big)+p_n\Big(\delta(iA_{jk}),\frac{1}{2}I,\dots,\frac{1}{2}I,P_k\Big)\\
&\quad+p_n\Big(\delta(iP_j),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{jk}\Big)+p_n\Big(\delta(iA_{jk}),\frac{1}{2}I,\dots,\frac{1}{2}I,B_{jk}\Big)\\
&\quad+p_n\Big(iP_j,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(P_k)\Big)+p_n\Big(iP_j,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(B_{jk})\Big)\\
&\quad+p_n\Big(iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(P_k)\Big)+p_n\Big(iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,\delta(B_{jk})\Big)\\
&=\delta\Big(p_n\Big(iP_j,\frac{1}{2}I,\dots,\frac{1}{2}I,P_k\Big)\Big)+\delta\Big(p_n\Big(iP_j,\frac{1}{2}I,\dots,\frac{1}{2}I,B_{jk}\Big)\Big)\\
&\quad+\delta\Big(p_n\Big(iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,P_k\Big)\Big)+\delta\Big(p_n\Big(iA_{jk},\frac{1}{2}I,\dots,\frac{1}{2}I,B_{jk}\Big)\Big)\\
&=\delta\Big(p_2(iP_j,B_{jk})\Big)+\delta\Big(p_2\Big(\frac{1}{2}i(A_{jk}+A^*_{jk}),P_k\Big)\Big)\\
&\quad+\delta\Big(p_2\Big(\frac{1}{2}i(A_{jk}+A^*_{jk}),B_{jk}\Big)\Big)\\
&=\delta(iB_{jk})+\delta\Big(\frac{1}{2}iA_{jk}+\frac{1}{2}iA^*_{jk}\Big)+\delta\Big(\frac{1}{2}iA^*_{jk}B_{jk}+\frac{1}{2}iB_{jk}A^*_{jk}\Big)\\
&=\delta(iB_{jk})+\delta\Big(\frac{1}{2}iA_{jk}\Big)+\delta\Big(\frac{1}{2}iA^*_{jk}\Big)+\delta\Big(\frac{1}{2}iA^*_{jk}B_{jk}\Big)+\delta\Big(\frac{1}{2}iB_{jk}A^*_{jk}\Big).
\end{aligned}$$

Comparing with the above equality, we now conclude that

$$\delta\Big(iB_{jk}+\frac{1}{2}iA_{jk}\Big)=\delta(iB_{jk})+\delta\Big(\frac{1}{2}iA_{jk}\Big). \tag{3.12}$$

Substituting B_{jk} by $\frac{1}{2}A_{jk}$ in (3.12) gives

$$\delta(iA_{jk}) = 2\delta\Big(\frac{1}{2}iA_{jk}\Big). \tag{3.13}$$

Combining equality (3.12) with (3.13), we see that

$$\delta(iB_{jk} + iA_{jk}\Big) = \delta\Big(iB_{jk} + \frac{1}{2}iA_{jk}\Big) + \delta\Big(\frac{1}{2}iA_{jk}\Big)$$
$$= \delta(iB_{jk}) + \delta\Big(\frac{1}{2}iA_{jk}\Big) + \delta\Big(\frac{1}{2}iA_{jk}\Big) = \delta(iB_{jk}) + \delta(iA_{jk}).$$

By Claim 5 it follows that

$$\delta(B_{jk} + A_{jk}) = \delta(B_{jk}) + \delta(A_{jk}).$$

Case 2. $j = k$. That is to prove that $\delta(A_{jj} + B_{jj}) = \delta(A_{jj}) + \delta(B_{jj})$.

Let us write $M = \delta(A_{jj} + B_{jj}) - \delta(A_{jj}) - \delta(B_{jj})$. Let us choose $l = 1, 2$, but $l \neq j$. Since

$$p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, A_{jj} + B_{jj}\Big) = p_2(iP_l, A_{jj} + B_{jj}) = 0,$$

we know that

$$0 = \delta\Big(p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, A_{jj} + B_{jj}\Big)\Big)$$
$$= p_n\Big(\delta(iP_l), \frac{1}{2}I, \dots, \frac{1}{2}I, A_{jj} + B_{jj}\Big) + p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(A_{jj} + B_{jj})\Big)$$
$$= p_n\Big(\delta(iP_l), \frac{1}{2}I, \dots, \frac{1}{2}I, A_{jj}\Big) + p_n\Big(\delta(iP_l), \frac{1}{2}I, \dots, \frac{1}{2}I, B_{jj}\Big)$$
$$+ p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(A_{jj} + B_{jj})\Big)$$
$$= \delta\Big(p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, A_{jj}\Big)\Big) - p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(A_{jj})\Big)$$
$$+ \delta\Big(p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, B_{jj}\Big)\Big) - p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(B_{jj})\Big)$$
$$+ p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(A_{jj} + B_{jj})\Big)$$
$$= p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, \delta(A_{jj} + B_{jj}) - \delta(A_{jj}) - \delta(B_{jj})\Big)$$
$$= p_n\Big(iP_l, \frac{1}{2}I, \dots, \frac{1}{2}I, M\Big) = p_2\Big(iP_l, M\Big) = [iP_l, M]_*.$$

We therefore have $M_{lj} = M_{jl} = M_{ll} = 0$ by Lemma 2.2.

Let us finish this proof by showing $M_{jj} = 0$. Note that

$$p_n\Big(\frac{1}{2}iP_j, A_{jj}+B_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big) = p_n\Big(\frac{1}{2}iP_j, A_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big)$$
$$+ p_n\Big(\frac{1}{2}iP_j, B_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big), \quad \forall\, C_{jl} \in \mathcal{A}_{jl}.$$

Adopting similar methods as the above claims, one can show that

$$\delta\Big(p_n\Big(\frac{1}{2}iP_j, A_{jj}+B_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big)\Big)$$
$$= \delta\Big(p_n\Big(\frac{1}{2}iP_j, A_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big)\Big) + \delta\Big(p_n\Big(\frac{1}{2}iP_j, B_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big)\Big).$$

By a direct calculation, we obtain

$$p_n\Big(\frac{1}{2}iP_j, \delta(A_{jj}+B_{jj}), \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big)$$
$$= p_n\Big(\frac{1}{2}iP_j, \delta(A_{jj}), \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big) + p_n\Big(\frac{1}{2}iP_j, \delta(B_{jj}), \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big).$$

That is

$$0 = p_n\Big(\frac{1}{2}iP_j, M, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big)$$
$$= p_{n-1}\Big(iM_{jj}, \frac{1}{2}I, \dots, \frac{1}{2}I, C_{jl}\Big) = [iM_{jj}, C_{jl}]_*.$$

This implies that $iM_{jj}C_{jl} = 0$. Since $\mathcal{A}$ is prime, we know that $M_{jj} = 0$.

CLAIM 10. *δ is an additive derivation with $\delta(A^*) = \delta(A)^*$.*

For any $A, B \in \mathcal{A}$, we have

$$A = \sum_{i,j=1}^{2} A_{ij}, \quad B = \sum_{i,j=1}^{2} B_{ij} \in \mathcal{A}.$$

Simulating the proof of [9, Lemma 2.12], one can show the additivity of δ

$$\delta(A+B) = \delta(A) + \delta(B).$$

and that $\delta(A^*) = \delta(A)^*$.

Let us now prove that δ is a derivation. Since

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \dots, \frac{1}{2}I, A, B\Big) = p_3\Big(\frac{1}{2}iI, A, B\Big) = p_2(iA, B) = iAB + iBA^*,$$

we get

$$\delta(iAB+iBA^*)=\delta\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\dots,\frac{1}{2}I,A,B\Big)\Big)$$
$$=p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\dots,\frac{1}{2}I,\delta\Big(A\Big),B\Big)+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\dots,\frac{1}{2}I,A,\delta\Big(B\Big)\Big)$$
$$=p_3\Big(\frac{1}{2}iI,\delta(A),B\Big)+p_3\Big(\frac{1}{2}iI,A,\delta(B)\Big)$$
$$=p_2(i\delta(A),B)+p_2(iA,\delta(B))=i\delta(A)B+iB\delta(A)^*+iA\delta(B)+i\delta(B)A^*.$$

It follows from Claim 5 that

$$\delta(AB)+\delta(BA^*)=\delta(A)B+B\delta(A)^*+A\delta(B)+\delta(B)A^*. \tag{3.14}$$

Replacing A (resp. B) by iA (resp. iB) in (3.14) gives

$$\delta(AB)-\delta(BA^*)=\delta(A)B-B\delta(A)^*+A\delta(B)-\delta(B)A^*. \tag{3.15}$$

Combining (3.14) with (3.15), we conclude

$$\delta(AB)=\delta(A)B+A\delta(B).$$

Let us end the proof of this theorem by using Theorem 2.3.

By Claim 10, we know that δ is an additive derivation with $\delta(A^*)=\delta(A)^*$ for all $A\in\mathcal{A}$. By invoking Theorem 2.3, we assert that δ is a linear inner derivation, i.e., there exists an operator $S\in\mathcal{B}(\mathcal{H})$ such that $\delta(A)=AS-SA$ for all $A\in\mathcal{A}$.

Using the fact $\delta(A^*)=\delta(A)^*$, we have

$$A^*S-SA^*=\delta(A^*)=\delta(A)^*=-A^*S^*+S^*A^*$$

for all $A\in\mathcal{A}$. This leads to $A^*(S+S^*)=(S+S^*)A^*$. Hence, $S+S^*=\lambda I$ for some $\lambda\in\mathbb{R}$. Let us set $T=S-\frac{1}{2}\lambda I$. One can check that $T+T^*=0$ and $\delta(A)=AT-TA$ for all $A\in\mathcal{A}$. □

As a direct consequence of Theorem 3.1, we have the following corollary.

Corollary 3.2 [9, Theorem 2.14]. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. If $\delta\colon\mathcal{A}\longrightarrow\mathcal{B}(\mathcal{H})$ is a nonlinear $*$-Lie derivation, then δ is a linear $*$-derivation. Moreover, there exists an operator $T\in\mathcal{B}(\mathcal{H})$ satisfying $T+T^*=0$ such that $\delta(A)=AT-TA$ for all $A\in\mathcal{A}$, i.e., δ is inner.*

In particular, when the standard operator algebra $\mathcal{A}$ is exactly the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators in Theorem 3.1, we have

COROLLARY 3.3. *Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$ and $\delta\colon \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a nonlinear $\ast$-Lie-type derivation. Then δ is an inner linear $\ast$-derivation and there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\delta(A) = AT - TA$ for all $A \in \mathcal{A}$.*

COROLLARY 3.4 [9, Corollary 2.15]. *Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\delta\colon \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a nonlinear $\ast$-Lie derivation. Then δ is an inner linear $\ast$-derivation and there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\delta(A) = AT - TA$ for all $A \in \mathcal{A}$.*

4. Topics for further research

The main aim of this paper is to concentrate on studying nonlinear $\ast$-Lie-type derivations on standard operator algebras. Note that, unlike von Neumann algebras which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that a *von Neumann algebra* or (*$\mathcal{W}^*$-algebra*) is a $\ast$-algebra of bounded operators on a Hilbert space $\mathcal{H}$ that is closed in the weak operator topology and contains the identity I. Roughly speaking, a von Neumann algebra $\mathcal{A}$ is a weakly closed and self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. Our present work together with [11,12] indicates that it is feasible to investigate $\ast$-Lie-type derivations on von Neumann algebras (or on factor von Neumann algebras) by moderate adaption of current methods. We have good reasons to believe that characterizing $\ast$-Lie-type derivations on von Neumann algebras is also of great interest. In the light of the motivation and contents of this article, we would like to end it by proposing two open questions.

CONJECTURE 4.1. *Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a factor von Neumann algebra. For a mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$, the following statements are equivalent.*

(1) *δ is a nonlinear $\ast$-Lie-type derivation,*

(2) *δ is an additive $\ast$-derivation.*

CONJECTURE 4.2. *Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra without central abelian projections. For a mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$, the statements of Conjecture* 4.1 *are equivalent.*

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