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PRESERVED UNDER SACKS FORCING AGAIN?

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Abstract. A hierarchy of topological Ramsey spaces $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$, generalizing the Ellentuck space, were built by Dobrinen and Todorcevic in order to completely classify certain equivalent classes of ultrafilters Tukey (resp. Rudin–Keisler) below $\mathcal{U}_\alpha$ $(\alpha < \omega_1)$, where $\mathcal{U}_\alpha$ are ultrafilters constructed by Laflamme satisfying certain partition properties and have complete combinatorics over the Solovay model. We show that Nash–Williams, or Ramsey ultrafilters in these spaces are preserved under countable-support side-by-side Sacks forcing. This is achieved by proving a parametrized theorem for these spaces, and showing that Nash–Williams ultrafilters localizes the theorem. We also show that every Nash–Williams ultrafilter in $\mathcal{R}_\alpha$ is selective.

1. Introduction

In [11] Laflamme constructed forcings $\mathbb{P}_\alpha$ to add the ultrafilters $\mathcal{U}_\alpha$ for $\alpha < \omega_1$ in order to obtain different combinatorics and related Rudin–Keisler ordering. These ultrafilters satisfy certain partition properties: $\mathcal{U}_1$ is weakly Ramsey; $\mathcal{U}_n$ $(n < \omega)$ is n Ramsey; $\mathcal{U}_\alpha$ $(\omega \le \alpha < \omega_1)$ satisfies analogous Ram sey partition properties. Inspired by the work of Laflamme, Dobrinen and Todorcevic [7,8] constructed a new hierarchy of topological Ramsey spaces $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$, which are modified versions of dense subsets of $\mathbb{P}_\alpha$, and proved extensions of the Pudlak–Rödl Theorem, canonizing equivalent relations on barriers of these spaces. This enabled their complete classification of the structure of the Tukey (resp. Rudin–Keisler) ultrafilters reducible to $\mathcal{U}_\alpha$, as well as the Rudin–Keisler structure of ultrafilters Tukey reducible to $\mathcal{U}_\alpha$.

Among other properties, each $\mathcal{U}_\alpha$ is Nash–Williams in the corresponding space $\mathcal{R}_\alpha$. We would like to show that $\mathcal{U}_\alpha$, and in fact every Nash–Williams ultrafilter, is preserved under countable support side by side Sacks forcing: the upward closure of the ultrafilter is still a Nash–Williams ultrafilter in the forcing extension.

Key words and phrases: Ramsey space, Sacks forcing, selective ultrafilter, Ramsey ultrafilter, parametrized Ramsey theory, local Ramsey thoery.
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THEOREM 1.1. *Let α be a countable ordinal, κ be an infinite cardinal and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding κ Sacks reals. Let $\mathcal{U}$ be a Nash–Williams ultrafilter in $\mathcal{R}_\alpha$ in the ground model, and $\dot{\mathcal{V}}$ a name for the upward closure $\{Y : (\exists [X] \in \mathcal{U})([X] \subseteq Y)\}$ of $\mathcal{U}$. Then $\Vdash_{\mathcal{P}_\kappa}$ ($\dot{\mathcal{V}}$ is a Nash–Williams ultrafilter in $\mathcal{R}_\alpha$).*

Mathias [15,16] introduced selective coideals and Louveau [13,14] was the first to consider the $\mathcal{U}$-topology on $\mathbb{N}^{[\infty]}$. This started the whole area of local Ramsey theory. Baumgartner and Laver [2], [1] showed that the selective ultrafilters on $\mathbb{N}$ are preserved under both side-by-side and iterated Sacks forcing. In the 1990s, Todorcevic conjectured that many topological Ramsey spaces have an ultrafilter associated to them analogous to the way selective ultrafilters on $\mathbb{N}$ are related to the Ellentuck space, when he proved using large cardinals that selective ultrafilters are generic over $L(\mathbb{R})$ in [9]. Mijares [17] defined (weakly) selective and Ramsey ultrafilters in general for topological Ramsey spaces, and showed that every Ramsey ultrafilter is (weakly) selective. The author [22] showed in the Milliken space that selective ultrafilters are preserved under Sacks forcing, and that *(weakly) selective* and *Ramsey* coincide. On the contrary, for the spaces $\mathcal{R}_\alpha$ under consideration here, Trujillo [21] showed that *Ramsey* is strictly stronger than *(weakly) selective* in $\mathcal{R}_1$. Motivated by Trujillo's work, a recent paper [5] of Di Prisco, Mijares and Nieto [22] has a new definition of *selective*. The two notions coincide in the Milliken space. The stronger definition is the one we will be using, and we refer to Mijares' [17] version as *weakly selective*.

In this paper, however, the *Ramsey* ultrafilters in [7,8] are renamed *Nash–Williams* ultrafilters since it is stronger, and may be strictly stronger in some spaces, than Ramsey ultrafilters in [17]. We will see that every Nash–Williams ultrafilter in $\mathcal{R}_\alpha$ is selective and that the ultrafilter being Nash–Williams is necessary for the Local Parametrized $\mathcal{R}_\alpha$ Theorem below to hold.

The proof of Theorem 1.1 is similar to that of [22, Theorem 0.3]. Firstly we prove the following Parametrized $\mathcal{R}_\alpha$ Theorem using the infinite version of the Hales–Jewett theorem in Laver [12]. The proof mimics the steps in [20, Section 9], where it is proved that the Ellentuck space $\mathbb{N}^{[\infty]}$ can be parametrized by infinite product trees of finite sets. The parametrized Ramsey theory was developed in the papers [4] of Di Prisco, Llopis and Todorcevic, and [6] of Di Prisco and Todorcevic.

THEOREM 1.2 (Parametrized $\mathcal{R}_\alpha$ Theorem). *Let $\alpha < \omega_1$. For every finite Souslin-measurable colouring of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ there exists $X \in \mathcal{R}_\alpha$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p]$ is monochromatic.*

Then we show that a Nash–Williams ultrafilter localizes the Parametrized $\mathcal{R}_\alpha$ Theorem, hence proving the Local Parametrized $\mathcal{R}_\alpha$ Theorem.

THEOREM 1.3 (Local Parametrized $\mathcal{R}_\alpha$ Theorem). *Let $\alpha < \omega_1$ and $\mathcal{U}$ be a Nash–Williams ultrafilter in $\mathcal{R}_\alpha$. For every finite Souslin-measurable colouring of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ there exists $X \in \mathcal{R}_\alpha$ with $[X] \in \mathcal{U}$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p]$ is monochromatic.*

Both proofs make essential use of $\mathcal{U}$-trees, introduced by Blass [3], and combinatorial forcing, introduced by Nash–Williams [18] and developed by Galvin and Prikry [10].

In Section 2, we remind ourselves of the topological Ramsey spaces $\mathcal{R}_\alpha$ and Sacks forcing. In Section 3 we prove the Parametrized $\mathcal{R}_\alpha$ Theorem. In Section 4 we recall the definition of Nash–Williams ultrafilters and selective ultrafilters. We check that every Nash–Williams ultrafilter in $\mathcal{R}_\alpha$ is selective and then show the Local Parametrized $\mathcal{R}_\alpha$ Theorem. In Section 5 we see that Nash–Williams ultrafilters are preserved under countable-support side-by-side Sacks forcing. Section 6 is a remark that the property of Nash–Williams is necessary for an ultrafilter to localize the Parametrized $\mathcal{R}_\alpha$ Theorem.

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2. Preliminaries

2.1. The topological Ramsey spaces $\mathcal{R}_\alpha$. In [8] Dobrinen and Todorcevic recursively designed the more and more elaborate tree structures $\mathbb{T}_\alpha$ for each $\alpha < \omega_1$. They then built the topological Ramsey spaces $\mathcal{R}_\alpha$ on each $\mathbb{T}_\alpha$. Intuitively, an element in $\mathcal{R}_\alpha$ is a subtree of $\mathbb{T}_\alpha$ which has the same shape as $\mathbb{T}_\alpha$. For example, $\mathbb{T}_0 = \{\langle\rangle\} \cup \{\langle n \rangle : n < \omega\}$ and $\mathcal{R}_0$ is the Ellentuck space $\mathbb{N}^{[\infty]}$. We have $\mathbb{T}_1$ as another example.

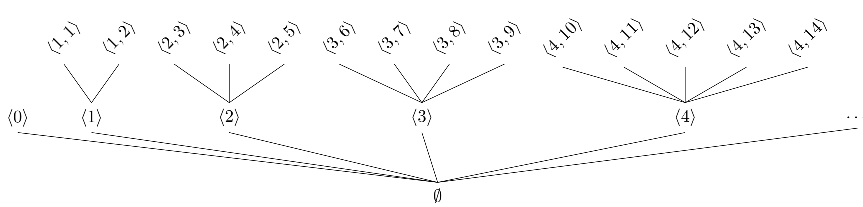


Fig. 1: $\mathbb{T}_1$

Definition 2.1 ($\mathcal{R}_1$) [7, Definition 3.7]. Let

$$\mathbb{T}_1 = \{\langle\rangle\} \cup \{\langle n\rangle : n < \omega\} \cup \bigcup_{n>0} \left\{ \langle n, i\rangle : \tfrac{1}{2}n(n+1) \le i < \tfrac{1}{2}n(n+1) + (n+1) \right\}.$$

We may think of $\mathbb{T}_1$ as an infinite sequence of finite trees of height 2, where the *nth subtree* of $\mathbb{T}_1$ is

$$\mathbb{T}_1(n) = \left\{ \langle\rangle, \langle n\rangle, \langle n, i\rangle : \tfrac{1}{2}n(n+1) \le i < \tfrac{1}{2}n(n+1) + (n+1) \right\}$$

for $n > 0$ and the *0th subtree* is $\mathbb{T}_1(0) = \{\langle\rangle, \langle 0\rangle\}$. A subtree $X \subseteq \mathbb{T}_1$ is in $\mathcal{R}_1$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega} \subseteq \omega$ such that $X \cap \mathbb{T}_1(k_n) \cong \mathbb{T}_1(n)$ for each $n < \omega$ and $X \cap \mathbb{T}_1(j) = \emptyset$ for $j \in \omega \setminus (k_n)_{n\in\omega}$.

In general, for $\alpha < \omega_1$, $\mathbb{T}_\alpha$ is always a union of an infinite sequence of finite trees: $\mathbb{T}_\alpha = \bigcup\{\mathbb{T}_\alpha(n) : n < \omega\}$, and $\mathbb{T}_\alpha(n)$ is called the *nth subtree* of $\mathbb{T}_\alpha$ for $n < \omega$. A subtree $X \subseteq \mathbb{T}_\alpha$ is in $\mathcal{R}_\alpha$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega} \subseteq \omega$ such that $X \cap \mathbb{T}_\alpha(k_n) \cong \mathbb{T}_\alpha(n)$ for each $n < \omega$ and $X \cap \mathbb{T}_\alpha(j) = \emptyset$ for $j \in \omega \setminus (k_n)_{n<\omega}$. We refer to [8, Section 2] for a detailed construction of $\mathbb{T}_\alpha$ and a rigorous definition of the relation $\cong$ involving a function ψ_α and an auxiliary structure $\mathbb{S}_\alpha$. However, the materials about $\mathcal{R}_\alpha$ presented here should be sufficient for the purpose of this paper.

Definition 2.2 [8, Definition 2.7]. Let $\alpha < \omega_1$, $X \in \mathcal{R}_\alpha$ and $(k_n)_{n<\omega} \subseteq \omega$ be the sequence associated with X as described above. For $n < \omega$, the *nth tree* of X is $X(n) = X \cap \mathbb{T}_\alpha(k_n)$. The *nth approximation* of X is $X \restriction n = \bigcup_{i<n} X(i)$. The *set of all nth approximations* is $\mathcal{AR}^\alpha_n = \{X \restriction n : X \in \mathcal{R}_\alpha\}$ and the *set of all finite approximations* is $\mathcal{AR}_\alpha = \bigcup_{n<\omega} \mathcal{AR}^\alpha_n$.

If $a = X \restriction m$ for some $m < \omega$, then we say a is an *initial segment* of X, and write $a \sqsubseteq X$. In this case, the *length* of a, denoted by $|a|$, is m. For $n < m$, the *nth tree* of a is $a(n) = X(n)$. Similarly, if $b \in \mathcal{AR}_\alpha$ satisfies $|a| < |b|$ and $\forall n < |a|$, $a(n) = b(n)$, then we write $a \sqsubseteq b$.

Since $\mathbb{T}_\alpha$ is fixed for each α, an element $X \in \mathcal{R}_\alpha$ is completely determined by the set of all maximal numbers in the top nodes of the tree X. For $X \in \mathcal{R}_\alpha \cup \mathcal{AR}_\alpha$, let $[X]$ denote the collection of all maximal numbers in the $\sqsubseteq$-maximal nodes in X, where we use $\sqsubseteq$ to denote end-extension of nodes in a tree. It is also useful to know which of the subtrees $\mathbb{T}_\alpha(n)$ a node in X belongs to. So for a node $t \in X \setminus \{\emptyset\}$, let $\|t\| = n$ if $t \in \mathbb{T}_\alpha(n)$. Similarly, for $i < \omega$, $\|X(i)\| = n$ if $X(i) \subseteq \mathbb{T}_\alpha(n)$ and $\|X\| = \{\|t\| : t \in X\}$.

In the following example of $X \in \mathcal{R}_2$ (Fig. 2), $\|\langle 3, 6, 23\rangle\| = 3$, $\|X\| = \{1, 2, 3, \ldots\}$, and $[X] = \{2, 3, 5, 16, 18, 19, 21, 23, 25, 27, \ldots\}$.

Definition 2.3. If $\alpha < \omega_1$ and $X, Y \in \mathcal{R}_\alpha \cup \mathcal{AR}_\alpha$, we write $X \le Y$ if $X \subseteq Y$. For $a \in \mathcal{AR}_\alpha$ and $X \in \mathcal{R}_\alpha$, $a \le X$ if there is $Y \le X$ and $n < \omega$ such that $a = Y \restriction n$. Let $[a, X] = \{Y \in \mathcal{R}_\alpha : (a \sqsubseteq Y) \wedge (Y \subseteq a \cup X)\}$, and $X/a =$

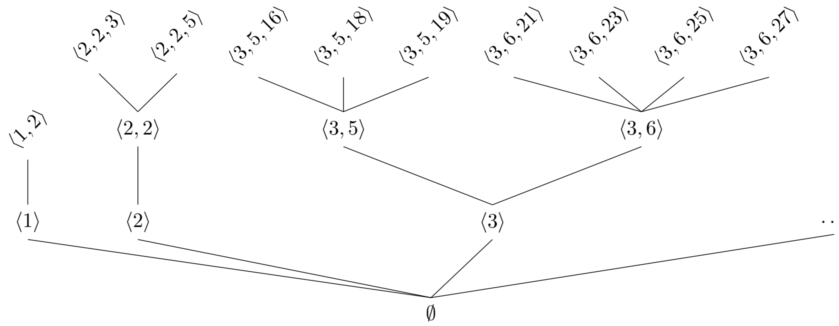


Fig. 2: $X \in \mathcal{R}_2$

$\bigcup\{X(n) : (n \in \omega) \wedge (\max \|a\| < \|X(n)\|)\}$. If $k \in \omega$, then $X/k = \bigcup\{X(n) : \|X(n)\| > k\}$.

The following notation is also used. Let $\alpha < \omega_1$, $X \in \mathcal{R}_\alpha$ and $a \in \mathcal{AR}_\alpha$. The set $\mathcal{AR}_\alpha(X) = \{b \in \mathcal{AR}_\alpha : b \leq X\}$ is the collection of all finite approximations of subtrees $Y \in \mathcal{R}_\alpha$ of X. We further define $\mathcal{AR}_\alpha[a, X] = \{b \in \mathcal{AR}_\alpha : a \sqsubseteq b \leq X\}$. Let $\mathcal{R}_\alpha(n) = \{X(n) : X \in \mathcal{R}_\alpha\}$ be the set of all finite subtrees of $\mathbb{T}_\alpha$ having the same shape as $\mathbb{T}_\alpha(n)$.

From now on we assume $\alpha < \omega_1$ and may omit the subscription α in $\mathbb{T}_\alpha$ when there is no confusion. Unless otherwise stated, we equip $\mathcal{R}_\alpha$ with the topology induced by the first-difference metric ρ , where ρ is defined as follows. For $X, Y \in \mathcal{R}_\alpha$, $\rho(X, Y) = \frac{1}{k}$ where $k = \min\{n < \omega : X(n) \neq Y(n)\}$. Therefore the basic open subsets of $\mathcal{R}_\alpha$ are of the form

$$[a, \mathbb{T}] = \{X \in \mathcal{R}_\alpha : a \sqsubseteq X\} \quad \text{for } a \in \mathcal{AR}_\alpha.$$

This metric topology is coarser than the Ellentuck topology usually associated with the spaces $\mathcal{R}_\alpha$.

2.2. Sacks forcing. Let 2^ω and $(2^\omega)^\omega$ be equipped with the product topology. We use 2^ω interchangeably with $\mathbb{R}$, and $(2^\omega)^\omega$ interchangeably with $\mathbb{R}^\mathbb{N}$. On the set of all finite 01-sequences $2^{<\omega}$, the symbols "$|\cdot|$", "$\sqsubseteq$" and "$\restriction$" respectively denote *length* of the sequence, *initial segment* and *restriction* to an initial segment of certain length as usual. Two finite 01-sequences are *comparable* if one is an initial segment of the other; otherwise they are *incomparable.*

Definition 2.4 [1]. We call a nonempty set $p \subseteq 2^{<\omega}$ a *tree* if it is $\sqsubseteq$-downwards closed. A tree p is *perfect* if every $s \in p$ has incomparable end-extensions $t, u \in p$. In particular, every perfect tree is infinite.

For a perfect tree p, let $[p] = \{f \in 2^\omega : (\forall n \in \omega)(f \restriction n \in p)\}$ be the set of all infinite branches of p. Then $[p] \subseteq 2^\omega$ is a perfect set.

Definition 2.5 [19]. Sacks forcing $\mathcal{P}$ is the set of all perfect trees, ordered by $p \le q$ if $p \subseteq q$.

Note that $p \le q$ if and only if $[p] \subseteq [q]$.

Definition 2.6 [1]. For $p \in \mathcal{P}$ and $s \in p$, let $p|s = \{t \in p : (t \sqsubseteq s) \vee (s \sqsubseteq t)\}$. The number of branchings below s in the tree p is called the *branching level* of s in p, which is

$$\Big| \Big\{ i < |s| : (\exists t \in p)((|t| > i) \wedge (t \restriction i = s \restriction i) \wedge (t \restriction (i+1) \neq s \restriction (i+1))) \Big\} \Big| .$$

The *nth branching level* $l(n,p)$ of the tree p is the set of all $s \in p$ which have branching level n and are $\sqsubseteq$-minimal with this property. If $p, q \in \mathcal{P}$, $q \subseteq p$, $n \in \omega$ and $l(n,p) = l(n,q)$, then we write $q \le^n p$.

Note that $l(n,p) \subseteq p$ is a collection of nodes in p.

Lemma 2.7 (Fusion 1) [1, Lemma 1.4]. *Suppose $(p_k)_{k\in\omega} \subseteq \mathcal{P}$ and $(m_k)_{k\in\omega} \subseteq \omega$ is unbounded and increasing such that $p_{k+1} \le^{m_k} p_k$ for all $k \in \omega$. Then $q = \bigcap_{k\in\omega} p_k \in \mathcal{P}$ and $q \le^{m_k} p_k$ for all $k \in \omega$. We call $(p_k)_{k\in\omega}$ a* fusion sequence *and q the* fusion *of the sequence.*

Now we are ready to define countable-support side-by-side Sacks forcing.

Definition 2.8 [1]. Let κ be an infinite cardinal. Let $\mathcal{P}_k$ be the set of all sequences $p = (p^i)_{i<\kappa}$ such that, for every $i < \kappa$, $p^i \in \mathcal{P}$ and for all but countably many $i < \kappa$, $p^i = 2^{<\omega}$. We say p^i is the *ith tree* of p. For $p = (p^i)_{i<\kappa}$ and $q = (q^i)_{i<\kappa}$ in $\mathcal{P}_\kappa$, $p \le q$ if $p^i \subseteq q^i$ for all $i < \kappa$.

For $p \in \mathcal{P}_\kappa$, let $[p] = \prod_{i<\kappa}[p^i]$. For $\varepsilon \in [p]$ and $i < \kappa$, let ε^i be the ith component in ε, so $\varepsilon^i \in [p^i]$. The *support* of p is $\operatorname{supp}(p) = \{i < \kappa : p^i \neq 2^{<\omega}\}$. So each $p \in \mathcal{P}_\kappa$ has countable support.

Notation. For a set S, $[S]^{<\omega}$ denotes the set of all finite subsets of S.

Definition 2.9. Let κ be an infinite cardinal. Let $F \in [\kappa]^{<\omega}$, $n \in \omega$ and $p \in \mathcal{P}_\kappa$. The set $l(F,n,p)$ is defined as follows

$$l(F,n,p) = \prod_{i\in F} l(n,p^i).$$

For $\sigma \in l(F,n,p)$ and $i \in F$, let σ^i denote the ith component of σ, so $\sigma^i \in p^i$. If there exists $k \in \omega$ such that $F = \{0, 1, \dots, k-1\}$, then we may write $l(k,n,p)$ for $l(F,n,p)$.

For $p, q \in \mathcal{P}_\kappa$, we write $q \le^{F,n} p$ if $q \le p$ and $q^i \le^n p^i$ for all $i \in F$.

For F, n, p, q as above and $\sigma \in l(F,n,p)$, we write $q \le_\sigma p$ if $q \le p$ and $\sigma^i \in q^i$ for every $i \in F$. We also define $p|\sigma = ((p|\sigma)^i)_{i<\kappa}$, where, for $i < \kappa$,

$$(p|\sigma)^i = \begin{cases} p^i|\sigma^i & \text{if } i \in F; \\ p^i & \text{otherwise.} \end{cases}$$

Moreover, let $\varepsilon \in [p]$ and $\sigma \in l(F,n,p)$. We say σ is a *pre-initial segment* of ε and ε is a *post-end-extension* of σ, and write $\sigma \sqsubseteq^* \varepsilon$, if $\sigma^i \sqsubseteq \varepsilon^i$ for every $i \in F$.

LEMMA 2.10 (Fusion 2) [1, Lemma 1.6]. *Let κ be an infinite cardinal. Suppose $(p_k)_{k\in\omega} \subseteq \mathcal{P}_\kappa$. Suppose also that $(F_k)_{k\in\omega} \subseteq [\kappa]^{<\omega}$ is an $\subseteq$-increasing sequence with $\bigcup_{k\in\omega} F_k \supseteq \bigcup\{\mathrm{supp}(p_k) : k \in \omega\}$ and that $(m_k)_{k\in\omega} \subseteq \omega$ is unbounded and increasing. Define $q = (q^i)_{i<\kappa}$ where $q^i = \bigcap_{k\in\omega} p^i_k$ for each $i < \kappa$. Then $q \in \mathcal{P}_\kappa$ and $q \leq^{F_k,m_k} p_k$ for all $k \in \omega$.*

Recall that 2^ω has the product topology. So it has basic open sets of the form $[s] = \{f \in 2^\omega : s \sqsubseteq f\}$, for $s \in 2^{<\omega}$. Let κ be an infinite cardinal. The set $(2^\omega)^\kappa$ also has the product topology, with basic open sets of the form $[\sigma] = \{\varepsilon = (\varepsilon^i)_{i\in\kappa} \in (2^\omega)^\kappa : \sigma \sqsubseteq^* \varepsilon\}$, where $\sigma^i \in 2^{<\omega}$ for every i in some $F \in [\kappa]^{<\omega}$. If $\kappa = \omega$, we may think of such σ as an element of $(2^{<\omega})^{<\omega}$. For $p \in \mathcal{P}_\kappa$, $[p]$ inherits the subspace topology from $(2^\omega)^\kappa$. Unless otherwise stated, $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$ has the product topology, with $\mathcal{R}_\alpha$ having the metric topology and $\mathbb{R}^\mathbb{N} = (2^\omega)^\omega$ having the product topology. Every subspace has the inherited subspace topology.

From now on, we fix an arbitrary $\alpha < \omega_1$ and an arbitrary infinite cardinal κ. We keep in mind that $\sqsubseteq$ denotes end-extension in different cases: Between finite approximations and elements in $\mathcal{AR}_\alpha \cup \mathcal{R}_\alpha$, we use $\sqsubseteq \subseteq \mathcal{AR}_\alpha \times (\mathcal{AR}_\alpha \cup \mathcal{R}_\alpha)$ to denote end-extension of a tree. The symbol is also used to denote end-extensions of a node inside a tree, e.g. $2^{<\omega}$ or $X \in \mathcal{R}_\alpha$.

3. Parametrized $\mathcal{R}_\alpha$ theorem

In this section, we aim to prove the Parametrized $\mathcal{R}_\alpha$ Theorem 1.2. The work in this section is an extension of an adaptation of the results for the Ellentuck space $\mathbb{N}^{[\infty]}$ in [20, §9] to $\mathcal{R}_\alpha$. Instead of parametrization with the infinite product trees $\bigcup_{k\in\omega}\prod_{i<k} H_i$ of finite sets (see [20, §3.3]), we consider parametrization with perfect trees $p \in \mathcal{P}$, and we extend the result to infinite sequences of perfect trees $p \in \mathcal{P}_\kappa$ using the Halpern–Läuchli theorem.

First we consider open subsets of $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$, then we generalize the result to all Souslin-measurable subsets.

3.1. Open subsets of $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$.

LEMMA 3.1. *For $p \in \mathcal{P}_\omega$ and $O \subseteq [p]$ open, there exists $q \leq p$ such that $[q] \subseteq O$ or $[q] \cap O = \emptyset$.*

PROOF. If there exists $\varepsilon \in (2^\omega)^\omega$ such that $\varepsilon \in O \cap [p]$, then since O is open, there exists a pre-initial segment $\sigma \in (2^{<\omega})^{<\omega}$ of ε such that $[p|\sigma] \subseteq O$. So let $q = p|\sigma$. Otherwise, there exists no such ε hence $[p] \cap O = \emptyset$. $\square$

Definition 3.2. Let $p \in \mathcal{P}_\omega$, $k \in \omega$, $F \in [\omega]^{<\omega}$ and $S \subseteq (2^\omega)^\omega$. Let Θ be a set of pre-initial segments of elements in $[p]$. We say $S \cap [p]$ *depends only on* Θ if, for each $\sigma \in \Theta$, either all or none of the post-end-extensions of σ are in S, i.e. either $[\sigma] \subseteq S$ or $[\sigma] \cap S = \emptyset$.

A finite number of applications of Lemma 3.1 gives the following.

Lemma 3.3. *Let $p \in \mathcal{P}_\omega$, $n \in \omega$ and $F \in [\omega]^{<\omega}$. Then for every open set $O \subseteq [p]$ there exists $q \leq^{F,n} p$ such that $O \cap [q]$ depends only on $l(F, n, q)$.*

Applying Lemma 3.3 and the method of fusion we have the following corollary.

Corollary 3.4. *Suppose $p \in \mathcal{P}_\omega$, $n \in \omega$, $F \in [\omega]^{<\omega}$ and O_l $(l \in \omega)$ is a family of open subsets of $(2^\omega)^\omega$. Suppose also that for each l, $n_l \in \omega$ and $F_l \in [\omega]^{<\omega}$ are such that $n < n_l < n_{l+1}, F \subseteq F_l \subseteq F_{l+1}$ and $\bigcup_l F_l = \omega$. Then there exists $q \leq^{F,n} p$ such that for every $l \in \omega$, $O_l \cap [q]$ depends only on $l(F_l, n_l, q)$.*

We will be using the infinite version of the Halpern–Läuchli theorem and its immediate corollary below.

Notation. For $p \in \mathcal{P}$ and $n \in \omega$, by $p(n)$ we denote the set of nodes in p with length n, i.e. $p(n) = \{s \in p : |s| = n\}$.

Theorem 3.5 (HL_ω) [12]. *If $p = (p^i)_{i\in\omega} \in \mathcal{P}_\omega$ and $\bigcup_{n<\omega} \prod_{i<\omega} p^i(n) = G_0 \cup G_1$, then there exists $j \in 2$, $A \in [\omega]^\omega$ and for each $i < \omega$ there exists $q^i \leq p^i$ such that $\bigcup_{n\in A} \prod_{i<\omega} q^i(n) \subseteq G_j$.*

Corollary 3.6. *For $p \in \mathcal{P}_\omega$, $n \in \omega$, $F \in [\omega]^{<\omega}$ and $\bigcup_{n<\omega} \prod_{i<\omega} p^i(n) = G_0 \cup G_1$ there exists $A \in [\omega]^\omega$ and $q \leq^{F,n} p$ such that*

$$\forall \sigma \in l(F, n, p)\ \exists j \in 2\ \bigcup_{n\in A} \prod_{i<\omega} (q|\sigma)^i(n) \subseteq G_j.$$

Lemma 3.7. *Suppose $M \in \mathbb{N}^{[\infty]}$ and O_l $(l \in M)$ is a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$, $n \in \omega$ and $F \in [\omega]^{<\omega}$ there exists $q \leq^{F,n} p$, an infinite subset $N \subseteq M$ and a clopen subset $G \subseteq [q]$ such that for every $l \in N$, $O_l \cap [q] = G$.*

Proof. Let $(n_l)_{l\in M}$ be an increasing sequence above n and $(F_l)_{l\in\omega} \subseteq [\omega]^{<\omega}$ be such that $F \subseteq F_l \subseteq F_{l+1}$ for every $l \in \omega$. Applying Corollary 3.4 to shrink p, we may assume that

$$\forall l \in M \quad O_l \cap [p] \text{ depends only on } l(F_l, n_l, p).$$

By increasing each n_l if necessary, we may assume that

$$\forall l \in M \quad O_l \cap [p] \text{ depends only on } \prod_{i\in F_l} \left\{t \in p^i : |t| = n_l\right\}.$$

We define a colouring $c : p \to 2$ as follows. Let $s = (s^i) \in p = (p^i)_{i<\omega}$. If there does not exist $k \in M$ such that $n_k \le |s^i|$ for each $i \in F_k$ then let $c(s) = 0$. Otherwise, let $k(s) = \max\{k : (\forall i \in F_k)(n_k \le |s^i|)\}$ and define

$$\bar{s} = (\bar{s}^i)_{i<\omega} \quad \text{where } \bar{s}^i = \begin{cases} s_i \restriction n_{k(s)} & \text{if } i \in F_{k(s)} \\ s_i & \text{if } i \in \omega \setminus F_{k(s)}. \end{cases}$$

Since $O_{k(s)} \cap [p]$ depends only on $\prod_{i \in F_{k(s)}} \{t \in p^i : |t| = n_{k(s)}\}$ either $[p|\bar{s}] \subseteq O_{k(s)}$ or $[p|\bar{s}] \cap O_{k(s)} = \emptyset$. We let $c(s) = 1$ if and only if $[p|\bar{s}] \subseteq O_{k(s)}$.

Take $\bar{n} > n$. By Corollary 3.6, there exist $N \in [M]^\omega$ and $q \le^{F,\bar{n}} p$ such that c is constant on $\bigcup_{k \in N} \prod_{i<\omega} (q|\sigma)^i(n_k)$ for every $\sigma \in l(F, \bar{n}, q)$. Without loss of generality, we may assume $n_k > \bar{n}$ for all $k \in N$. Now we check that the map $l \mapsto O_l \cap [q]$ is constant on N, hence q, N satisfy the lemma: Let $l, l' \in N$. Suppose $\varepsilon \in O_l \cap [q]$. Let $t = (\varepsilon^i \restriction n_l)_{i \in \omega}$. As $\varepsilon \in O_l$, by the definition of c, $c(t) = 1$. There exists a unique $\sigma \in l(F, \bar{n}, q)$ with $\sigma \sqsubseteq^* \varepsilon$. Since $t \in \prod_{i<\omega} (q|\sigma)^i(n_l)$ and $c(t) = 1$, it must be the case that c is constantly 1 on $\bigcup_{k \in N} \prod_{i<\omega} (q|\sigma)^i(n_k)$. Therefore, as $l' \in N$, $c \restriction \prod_{i<\omega} (q|\sigma)^i(n_{l'}) \equiv 1$. Let $t' = (\varepsilon^i \restriction n_{l'})_{i<\omega}$. Then $c(t') = 1$, so $\varepsilon \in O_{l'}$. Thus we have proved that, for $l, l' \in N$ and $\varepsilon \in [q]$, if $\varepsilon \in O_l$ then $\varepsilon \in O_{l'}$. Hence by symmetry, $O_l \cap [q] = O_{l'} \cap [q]$ as required. Moreover, since $O_l \cap [q]$ depends only on $\prod_{i \in F_l} \{t \in q^i : |t| = n_l\}$, the set $O_l \cap [q]$ is clopen. $\square$

Notation. For a family $\mathcal{F}$ and a set X, let $\mathcal{F}|X = \{Y \in \mathcal{F} : Y \subseteq X\}$.

Theorem 3.8 (finite version of the pigeonhole principle for $\mathcal{R}_\alpha(n)$) [8, Theorem 3.13]. *Let $n \le k < \omega$ and $X \in \mathcal{R}_\alpha$ be given. Then there is an l such that for each 2-colouring $f : \mathcal{R}_\alpha(n)|X(l) \to 2$ there is a $\zeta \in \mathcal{R}_\alpha(k)|X(l)$ such that f is monochromatic on $\mathcal{R}_\alpha(n)|\zeta$.*

Lemma 3.9. *Let $m \in \omega$ and $A \in \mathcal{R}_\alpha$. Let O_b $(b \in \mathcal{R}_\alpha(m)|A)$ be a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$ there exist $q \le p, B \le A$ and a clopen subset $G \subseteq [q]$ such that $O_b \cap [q] = G$ for every $b \in \mathcal{R}_\alpha(m)|B$.*

Proof. For $k \in \omega$, the set $\mathcal{R}_\alpha(m)|A(m+k)$ is finite. So by Corollary 3.4, we may assume

$$(1) \qquad \begin{cases} \forall k \in \omega \; \forall b \in \mathcal{R}_\alpha(m)|A(m+k) \; \exists F \in [\omega]^{<\omega} \\ \exists n \in \omega \; O_b \text{ depends only on } l(F, n, p). \end{cases}$$

Starting from $A_0 = A$ and $p_0 = p$, we construct decreasing sequences $(A_k)_{k \in \omega}$ and $(p_k)_{k \in \omega}$ together with $(F_k)_{k \in \omega} \subseteq [\omega]^{<\omega}$ and $(m_k)_{k \in \omega} \subseteq \omega$ such that for every $k \in \omega$,

(i) $A_{k+1} \restriction (m+k+1) = A_k \restriction (m+k+1)$ and $p^i_{k+1} \le^{k-i} p^i_k$ for every $i \le k$; and

(ii) the mapping $b \mapsto O_b \cap [p_k]$ is constant on $\mathcal{R}_\alpha(m)|A_k(m+k)$ and the constant value is a clopen subset of $[p_k]$.

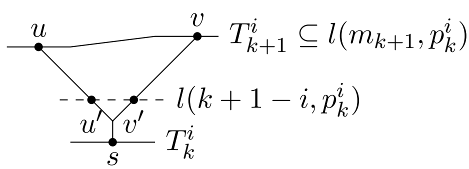
We also define auxiliary sets T_k at each step k, where $T_k = \prod_{i<\omega} T_k^i$ and $T_k^i \subseteq p_k^i$ for every $i < \omega$. Moreover, we require that, for $i \le k$ and $s \in T_k^i$ there are exactly $k-i$ branchings below s in p_k^i, and $|T_k^i| = 2^{k-i}$. For $i \ge k$, T_k^i is a singleton.

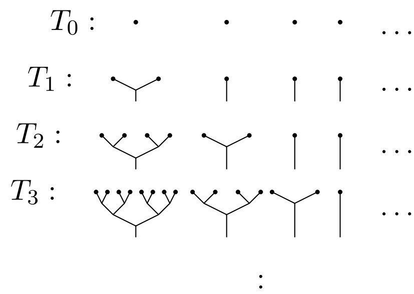
First notice that (ii) holds for $k = 0$ since $\mathcal{R}_\alpha(m)|A_0(m)$ is a singleton. Let $F_0 = \emptyset$, $m_0 = 0$ and $T_0 = \prod_{i<\omega}\{\emptyset\}$. Suppose we have constructed A_k, p_k, F_k and m_k. By Theorem 3.8, there exists $l \in \omega$ such that for every colouring $\mathcal{R}_\alpha(m)|A_k(l) \to 2^{2^{\frac{1}{2}(k+1)(k+2)}}$ there exists $\zeta \in \mathcal{R}_\alpha(m+k+1)|A_k(l)$ with $\mathcal{R}_\alpha(m)|\zeta$ monochromatic. By assumption (1), we can find $F_{k+1} \supseteq (k+1) \cup F_k$ and $m_{k+1} > m_k$ such that

$$\forall b \in \mathcal{R}_\alpha(m)|A_k(l) \quad O_b \text{ depends only on } l(F_{k+1}, m_{k+1}, p_k).$$

We define $T_{k+1} = \prod_{i<\omega} T_{k+1}^i$ as follows:

• For $i \le k$, by construction, every $s \in T_k^i$ has exactly $k-i$ branchings below it in p_k^i. For each $s \in T_k^i$, choose $u, v \in l(m_{k+1}, p_k^i)$ such that u, v are respectively end-extensions of two distinct elements $u', v' \in l(k+1-i, p_k^i)$ end-extending s, as shown in the left figure below. Thus, T_{k+1}^i is a set of end-extensions of elements in T_k^i and $|T_{k+1}^i| = 2|T_k^i|$. The idea is to construct a sequence (T_k) as shown in the right figure below.





• If $i > k$ then $T_k^i = \{s\}$ for some $s \in p_k^i$. Pick an arbitrary $t \in l(m_{k+1}, p_k^i)$ end-extending s, and let $T_{k+1}^i = \{t\}$. So $|T_{k+1}^i| = 1$.

Then

$$\Bigg| \prod_{i \in F_{k+1}} T_{k+1}^i \Bigg| = \prod_{i \le k} 2|T_k^i| = \prod_{i<k+1} 2^{k+1-i} = 2^{\frac{1}{2}(k+1)(k+2)}.$$

By the choice of F_{k+1} and m_{k+1}, for every $b \in \mathcal{R}_\alpha(m)|A_k(l)$ and every $\sigma \in \prod_{i \in F_{k+1}} T_{k+1}^i$, either $[p_k|\sigma] \subseteq O_b$ or $[p_k|\sigma] \cap O_b = \emptyset$. For each b there are $2^{2^{\frac{1}{2}(k+1)(k+2)}}$ possibilities for $[p_k|\sigma]$ $\big(\sigma \in \prod_{i \in F_{k+1}} T_{k+1}^i\big)$ to be inside or disjoint from O_b. Thus, by the choice of l, there exists $\zeta \in \mathcal{R}_\alpha(m+k+1)|A_k(l)$ such that for each $\sigma \in \prod_{i\in F_{k+1}} T^i_{k+1}$,

$$\text{either } \forall b \in \mathcal{R}_\alpha(m)|\zeta \quad [p_k|\sigma] \subseteq O_b, \qquad \text{or } \forall b \in \mathcal{R}_\alpha(m)|\zeta \quad [p_k|\sigma] \cap O_b = \emptyset.$$

Now let $A_{k+1} \leq A_k$ be such that $A_{k+1} \restriction (m+k+1) = A_k \restriction (m+k+1)$ and $A_{k+1}(m+k+1) = \zeta$. For $i \in \omega$, let $p^i_{k+1} = \bigcup_{s\in T^i_{k+1}} p^i_k|s$, so $p^i_{k+1} \leq^{k-i} p^i_k$ for $i \leq k$. Let $p_{k+1} = (p^i_{k+1})_{i<\omega}$. Then the mapping $b \mapsto O_b \cap [p_{k+1}]$ is constant on $\mathcal{R}_\alpha(m)|A_{k+1}(m+k+1)$ and the constant value is a clopen subset of $[p_{k+1}]$, hence (ii) is satisfied.

This finishes the construction of the sequences $(A_k)_{k\in\omega}$ and $(p_k)_{k\in\omega}$. Let

$$A_\infty = A \restriction m \cup \bigcup_{k\in\omega} A_k(m+k), \text{ and } p_\infty = (p^i_\infty)_{i<\omega} = \left(\bigcap_{k'<\omega} p^i_{i+k'} \right)_{i<\omega}.$$

Clearly, $A_\infty \in \mathcal{R}_\alpha$. By (i), for every $i, k' \in \omega$, $p^i_{i+k'+1} \leq^{k'} p^i_{i+k'}$, so $\bigcap_{k'<\omega} p^i_{i+k'} \in \mathcal{P}$. Hence $p_\infty \in \mathcal{P}_\omega$. Then for $k \in \omega$, the mapping $b \mapsto O_b \cap [p_\infty]$ is constant on $\mathcal{R}_\alpha(m)|A_\infty(m+k)$. Let the constant value be denoted by $O^*_{\|A_\infty(m+k)\|}$. Note that $O^*_{\|A_\infty(m+k)\|}$ is clopen in $[p_\infty]$. Now we have $\|A_\infty\| \in \mathbb{N}^{[\infty]}$ and a family O^*_j $(j \in \|A_\infty\|)$ of open subsets of $[p_\infty]$. By Lemma 3.7, there exists an infinite $N \subseteq \|A_\infty\|$, $q \leq p_\infty$ and a clopen $G \subseteq [q]$ such that $O^*_j \cap [q] = G$ for every $j \in N$. Then we can find $B \in \mathcal{R}_\alpha$ with $B \leq A_\infty$ and $\|B\| = N$. Thus $O_b \cap [q] = G$ for all $b \in \mathcal{R}_\alpha(m)|B$ as required. □

We would like to generalize Lemma 3.9 to Lemma 3.13 below.

DEFINITION 3.10. Let $\mathcal{F} \subseteq \mathcal{AR}_\alpha$. We say that $\mathcal{F}$ is *Nash–Williams* if $s \not\sqsubseteq t$ for distinct $s, t \in \mathcal{F}$; $\mathcal{F}$ is *Sperner* if $s \not\leq t$ for distinct $s, t \in \mathcal{F}$.

DEFINITION 3.11. Let $a \in \mathcal{AR}_\alpha$ and $A \in \mathcal{R}_\alpha$. We say that $\mathcal{F} \subseteq \mathcal{AR}_\alpha$ is a *barrier on* $[a, A]$ if $\mathcal{F}$ is Sperner and every $X \in [a, A]$ has an initial segment in $\mathcal{F}$. We say $\mathcal{F}$ is a *barrier on* A if it is a barrier on $[\emptyset, A]$.

DEFINITION 3.12 (rank of barriers). Let $\mathcal{F}$ be a barrier on $[a, A]$. Consider $T(\mathcal{F}) := \{s \in \mathcal{AR}_\alpha : (a \sqsubseteq s \leq A) \wedge (\exists t \in \mathcal{F})(s \sqsubseteq t)\}$ as a tree ordered by $\sqsubseteq$. We define a strictly decreasing map $\rho_{T(\mathcal{F})}$ as follows.

$$\rho_{T(\mathcal{F})} : T(\mathcal{F}) \to \mathrm{Ord};$$
$$s \mapsto \sup\{\rho_{T(\mathcal{F})}(t) + 1 : t \in T(\mathcal{F}) \wedge (s \sqsubset t)\},$$

where $\sup \emptyset = 0$. The *rank* of $\mathcal{F}$ on $[a, A]$ is defined to be $\mathrm{rk}(\mathcal{F}) = \rho_{T(\mathcal{F})}(a)$.

LEMMA 3.13. *Let $a \in \mathcal{AR}_\alpha$ and $A \in \mathcal{R}_\alpha$. Suppose $\mathcal{F}$ is a barrier on $[a, A]$ and O_b $(b \in \mathcal{F})$ is a family of open subsets of $(2^\omega)^\omega$. Then for every*

$p \in \mathcal{P}_\omega$, $n \in \omega$ and $F \in [\omega]^{<\omega}$, there exist $q \leq^{F,n} p$, $B \in [a, A]$ and a clopen subset $G \subseteq [q]$ such that $O_b \cap [q] = G$ for every $b \in \mathcal{F} \cap \mathcal{AR}_\alpha[a, B]$.

Proof. We prove it by induction on the rank of $\mathcal{F}$. The base case, where $\mathrm{rk}(\mathcal{F}) = 0$ and $\mathcal{F} = \{a\}$, is trivial. If $\mathrm{rk}(\mathcal{F}) = 1$ then $\mathcal{F} = \mathcal{R}_\alpha(|a|)|A$ and the statement holds by Lemma 3.9. So we assume $\mathrm{rk}(\mathcal{F}) > 1$ and the statement holds for barriers of smaller ranks.

For $\zeta \in \mathcal{R}_\alpha(|a|)|A$, let $\mathcal{F}_\zeta = \{b \in \mathcal{F} : a \sqcup \zeta \sqsubseteq b\}$. Note that $\mathcal{F}_\zeta$ is a barrier on $[a \sqcup \zeta, A]$ of smaller rank than $\mathcal{F}$. We construct sequences $(p_k)_{k\in\omega}$ and $(A_k)_{k\in\omega}$ as follows: Let $\zeta_0 = A(|a|)$. By the induction hypothesis, for O_b ($b \in \mathcal{F}_{\zeta_0} \subseteq \mathcal{F}$) and p, F, n given,

$$\exists p_0 \leq^{F,n} p \; \exists A_0 \in [a \sqcup \zeta_0, A] \quad \text{such that } b \mapsto O_b \cap [p_0] \text{ is constant on } \mathcal{F} \cap \mathcal{AR}_\alpha[a \sqcup \zeta_0, A_0],$$

and the constant value is clopen in $[p_0]$. Suppose we have constructed p_k and A_k. Let $F_{k+1} = F_k \cup k$. By applying the induction hypothesis finitely many times, we find $p_{k+1} \leq^{F_k, n+k} p_k$ and $A_{k+1} \leq A_k$ such that $A_{k+1} \restriction (|a| + k + 2) = A_k \restriction (|a| + k + 2)$ and

$$\forall \zeta \in \mathcal{R}_\alpha(|a|)|A_{k+1}(|a| + k + 1) \quad b \mapsto O_b \cap [p_{k+1}] \text{ is constant on } \mathcal{F} \cap \mathcal{AR}_\alpha[a \sqcup \zeta, A_{k+1}],$$

and the constant value is clopen in $[p_{k+1}]$. Let $p_\infty = \left(\bigcap p^i_k\right)_{i<\omega}$ and $A_\infty = a \cup \bigcup_{k\in\omega} A_k(|a| + k)$.

By construction, the set $b \mapsto O_b \cap [p_\infty]$ for $b \in \mathcal{F} \cap \mathcal{AR}_\alpha[a, A_\infty]$ depends only on $b(|a|)$. For $\zeta \in \mathcal{R}_\alpha(|a|)|A_\infty$, let O^*_ζ be the constant value of the mapping on $\mathcal{F} \cap \mathcal{AR}_\alpha[a \sqcup \zeta, A_\infty]$, which is clopen in $[p_\infty]$. Now we have $A_\infty \in \mathcal{R}_\alpha$ and O^*_ζ ($\zeta \in \mathcal{R}_\alpha(|a|)|A_\infty$) open in $[p_\infty]$. By Lemma 3.9,

$$\exists q \leq^{F,n} p_\infty \; \exists B \leq A_\infty \exists G \subseteq [q] \text{ clopen} \quad \text{such that } O^*_\zeta \cap [q] = G \text{ for every } \zeta \in \mathcal{R}_\alpha(|a|)|B,$$

where we may assume that $B \in [a, A_\infty]$. Thus $O_b \cap [q] = G$ for every $b \in \mathcal{F} \cap \mathcal{AR}_\alpha[a, B]$. $\square$

Corollary 3.14. *Let $a \in \mathcal{AR}_\alpha$ and $A \in \mathcal{R}_\alpha$. Suppose $\mathcal{F}$ is a barrier on $[a, A]$ and O_b ($b \in \mathcal{F}$) are open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$ there exists $q \leq p$ and $B \in [a, A]$ such that either $[q] \subseteq O_b$ for all $b \in \mathcal{F} \cap \mathcal{AR}_\alpha[a, B]$, or $[q] \cap O_b = \emptyset$ for every $b \in \mathcal{F} \cap \mathcal{AR}_\alpha[a, B]$.*

3.1.1. Combinatorical forcing parametrized. In this subsubsection, let $\mathcal{O} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ be an open subset.

DEFINITION 3.15. Let $A \in \mathcal{R}_\alpha, a \in \mathcal{AR}_\alpha, p \in \mathcal{P}_\omega$ and $\sigma \in l(F, n, p)$ for some $n \in \omega$ and $F \in [\omega]^{<\omega}$. We say (A, p) *accepts* (a, σ) if $[a, A] \times [p|\sigma] \subseteq \mathcal{O}$; (A, p) *rejects* (a, σ) if there does not exist $q \leq_\sigma p$ and $B \leq A$ such that (B, q) accepts (a, σ). We say (A, p) *decides* (a, σ) if it either accepts or rejects (a, σ).

The following are immediate facts.

LEMMA 3.16. *Let $A \in \mathcal{R}_\alpha, a \in \mathcal{AR}_\alpha, p \in \mathcal{P}_\omega$ and $\sigma \in l(F, n, p)$ for some $n \in \omega$ and $F \in [\omega]^{<\omega}$.*

(a) *If (A, p) accepts (a, σ) then for every $q \leq p$ and $B \leq A$, (B, q) accepts (a, σ).*

(b) *If (A, p) rejects (a, σ) then for every $q \leq_\sigma p$ and $B \leq A$, (B, q) rejects (a, σ).*

(c) *For every pair (A, p) and (a, σ) there exists $q \leq_\sigma p$ and $B \leq A$ such that (B, q) decides (a, σ).*

(d) *If (A, p) decides (a, σ), $B/a \subseteq A$ and $q \leq_\sigma p$, then (B, q) decides (a, σ).*

(e) *If (A, p) decides (a, τ), where $\tau \in l(F, m, p)$ for some F, m, and $\tau_i \sqsubseteq \sigma_i$ for every $i \in F$, then (A, p) decides (a, σ).*

Let us recall the symbol $\| \cdot \|$ from Section 2.1. For $X \in \mathcal{R}_\alpha \cup \mathcal{AR}_\alpha$ and a node $t \in X$, $\|t\| = n$ if t is in the nth subtree $\mathbb{T}(n)$ of $\mathbb{T}$. We let $\|X\| = \{\|t\| : t \in X\}$.

LEMMA 3.17. *Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{R}_\alpha$, $n \in \omega$ and $F \in [\omega]^{<\omega}$. Then there exist $q \leq^{F,n} p$ and $B \leq A$ such that*

$$\forall a \in \mathcal{AR}_\alpha \; \forall l \in \|B\| \text{ with } l \geq \max \|a\| \; \forall \sigma \in l(l, l, q) \quad (B, q) \text{ decides } (a, \sigma).$$

PROOF. By shrinking A, we may assume $n, \max F < \min \|A\|$. We build $(p_k)_{k\in\omega}$, $(A_k)_{k\in\omega}$ and $(n_k)_{k\in\omega}$ recursively, starting from $p_0 = p$, $A_0 = A$ and $n_0 = \min \|A\|$. Suppose we have constructed p_k, A_k. Let $n_k = \|A_k(k)\|$. Applying Lemma 3.16(c) and (d) finitely many times, we have $p_{k+1} \leq^{n_k,n_k} p_k$ and $A_{k+1} \subseteq A_k/n_k$ such that (A_{k+1}, p_{k+1}) decides (a, σ) for every $a \leq \mathbb{T}(0) \cup \cdots \cup \mathbb{T}(n_k)$ and every $\sigma \in l(n_k, n_k, p_k)$.

Let $q = \big(\bigcap_{k\in\omega} p^i_k\big)_{i<\omega}$ and $B = \bigcup_{k\in\omega} A_k(k)$. So $q \leq^{n_k,n_k} p_k$ and $B/n_k \subseteq A_{k+1}$ for every $k \in \omega$ and $\|B\| = \{n_k : k \in \omega\}$. We check that q and B satisfy the lemma: Let $a \in \mathcal{AR}_\alpha$ and $n_l \in \|B\|$ with $n_l \geq \max \|a\|$ and $\sigma \in l(n_l, n_l, q)$. Let $n_k \in \|B\|$ be minimal such that $\max \|a\| \leq n_k$. So $n_k \leq n_l$. Let $\tau \in l(n_k, n_k, q)$ be such that $\tau_i \sqsubseteq \sigma_i$ for every $i < n_k$. By construction, (A_{k+1}, p_{k+1}) decides (a, τ). As $B/a = B/n_k \subseteq A_{k+1}$ and $q \leq_\tau p_{k+1}$, by Lemma 3.16(d), (B,q) decides (a,τ). Then by Lemma 3.16(e), (B,q) decides (a,σ) as required. $\square$

We would like to apply the following abstract Galvin lemma to the space $\mathcal{R}_\alpha^{(a)}$ defined below, in order to obtain a desired result for the space $\mathcal{R}_\alpha$.

Theorem 3.18 (Abstract Galvin Lemma) [20, Theorem 5.15]. *Suppose $\mathcal{R}$ is a topological Ramsey space. Then for every family $\mathcal{F} \subseteq \mathcal{AR}$ and every $X \in \mathcal{R}$ there is $Y \le X$ such that either $\mathcal{F}|Y = \emptyset$ or every $B \le Y$ has an initial segment in $\mathcal{F}$.*

Recall that for $a \in \mathcal{AR}_\alpha$ $|a|$ is the length of a, i.e. $|a| = m$ if $a = X \restriction m$ for some $X \in \mathcal{R}_\alpha$. Note that $|a|$ is different from $\|a\|$.

Definition 3.19. For $a \in \mathcal{AR}_\alpha$, let $\mathcal{R}_\alpha^{(a)} = \{Y \setminus a : Y \in [a,\mathbb{T}]\}$. So members of $\mathcal{R}_\alpha^{(a)}$ are tails of members of $[a,\mathbb{T}]$ above a. Let $k = \max\|a\| + 1$. If $X = Y \setminus a \in \mathcal{R}_\alpha^{(a)}$ and $n \in \omega$, then we define $X(n) = Y(n) - k$.

More precisely, we define the space $\mathcal{R}_\alpha^{(a)}$ as follows. Let $\mathbb{T}^{(k)} = \bigcup_{n\ge k} \mathbb{T}_\alpha(n)$. Then the *nth subtree of* $\mathbb{T}^{(k)}$ is $\mathbb{T}^{(k)}(n) = \mathbb{T}_\alpha(n+k)$. The members of $\mathcal{R}_\alpha^{(a)}$ are infinite subtrees X of $\mathbb{T}^{(k)}$ with the same structure as $\mathbb{T}^{(|a|)}$, that is, $X \in \mathcal{R}_\alpha^{(a)}$ if $X \subseteq \mathbb{T}^{(k)}$ such that there exists a strictly increasing sequence $(k_n)_{n<\omega}$ such that $X \cap \mathbb{T}^{(k)}(k_n) \cong \mathbb{T}^{(|a|)}(n)$ for all $n \in \omega$, and $X \cap \mathbb{T}^{(k)}(j) = \emptyset$ for all $j \in \omega \setminus (k_n)_{n\in\omega}$. Let the *nth tree of* X be $X(n) = X \cap \mathbb{T}^{(k)}(k_n)$. For $n < \omega$, $\restriction$, $\mathcal{AR}_\alpha^{(a)}$, $\mathcal{AR}_n^{\alpha(a)}$, $\le^{(a)}$, are defined in the same way as those for $\mathcal{R}_\alpha$. Basic open sets of $\mathcal{R}_\alpha^{(a)}$ are of the form $[b,\mathbb{T}^{(k)}]$ with $b \in \mathcal{AR}_\alpha^{(a)}$.

It follows from the fact that $(\mathcal{R}_\alpha,\le,r)$ is a topological Ramsey space that $(\mathcal{R}_\alpha^{(a)},\le^{(a)},r)$ is also a topological Ramsey space.

Theorem 3.20. *The space $(\mathcal{R}_\alpha^{(a)},\le^{(a)},r)$ is a topological Ramsey space.*

Corollary 3.21 ($\mathcal{R}_\alpha^{(a)}$-Galvin). *For $a \in \mathcal{AR}_\alpha$, $\mathcal{F} \subseteq \mathcal{AR}_\alpha[a,\mathbb{T}]$ and $X \in \mathcal{R}_\alpha$, there exists $Y \in [a,X]$ such that either $\mathcal{F}|Y = \emptyset$ or $\mathcal{F}|Y$ contains a barrier on $[a,Y]$.*

Proof. If $\mathcal{F} = \{a\}$, then the second alternative must hold. So we assume $\mathcal{F} \ne \{a\}$. Let $\mathcal{G} = \{c \in \mathcal{AR}_\alpha^{(a)} : a \cup c \in \mathcal{F}\} \subseteq \mathcal{AR}_\alpha^{(a)}$. Since $X/a \in \mathcal{R}_\alpha^{(a)}$, by Theorem 3.18, there exists $Y' \in \mathcal{R}_\alpha^{(a)}$ with $Y' \le X/a$ such that one of the following two cases holds: If $\mathcal{G}|Y' = \emptyset$, then for every $c \le^{(a)} Y'$, $c \notin \mathcal{G}$ so $a \cup c \notin \mathcal{F}$. Let $Y = a \cup Y'$. Then $\mathcal{F}|Y = \emptyset$. Otherwise, every $c \le^{(a)} Y'$ has an initial segment $\bar{c} \in \mathcal{G}$. Again let $Y = a \cup Y'$. Then every $c \in \mathcal{AR}[a,Y]$ has an initial segment $a \cup (\overline{c \setminus a}) \in \mathcal{F}$. Thus $\mathcal{F}|Y$ contains a barrier on $[a,Y]$. $\square$

Now we are ready to prove the main theorem of this subsection.

THEOREM 3.22. *Let $\mathcal{O} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ be open. For $p \in \mathcal{P}_\omega, A \in \mathcal{R}_\alpha$ and $a \in \mathcal{AR}_\alpha$, there exist $q \leq p$ and $B \leq A$ such that $[a, B] \times [q] \subseteq \mathcal{O}$ or $[a, B] \times [q] \cap \mathcal{O} = \emptyset$.*

PROOF. We concentrate on the case $a = \emptyset$. The proof for $a \neq \emptyset$ is similar and utilizes the space $\mathcal{R}_\alpha^{(a)}$.

By Lemma 3.17, shrinking p and A if necessary, we assume

(2)
$$\forall a \in \mathcal{AR}_\alpha \; \forall l \in \|A\| \text{ with } l \geq \max \|a\| \; \forall \sigma \in l(l, l, p) \quad (A, p) \text{ decides } (a, \sigma).$$

For $b \in \mathcal{AR}_\alpha$, let

$$\mathcal{O}_b = \bigcup \{[p|\tau] : ([b, A] \times [p|\tau] \subseteq \mathcal{O}) \wedge (\exists F \in [\omega]^{<\omega})(\exists n \in \omega)(\tau \in l(F, n, p))\}$$
$$\mathcal{B} = \{b \in \mathcal{AR}_\alpha : (\mathcal{O}_b \neq \emptyset) \wedge (b \neq \emptyset)\}.$$

Applying Corollary 3.21 to $\mathcal{B}$ and $[\emptyset, A]$, we consider two cases.

Case 1: $\exists B \leq A$ with $\mathcal{B}|B = \emptyset$. We check that $[\emptyset, B] \times [p] \cap \mathcal{O} = \emptyset$: Otherwise, there is $(X, \varepsilon) \in [\emptyset, B] \times [p] \cap \mathcal{O}$. Since $\mathcal{O}$ is open, there is $b \sqsubseteq X$ and a pre-initial segment σ of ε such that $[b, \mathbb{T}] \times [p|\sigma] \subseteq \mathcal{O}$. Then $[b, A] \times [p|\sigma] \subseteq \mathcal{O}$ and $\mathcal{O}_b \neq \emptyset$. Thus $b \in \mathcal{B}|B$, a contradiction.

Case 2: $\exists B \leq A$ such that $\mathcal{B}|B$ contains a barrier $\mathcal{F}$ on B. By Corollary 3.14, there exist $C \leq B$ and $q \leq p$ such that either

(i) $[q] \subseteq \mathcal{O}_b$ for every $b \in \mathcal{F}|C$, or

(ii) $[q] \cap \mathcal{O}_b = \emptyset$ for every $b \in \mathcal{F}|C$.

If (i) holds, then by the definition of $\mathcal{O}_b$ and that $\mathcal{F}$ contains a barrier on C, $[\emptyset, C] \times [q] \subseteq \bigcup_{b \in \mathcal{F}|C} [b, C] \times [q] \subseteq \mathcal{O}$ as required. So we assume (ii) holds.

CLAIM 3.22.1. *If* (ii) *holds, then for every $F \in [\omega]^{<\omega}$, $n \in \omega$, $b \in \mathcal{F}|C$ and $D \leq C$, there exists $\bar{q} \leq^{F,n} q$ such that $[b, E] \times [\bar{q}] \cap \mathcal{O} = \emptyset$.*

We postpone the proof of this claim to the end of this proof of the theorem. Now using the claim, we can construct decreasing sequences $(C_k)_{k \in \omega}$ and $(q_k)_{k \in \omega}$, with $C_0 = C$, $q_0 = q$ and $q_{k+1} \leq^{k,k} q_k$ for $k \in \omega$, such that

$$\forall k \in \omega \; \forall b \in \mathcal{F} \quad \text{with } b \subseteq \bigcup_{i < k+1} C_i(i) \quad [b, C_{k+1}] \times [q_{k+1}] \cap \mathcal{O} = \emptyset.$$

Let $C_\infty = \bigcup_{k \in \omega} C_k(k)$ and $q_\infty = \bigcap_{k \in \omega} q_k$. As $\mathcal{F}$ contains a barrier on C_∞, $[\emptyset, C_\infty] \times [q_\infty] \subseteq \bigcup \{[b, C_k] \times [q_k] : b \in \mathcal{F}|(C_\infty \restriction k), k \in \omega\}$ which is disjoint from $\mathcal{O}$, as required.

Now it is sufficient to prove the claim. Fix $F \in [\omega]^{<\omega}, n \in \omega,\ b \in \mathcal{F}|C$ and $D \leq C$. Let

$$\mathcal{B}_b = \{c \in \mathcal{AR}_\alpha[b, C] : \mathcal{O}_c \cap [q] \neq \emptyset\}.$$

Applying Corollary 3.21 to $\mathcal{B}_b$ and $[b, D]$, we have two cases.

Case 2.1: $\exists E \leq D$ such that $\mathcal{B}_b|E = \emptyset$. Then similar to Case 1, $[b, E] \times [q] \cap \mathcal{O} = \emptyset$.

Case 2.2: $\exists E \leq D$ such that $\mathcal{B}_b|E$ contains a barrier $\mathcal{F}_b$ on $[b, E]$. By Lemma 3.13 applied to $\mathcal{F}_b$ on $[b, E]$ and $\mathcal{O}_c(c \in \mathcal{F}_b)$, we find $\bar{q} \leq^{F,n} q$, $E' \in [b, E]$ and a clopen $G \subseteq [\bar{q}]$ such that $\mathcal{O}_c \cap [\bar{q}] = G$ for every $c \in \mathcal{F}_b \cap \mathcal{AR}_\alpha[b, E']$.

If $G = \emptyset$, then $[b, E'] \times [\bar{q}] \cap \mathcal{O} = \emptyset$ as required. So we assume $G \neq \emptyset$ and show that $\mathcal{O}_b \cap [q] \neq \emptyset$, contradicting (ii). Pick $l \in \|A\|$ with $l \geq \max \|b\|$ and l large enough that there exists $\sigma \in l(l, l, p) \cap l(l, l', \bar{q})$, for some $l' < l$, such that $[\bar{q}|\sigma] \subseteq G$. So $[\bar{q}|\sigma] \subseteq \mathcal{O}_c$ for every $c \in \mathcal{F}_b \cap \mathcal{AR}_\alpha[b, E']$. Therefore

$$[b, E'] \times [\bar{q}|s] \subseteq \bigcup \big\{ [c, E'] \times [\bar{q}|\sigma] : c \in \mathcal{F}_b \cap \mathcal{AR}_\alpha[b, E'] \big\} \subseteq \mathcal{O}_c.$$

By assumption (2) and the choice of σ, (A, p) decides (b, σ). As $\bar{q} \leq q$, $\sigma \in l(l, l', \bar{q})$ and $E' \leq A$ are such that $[b, E'] \times [\bar{q}|\sigma] \subseteq \mathcal{O}$, (A, p) must accept (b, σ). So $[p|\sigma] \subseteq \mathcal{O}_b$. Hence $[\bar{q}|\sigma] \subseteq \mathcal{O}_b \cap [q]$ and the intersection is non-empty as required. $\square$

3.2. Souslin-measurable subsets of $\boldsymbol{\mathcal{R}_\alpha} \times \mathbb{R}^{\mathbb{N}}$. As in the previous subsection, this subsection adapts the results in [20, §9] to the topological Ramsey space $\mathcal{R}_\alpha$ parametrized by infinite sequences of perfect trees.

Definition 3.23. A subset $\mathcal{X} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is *perfectly Ramsey* if for every $a \in \mathcal{AR}_\alpha$, $A \in \mathcal{R}_\alpha$ and $p \in \mathcal{P}_\omega$, there exists $B \in [a, A], q \leq p$ such that $[a, B] \times [q] \subseteq \mathcal{X}$ or $[a, B] \times [q] \cap \mathcal{X} = \emptyset$.

So we can rephrase Lemma 3.22 as follows.

Lemma 3.24. *Every open subset of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is perfectly Ramsey.*

Lemma 3.25. *The perfectly Ramsey subsets of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ form a σ-field.*

Proof. The perfectly Ramsey subsets clearly form a field. We check only that it is closed under countable union. Let $(\mathcal{X}_k)_{k\in\omega}$ be a given sequence of perfectly Ramsey sets. Let $a \in \mathcal{AR}_\alpha$, $A \in \mathcal{R}_\alpha$ and $p \in \mathcal{P}_\omega$ be given. We show that $\mathcal{X} := \bigcup_{k\in\omega} \mathcal{X}_k$ is perfectly Ramsey.

We build a sequence (A_k, p_k) recursively, starting from $(A_0, p_0) = (A, p)$. Assuming (A_k, p_k) built, we construct (A_{k+1}, p_{k+1}). By the fact that $\mathcal{X}_k$

is perfect Ramsey and that the set $a \cup \bigcup_{i\leq k} A_i(|a|+i)$ is finite, we obtain $p_{k+1} \leq^{k,k} p_k$ and $A_{k+1} \subseteq \bigcup_{i\geq k} A_k(|a|+i)$ such that for every $b \leq a \cup \bigcup_{i\leq k} A_i(|a|+i)$ and every $\sigma \in l(k,k,p_k)$, either $[b, A_{k+1}] \times [p_{k+1}|\sigma] \subseteq \mathcal{X}_k$ or $[b, A_{k+1}] \times [p_{k+1}|\sigma] \cap \mathcal{X}_k = \emptyset$.

Let $A_\infty = a \cup \bigcup_k A_k(|a|+k)$ and $p_\infty = (\bigcap_k p_k^i)_{i<\omega}$. Note that $[a, A_\infty] \times [p_\infty] \cap \mathcal{X}$ is open in $[a, A_\infty] \times [p_\infty]$. Thus by Lemma 3.24, the conclusion holds. $\square$

THEOREM 3.26. *The field of perfectly Ramsey subsets of $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$ is closed under the Souslin operation.*

PROOF. Let $\mathcal{X}_v$ $(v \in \omega^{<\omega})$ be a given Souslin scheme of perfectly Ramsey subsets of $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$. Without loss of generality, we assume that $\mathcal{X}_u \supseteq \mathcal{X}_v$ if $u \sqsubseteq v$. Let $a \in \mathcal{AR}_\alpha$, $A \in \mathcal{R}_\alpha$ and $p \in \mathcal{P}_\omega$ be given. We aim to show that $\mathcal{X} := \bigcup_{f\in\omega^\omega} \bigcap_{n\in\omega} \mathcal{X}_{f\restriction n}$ is also perfectly Ramsey. For each $v \in \omega^{<\omega}$, let $\mathcal{X}_v^* = \bigcup_{v\sqsubseteq f\in\omega^\omega} \bigcap_{n\in\omega} \mathcal{X}_{f\restriction n}$. Note that $\mathcal{X}_v^* \subseteq \mathcal{X}_v$.

We build a sequence (A_k, p_k) such that for every $k < \omega$, $\sigma \in l(k,k,p_k)$, $b \subseteq a \cup \bigcup_{i\leq k} A_i(|a|+i)$ and $v \in \omega^{<\omega}$ with $\max(v) < k$, either

(1) $[b, A_{k+1}] \times [p_{k+1}|\sigma] \cap \mathcal{X}_v^* = \emptyset$, or

(2) there does not exist $q \leq_\sigma p_{k+1}, B \leq A_{k+1}$ with $[b, B] \times [q|\sigma] \cap \mathcal{X}_v^* = \emptyset$.

Let $A_0 = A$ and $p_0 = p$. Let $(\sigma_l, b_l, v_l)_{l\leq m}$ be an enumeration of

$$l(k,k,p_k) \times \Big\{ b \in \mathcal{AR}_\alpha : b \subseteq a \cup \bigcup_{i\leq k} A_i(|a|+i) \Big\} \times \big\{ v \in \omega^{<\omega} : \max(v) < k \big\}.$$

(i) Suppose there exist $q \leq_{\sigma_{l+1}} q_l$ and $B \leq B_l$ such that $[b_{l+1}, B] \times [q|\sigma_{l+1}] \cap \mathcal{X}_{v_{l+1}}^* = \emptyset$. Let q_{l+1} be the $\subseteq$-maximal such q and B_{l+1} the corresponding B. Then $q_{l+1} \leq^{k,k} p_k$ and $[b_{l+1}, B_{l+1}] \times [q_{l+1}|\sigma_{l+1}] \cap \mathcal{X}_{v_{l+1}}^* = \emptyset$.

(ii) If there does not exist $q \leq_{\sigma_{l+1}} q_l$, and $B \leq B_l$ such that $[b_{l+1}, B] \times [q|\sigma_{l+1}] \cap \mathcal{X}_{v_{l+1}}^* = \emptyset$, let $q_{l+1} = q_l$ and $B_{l+1} = B_l$.

Proceeding this way for all $l \leq m$, we arrive at $q_m \leq^{k,k} p_k$ and $B_m \leq A_k$. Let $p_{k+1} = q_m$, $A_{k+1} = B_m$. This finishes the construction of $(A_k, p_k)_{k\in\omega}$. Let $p_\infty = (\bigcap_{k\in\omega} p_k^i)_{i<\omega}$ and $A_\infty = a \cup \bigcup\{A_k(|a|+k) : k \in \omega\}$. So for every $k < \omega$, $\sigma \in l(k,k,p_\infty)$, $b \leq A_\infty \restriction (|a|+k)$ and $v < \omega^{<\omega}$ with $\max(v) < k$, either $[b, A_\infty] \times [p_\infty|\sigma] \cap \mathcal{X}_v^* = \emptyset$ or there does not exist $q \leq p_\infty$, $B \leq A_\infty$ with $[b, B] \times [q|\sigma] \cap \mathcal{X}_v^* = \emptyset$.

For $v \in \omega^{<\omega}$, let

$$\Psi(\mathcal{X}_v^*) = \bigcup \{[b, A_\infty] \times [p_\infty|\sigma] : [b, A_\infty] \times [p_\infty|\sigma] \cap \mathcal{X}_v^* = \emptyset\}, \text{ and}$$

$$\Phi(\mathcal{X}_v^*) = (\mathcal{X}_v \cap [a, A_\infty] \times [p_\infty]) \setminus \Psi(\mathcal{X}_v^*).$$

By Lemma 3.25, $\Phi(\mathcal{X}_v^*)$ is perfectly Ramsey. The set $\mathcal{M}_v := \Phi(\mathcal{X}_v^*) \setminus \bigcup_{l<\omega} \Phi(\mathcal{X}_{v^\frown l}^*)$ is also perfectly Ramsey, and $\mathcal{M}_v \subseteq \Phi(\mathcal{X}_v^*) \setminus \mathcal{X}_v^*$. Since $\mathcal{M}_v$

is perfectly Ramsey, starting from $B_0 = A_\infty$ and $q_0 = p_\infty$, we can build $B_{k+1} \in [|a|+k, B_k]$ and $q_{k+1} \leq^{k,k} q_k$ such that for every $\sigma \in l(k,k,q_k)$, $b \subseteq a \cup \bigcup_{i<k} B_i(|a|+i)$ and $v \in \omega^{<\omega}$ with $\max(v) < k$, either

(a) $[b, B_{k+1}] \times [q_{k+1}|\sigma] \subseteq \mathcal{M}_v$, or

(b) $[b, B_{k+1}] \times [q_{k+1}|\sigma] \cap \mathcal{M}_v = \emptyset$.

We check that alternative (b) always holds: Suppose not. We assume, for some k, there are σ, b, v as above with $[b, B_{k+1}] \times [q_{k+1}|\sigma] \subseteq \mathcal{M}_v$. Note $q_{k+1} \leq p_{k+1}$, $B_{k+1} \leq A_{k+1}$ are such that $[b, B_{k+1}] \times [q_{k+1}|\sigma] \subseteq \Phi(\mathcal{X}^*_v) \setminus \mathcal{X}^*_v$ disjoint from $\mathcal{X}^*_v$, i.e. alternative (2) above does not hold. So $[b, A_{k+1}] \times [p_{k+1}|\sigma] \cap \mathcal{X}^*_v = \emptyset$. Then by definition, $[b, A_{k+1}] \times [p_{k+1}|\sigma] \subseteq \Psi(\mathcal{X}^*_v)$. So $\mathcal{M}_v \cap \Psi(\mathcal{X}^*_v) \neq \emptyset$, contradicting that $\mathcal{M}_v \subseteq \Phi(\mathcal{X}^*_v)$ is disjoint from $\Psi(\mathcal{X}^*_v)$.

Let $B_\infty = a \cup \{B_k(|a|+k) : k \in \omega\}$ and $q_\infty = (\bigcap_k q^i_k)_{i<\omega}$. As alternative (b) always holds, $[a, B_\infty] \times [q_\infty] \cap \mathcal{M}^c_v = \emptyset$ for all $v \in \omega^{<\omega}$. Then it is straightforward to check $\big([a, B_\infty] \times [q_\infty]\big) \cap \mathcal{X}^*_\emptyset = \big([a, B_\infty] \times [q_\infty]\big) \cap \Phi(\mathcal{X}^*_\emptyset)$. As $\Phi(\mathcal{X}^*_\emptyset)$ is perfectly Ramsey, $\exists B \leq B_\infty \leq A$ $\exists q \leq q_\infty \leq p$ such that $[a, B] \times [q] \subseteq [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}^*_\emptyset)$ or $[a, B] \times [q] \cap \Phi(\mathcal{X}^*_\emptyset) = \emptyset$, so $[a, B] \times [q] \subseteq \mathcal{X}$ or $[a, B] \times [q] \cap \mathcal{X} = \emptyset$ as required. □

Thus we have proved the main theorem of this section, Theorem 1.2.

4. Local parametrized $\mathcal{R}_\alpha$ theorem

In this section, we recall the definition of selective [5] and Nash–Williams [8] ultrafilters, and aim to prove the Local Parametrized $\mathcal{R}_\alpha$ Theorem 1.3.

Definition 4.1. Let $\mathcal{U}$ be an ultrafilter on the base set $[\mathbb{T}]$ of all maximal nodes of the tree $\mathbb{T}$. We say $\mathcal{U}$ is *Nash–Williams* if $\mathcal{U}$ is generated by sets of the form $[X]$, $X \in \mathcal{R}_\alpha$, and for every Nash–Williams set $\mathcal{G} \subseteq \mathcal{AR}_\alpha$ and every partition $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$, there exist $[X] \in \mathcal{U}$ with $X \in \mathcal{R}_\alpha$ and $i \in 2$ such that $\mathcal{G}_i|X = \emptyset$. We say $\mathcal{U}$ is *selective* if it is generated by sets of the form $[X]$, $X \in \mathcal{R}_\alpha$, and for every family $\{[X_a] : a \in \mathcal{AR}_\alpha\} \subseteq \mathcal{U}$ such that $X_a \in \mathcal{R}_\alpha$ for all $a \in \mathcal{AR}_\alpha$ there exists $[X] \in \mathcal{U}$ with $X \in \mathcal{R}_\alpha$ such that $X/a \subseteq X_a$ for all $a \leq X$.

When we write $[X] \in \mathcal{U}$, we tacitly assume $X \in \mathcal{R}_\alpha$. Let us first check that every Nash–Williams ultrafilter is selective in $\mathcal{R}_\alpha$.

Definition 4.2 [17, Definition 3.4]. An ultrafilter $\mathcal{U}$ on $[\mathbb{T}]$ is *weakly selective* if for every family $\{[A_b] : b \in \mathcal{AR}^\alpha_1\} \subseteq \mathcal{U}$ there exists $[B] \in \mathcal{U}$ such that $B/b \subseteq A_b$ for each $b \in \mathcal{AR}^\alpha_1(B)$.

Lemma 4.3 [17, Lemma 3.8]. *In a topological Ramsey space every Ramsey ultrafilter is weakly selective.*

The definition of *Ramsey* ultrafilter ([17, Definition 3.2]) is not crucial here. It is straightforward to check that every Nash–Williams ultrafilter is

Ramsey. From Lemma 4.3 and the lemma below, we can easily conclude that every Nash–Williams ultrafilter, as well as every Ramsey ultrafilter, is selective in $\mathcal{R}_\alpha$.

LEMMA 4.4. *Every weakly selective ultrafilter on* $[\mathbb{T}]$ *is selective.*

PROOF. Let $\mathcal{U}$ be a weakly selective ultrafilter on $[\mathbb{T}]$. Let $\{[X_a] : a \in \mathcal{AR}_\alpha\} \subseteq \mathcal{U}$. We aim to find $[X] \in \mathcal{U}$ such that $X/a \subseteq X_a$ for all $a \le X$.

We recursively construct a $\le$-decreasing sequence $([Y_n])_{n<\omega} \subseteq \mathcal{U}$. For $n = 0$, consider the set $S_0 = \{b \in \mathcal{AR}_\alpha : \max[b] = 0\}$. Since S_0 is finite and $\mathcal{U}$ is an ultrafilter, we can find $[Y_0] \in \mathcal{U}$ such that $[Y_0] \subseteq \bigcap_{b\in S_0}[X_b]$. Suppose we have constructed $Y_0, \dots, Y_n$. The set $S_{n+1} = \{b \in \mathcal{AR}_\alpha : \max[b] = n+1\}$ is finite. We can find $[Y_{n+1}] \in \mathcal{U}$ such that $[Y_{n+1}] \subseteq \bigcap_{b\in S_{n+1}}[X_b] \cap [Y_n]$. This finishes the construction of the sequence $([Y_n])_{n<\omega}$. In particular, for all $n \in \omega$, $[Y_n] \in \mathcal{U}$ and $Y_n \le X_b$ for all $b \in \mathcal{AR}_\alpha$ with $\max[b] = n$.

In order to apply the property that $\mathcal{U}$ is weakly selective, for each $b \in \mathcal{AR}^\alpha_1$, let $A_b = Y_n$ where $n = \max[b]$. Then there exists $[X] \in \mathcal{U}$ such that $X/b \subseteq A_b$ for all $b \in \mathcal{AR}^\alpha_1(X)$. Let us check that X is a witness to the selectivity of $\mathcal{U}$. Suppose $a \le X$. Let $b \in \mathcal{AR}^\alpha_1$ be such that $\max[b] = \max[a]$. So $b \le X$. Thus

$$X/a = X/b \subseteq A_b = Y_{\max[b]} = Y_{\max[a]} \le X_a$$

as required. $\square$

From now on in this section, we fix a Nash–Williams ultrafilter $\mathcal{U}$ in $\mathcal{R}_\alpha$.

4.1. Open subsets of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$. Firstly we relativise the ultra-Ramsey theory. The definitions in this subsection are adapted to $\mathcal{R}_\alpha$ from those in [20, §7]. Consider $\mathcal{AR}_\alpha$ as a tree ordered by $\sqsubseteq$ with root $\emptyset$. From now on by a *tree* we mean a downward closed subtree of $\mathcal{AR}_\alpha$.

NOTATION. For a tree T, let $[T]$ denote the set of all infinite branches of T, that is, $[T] = \{X \in \mathcal{R}_\alpha : X \restriction n \in T \text{ for all } n \in \omega\}$. Let the *stem of* T, denoted by $\mathrm{st}(T)$, be the maximal node of T that is $\sqsubseteq$-comparable with every node in T. For $s \in T$, by T/s we denote the set of nodes in T above s, i.e. $T/s = \{t \in T : s \sqsubseteq t\}$.

In [7,8], $\mathcal{U}$-trees for the spaces $\mathcal{R}_\alpha$ are seen in well-founded form, and used together with the Ramsey-classification Theorem canonizing the equivalent relations on fronts, to determine the Rudin–Keisler structure inside each of the Tukey types of ultrafilters Tukey reducible to $\mathcal{U}$. We recall the definition of a $\mathcal{U}$-tree:

DEFINITION 4.5. A $\mathcal{U}$*-tree* T is a tree such that for all $t \in T$ with $\mathrm{st}(T) \sqsubseteq t$ there exists $[X] \in \mathcal{U}$ such that $t \cup a \in T$ for every $a \in \mathcal{R}_\alpha(|t|)|X$. For two

$\mathcal{U}$-trees T and T', we say T' is a *pure refinement* of T, and write $T' \leq^0 T$, if $T' \subseteq T$ and $\mathrm{st}(T') = \mathrm{st}(T)$. Similarly, $T' \leq^n T$ if $T' \leq^0 T$ and T agrees with T' on the first n levels above the common stem.

Definition 4.6. A sequence (T_n) of $\mathcal{U}$-trees is a *fusion sequence* if $T_{n+1} \leq^n T_n$ for all $n \in \omega$. In this case $T_\infty := \bigcap_{n\in\omega} T_n$ is also a $\mathcal{U}$-tree and is called the *fusion* of the sequence.

Lemma 4.7. *Suppose T, T' are $\mathcal{U}$-trees such that $\mathrm{st}(T') \in T/\mathrm{st}(T)$. Then $T \cap T'$ is also a $\mathcal{U}$-tree.*

Proof. Let $t \sqsupseteq \mathrm{st}(T') \sqsupseteq \mathrm{st}(T)$ be such that $t \in T \cap T'$. By assumption, there exist $[X]$ and $[X']$ in $\mathcal{U}$ such that $t \cup a \in T$ for every $a \in \mathcal{R}_\alpha(|t|)|X$ and $t \cup a \in T'$ for every $a \in \mathcal{R}_\alpha(|t|)|X'$. Since $[X], [X'] \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, there exists $[Y] \in \mathcal{U}$ such that $[Y] \subseteq [X] \cap [X']$. Then $t \cup a \in T \cap T'$ for every $a \in \mathcal{R}_\alpha(|t|)|Y$ as required. $\square$

Definition 4.8. A subset $G \subseteq \mathcal{AR}_\alpha$ is *$\mathcal{U}$-open* if for every $t \in G$ there exists a $\mathcal{U}$-tree T such that $\mathrm{st}(T) = t$ and $T/t \subseteq G$.

Lemma 4.9. *For every subset $G \subseteq \mathcal{AR}_\alpha$ and every $s \in \mathcal{AR}_\alpha$, since $\mathcal{U}$ is* Nash–Williams, *there exists $[X] \in \mathcal{U}$ with $s \leq X$ such that either*

$$(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \in G) \quad \textit{or} \quad (\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \notin G).$$

Proof. For $G \subseteq \mathcal{AR}_\alpha$, consider $\mathcal{G} = \{b \in \mathcal{AR}^\alpha_{|s|+1} : s \sqsubseteq b\} \subseteq \mathcal{AR}_\alpha$ and the partition $\mathcal{G} = G_0 \cup G_1$ where $G_0 = G \cap \mathcal{G}$. Since $\mathcal{U}$ is *Nash–Williams*, there exist $[X] \in \mathcal{U}$ and $i \in 2$ such that $G_i|X = \emptyset$. Thus, if $i = 1$, then the first alternative in the lemma holds; if $i = 0$, then the second alternative holds. $\square$

The lemma below follows immediately from the definition of $\mathcal{U}$-open sets and Lemma 4.9.

Lemma 4.10. *A subset $G \subseteq \mathcal{AR}_\alpha$ is $\mathcal{U}$-open if and only if $(\exists [X] \in \mathcal{U})(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \in G)$ holds for every $s \in G$.*

Definition 4.11. A subset $\mathcal{G} \subseteq \mathcal{R}_\alpha$ is *$\mathcal{U}$-open* if for every $X \in \mathcal{G}$ there exists a $\mathcal{U}$-tree T such that $X \in [T] \subseteq \mathcal{G}$.

By Lemma 4.7, the basic $\mathcal{U}$-open sets $[T]$ (T a $\mathcal{U}$-tree) generates a topology on $\mathcal{R}_\alpha$, containing the metric topology.

Definition 4.12. A subset $\mathcal{X} \subseteq \mathcal{R}_\alpha$ is *$\mathcal{U}$-Ramsey* if for every $\mathcal{U}$-tree T there exists $T' \leq^0 T$ such that $[T'] \subseteq \mathcal{X}$ or $[T'] \cap \mathcal{X} = \emptyset$. It is *$\mathcal{U}$-Ramsey null* if the second alternative always holds.

Lemma 4.13. *Every $\mathcal{U}$-open set is $\mathcal{U}$-Ramsey.*

Proof. Let $\mathcal{X} \subseteq \mathcal{R}_\alpha$ be a given $\mathcal{U}$-open set. Let $G = \{s \in \mathcal{AR}_\alpha : (\exists \mathcal{U}\text{-tree } T)((\mathrm{st}(T) = s) \wedge ([T] \subseteq \mathcal{X}))\}$. Note G is a $\mathcal{U}$-open subset of $\mathcal{AR}_\alpha$. Also note that if $s \notin G$ then there does not exist $[X] \in \mathcal{U}$ such that $s \cup a \in G$ for all $a \in \mathcal{R}_\alpha(|s|)|X$: If every $s \cup a$ has a $\mathcal{U}$-tree in $\mathcal{X}$ above it, then putting them together we would get a $\mathcal{U}$-tree in $\mathcal{X}$ above s. By Lemma 4.9, $s \notin G$ implies that $(\exists [X] \in \mathcal{U})(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \notin G)$. Let $F = \mathcal{AR}_\alpha \setminus G$, so F satisfies the criterion of Lemma 4.10 for being $\mathcal{U}$-open. It follows that F is $\mathcal{U}$-clopen.

We prove that for every $t \in F$ there exists a $\mathcal{U}$-tree T' with stem t such that $[T'] \cap \mathcal{X} = \emptyset$. Then given T with stem s: either $s \in G$, so there exists $\mathcal{U}$-tree T' such that $[T' \cap T] \subseteq \mathcal{X}$ by the definition of G; or $s \in F$, so there exists T' such that $[T' \cap T] \cap \mathcal{X} = \emptyset$.

Claim 4.13.1. *If T is a $\mathcal{U}$-tree with* $\mathrm{st}(T) = t$ *such that $T/t \subseteq F$, then $[T] \cap \mathcal{X} = \emptyset$.*

First note that such tree exists since F is $\mathcal{U}$-open. Suppose $X \in [T] \cap \mathcal{X}$. Since $\mathcal{X}$ is $\mathcal{U}$-open, there exists T' such that $X \in [T'] \subseteq \mathcal{X}$. Let $s = \mathrm{st}(T')$. Then $s \in G$. But this contradicts that $s \in T/t \subseteq F$. $\square$

The proofs of the following three lemmas below closely follow those of Lemma 7.40, 7.41 and Theorem 7.42 in [20], respectively. The corollary then easily follows.

Lemma 4.14. *Every $\mathcal{U}$-nowhere dense sets is $\mathcal{U}$-Ramsey null.*

Lemma 4.15. *The $\mathcal{U}$-Ramsey null sets form a σ-ideal.*

Lemma 4.16. *A subset of $\mathcal{R}_\alpha$ has the property of Baire with respect to the $\mathcal{U}$-topology if and only if it is $\mathcal{U}$-Ramsey.*

Corollary 4.17. *If $\mathcal{X} \subseteq \mathcal{R}_\alpha$ is Souslin-measurable, then it is $\mathcal{U}$-Ramsey.*

This finishes the relativization of ultra-Ramsey theory. Now we show that selectivity helps us obtain a $\mathcal{U}$-tree from an element $[X] \in \mathcal{U}$, and vice versa.

Lemma 4.18. *For a $\mathcal{U}$-tree T with stem s there is $[X] \in \mathcal{U}$ such that $[s, X] \subseteq [T]$. Conversely for $s \sqsubseteq X$ with $[X] \in \mathcal{U}$ there is a $\mathcal{U}$-tree T with stem s such that $[T] \subseteq [s, X]$.*

Proof. Consider $t \in \mathcal{AR}_\alpha$. If $t \in T/s$, then there exists $[X_t] \in \mathcal{U}$ such that $t \cup a \in T$ for every $a \in \mathcal{R}_\alpha(|t|)|X_t$. If $t \in \mathcal{AR}_\alpha \setminus (T/s)$ is such that

$|t| > |s|$ and $s \cup t \setminus (t(0) \cup \cdots \cup t(|s|-1)) \in T/s$, then we can find $[X_t] \in \mathcal{U}$ such that

$$\big[\, s \cup t \setminus \big(t(0) \cup \cdots \cup t(|s|-1)\big)\big] \cup a \in T \quad \text{for every } a \in \mathcal{R}_\alpha(|t|)|X_t.$$

Otherwise, let $X_t = \mathbb{T}$. (Note that $[\mathbb{T}] \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter on the base set $[\mathbb{T}]$.) Applying *selectivity* of $\mathcal{U}$ to $\{[X_t] : t \in \mathcal{AR}_\alpha\} \subseteq \mathcal{U}$, we get $[X] \in \mathcal{U}$ such that $X/t \subseteq X_t$ for all $t \leq X$.

We check that $[s, X] \subseteq [T]$: Suppose $A \in [s, X]$; we prove $A \restriction n \in T$ for all n by induction. Firstly, $A \restriction |s| = s \in T$ by assumption. Now consider $A \restriction (n+1) = A \restriction n \cup A(n)$, where $n \geq |s|$. By the induction hypothesis, $A \restriction n \in T$. As $s \sqsubseteq A \leq a \cup X$, there exists $t \leq X$ such that

$$A \restriction n = s \cup t \setminus \big(t(0) \cup \cdots \cup t(|s|-1)\big).$$

So

$$A(n) \in \mathcal{R}_\alpha(n)|(X/t) \subseteq \mathcal{R}_\alpha(n)|X_t.$$

Then by the definition of X_t, $A \restriction n \cup A(n) \in T$, that is, $A \restriction (n+1) \in T$ as required.

Now suppose $s \sqsubseteq X$ and $[X] \in \mathcal{U}$. Starting with the stem s we construct T recursively. For $t \in T$, let the set of the immediate descendants of t in T be $\{t \cup b : b \in \mathcal{R}_\alpha(|t|)|X\}$. Note that $[X] \in \mathcal{U}$ implies that T is a $\mathcal{U}$-tree. $\square$

THEOREM 4.19. *For every finite Souslin-measurable colouring of $\mathcal{R}_\alpha$ there exists $[X] \in \mathcal{U}$ such that $[\emptyset, X]$ is monochromatic.*

PROOF. Without loss of generality, consider a two colouring of $\mathcal{R}_\alpha$ given by $\mathcal{R}_\alpha = \mathcal{X} \cup \mathcal{X}^c$ where $\mathcal{X}$ is Souslin-measurable. By Corollary 4.17, $\mathcal{X}$ is $\mathcal{U}$-Ramsey, so there is a $\mathcal{U}$-tree T such that $[T] \subseteq \mathcal{X}$ or $\mathcal{X}^c$. Then by Lemma 4.18, there exists $[X] \in \mathcal{U}$ such that $[\emptyset, X] \subseteq [T] \subseteq \mathcal{X}$ or $\mathcal{X}^c$ as required. $\square$

Now, as in [22], we define uniform families in $\mathcal{R}_\alpha$, in order to obtain Theorem 4.22 below.

NOTATION. For $\mathcal{S} \subseteq \mathcal{AR}_\alpha$ and $a \in \mathcal{AR}_\alpha$, let $\mathcal{S}_{[a]} = \{y : a \sqsubseteq y \in \mathcal{S}\}$.

DEFINITION 4.20. Let γ be a countable ordinal and $\mathcal{S} \subseteq \mathcal{AR}_\alpha$. Let $X \in \mathcal{R}_\alpha$ and $b \in \mathcal{AR}_\alpha$. We say $\mathcal{S}$ is a γ*-uniform family* on $[b, X]$ if

- $\gamma = 0$ and $\mathcal{S} = \{b\}$; or
- $\gamma = \beta + 1$, $b \notin \mathcal{S}$ and $\mathcal{S}_{[a]}$ is β-uniform on $[a, X]$ for every $a \in \mathcal{AR}^\alpha_{|b|+1}[b, X]$; or
- γ is a limit ordinal, $b \notin \mathcal{S}$ and there exists a sequence $(\gamma_a)_{a \in \mathcal{AR}^\alpha_{|b|+1}[b,X]}$ of ordinals, with $\bigcup\{\gamma_a : a \in \mathcal{AR}^\alpha_{|b|+1}[b, X]\} = \gamma$, such that $\mathcal{S}_{[a]}$ is γ_a-uniform on $[a, X]$ for every $a \in \mathcal{AR}^\alpha_{|b|+1}[b, X]$.

We say $\mathcal{S}$ is a *uniform family* on $[b, X]$ if it is γ-uniform on $[b, X]$ for some $\gamma < \omega_1$.

For example, if $n < \omega$, the only n-uniform family on $[b,X]$ is $\mathcal{AR}^{\alpha}_{|b|+n}[b,X]$. For every $k \in \omega$, the family $\mathcal{S} = \{y : (b \sqsubseteq y \leq X) \wedge (|y| = \|y(|b|+1)\| + k)\}$ is ω-uniform, and the family $\mathcal{T} = \{y : (b \sqsubseteq y \leq X) \wedge (|y| = \|y(|b|+2)\| + k)\}$ is $(\omega+1)$-uniform.

We have the following lemma relating fronts and uniform families.

LEMMA 4.21 [22, Lemmas 2.13, 2.14]. *Let $b \in \mathcal{AR}_\alpha$ and $X \in \mathcal{R}_\alpha$. If $\mathcal{S}$ is a uniform family on $[b, X]$, then it is a front on $[b, X]$. Conversely, if $\mathcal{F}$ is a front on $[b, X]$, then there exists a uniform family $\mathcal{S}$ on $[b, X]$ such that every $s \in \mathcal{S}$ has an initial segment in $\mathcal{F}$.*

A proof similar to that of Theorem 2.15 in [22] gives the following theorem.

THEOREM 4.22. *For every open set $\mathcal{O} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ there exist $[X] \in \mathcal{U}$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p] \subseteq \mathcal{O}$ or $[\emptyset, X] \times [p] \cap \mathcal{O} = \emptyset$.*

4.2. Perfectly $\mathcal{U}$-Ramsey sets. In this subsection, we strengthen Theorem 4.22 from open sets $\mathcal{O}$ to all Souslin-measurable sets $\mathcal{B}$.

DEFINITION 4.23 [22, Definition 2.18]. A subset $\mathcal{B} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is *perfectly $\mathcal{U}$-Ramsey* if for all $\mathcal{U}$-tree T and $p \in \mathcal{P}_\omega$ there exist $T' \leq^0 T$ and $p' \leq p$ such that $[T'] \times [p'] \subseteq \mathcal{B}$ or $[T'] \times [p'] \cap \mathcal{B} = \emptyset$. We say $\mathcal{B}$ is *perfectly $\mathcal{U}$-Ramsey null* if the second alternative always holds.

The following lemma is immediate.

LEMMA 4.24 [22, Lemma 2.20]. *A subset $\mathcal{B} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is perfectly $\mathcal{U}$-Ramsey if and only if for arbitrary $F \in [\omega]^{<\omega}$, $n \in \omega$, $\mathcal{U}$-tree T and $p \in \mathcal{P}_\omega$,*

$$\exists T' \leq^0 T \; \exists p' \leq^{F,n} p \; \forall \sigma \in l(F, n, p') \; [T'] \times [p'|\sigma] \subseteq \mathcal{B} \text{ or } [T'] \times [p'|\sigma] \cap \mathcal{B} = \emptyset.$$

For $a \in \mathcal{AR}_\alpha$, recall the space $\mathcal{R}^{(a)}_\alpha$ from Definition 3.19. Exactly the same as the case for $\mathcal{R}_\alpha$, we have the following.

DEFINITION 4.25. An ultrafilter $\mathcal{U}_a$ on the base set $[\mathbb{T}^{(k)}]$ is *selective* if it is generated by sets of the form $[X]$ with $X \in \mathcal{R}^{(a)}_\alpha$ such that for every family $\{[X_b] : b \in \mathcal{AR}^{(a)}_\alpha\} \subseteq \mathcal{U}_a$ there exists $[X] \in \mathcal{U}_a$ such that $X/b \subseteq X_b$ for all $b \leq^{(a)} X$. Moreover, it is *Nash–Williams* if, in addition, for every Nash–Williams subset $\mathcal{G} \subseteq \mathcal{AR}^{(a)}_\alpha$ and partition $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ there exist $[X] \in \mathcal{U}_a$ and $i \in 2$ such that $\mathcal{G}_i | X = \emptyset$.

THEOREM 4.26. *If $\mathcal{U}_a$ is a Nash–Williams ultrafilter on $[\mathbb{T}^{(k)}]$ then for every open set $\mathcal{B} \subseteq \mathcal{R}^{(a)}_\alpha \times \mathbb{R}^{\mathbb{N}}$ there exist $[X] \in \mathcal{U}_a$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p] \subseteq \mathcal{B}$ or $[\emptyset, X] \times [p] \cap \mathcal{B} = \emptyset$.*

The proof closely follows the procedure in Section 4.1 leading to Theorem 4.22.

Lemma 4.27. *If $\mathcal{B} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is open, then $\mathcal{B}$ is perfectly $\mathcal{U}$-Ramsey.*

Proof. Let a $\mathcal{U}$-tree T and $p \in \mathcal{P}_\omega$ be given. Let $a = \mathrm{st}(T)$. Let $\mathcal{B}_a = \{(X/a, \varepsilon) : (a \sqsubseteq X) \wedge ((X, \varepsilon) \in \mathcal{B})\} \subseteq \mathcal{R}_\alpha^{(a)} \times \mathbb{R}^{\mathbb{N}}$, and $\mathcal{U}_a$ be the ultrafilter generated by $\{[X/a] : (a \sqsubseteq X) \wedge ([X] \in \mathcal{U})\}$. Note that $\mathcal{B}_a$ is open in $\mathcal{R}_\alpha^{(a)} \times \mathbb{R}^{\mathbb{N}}$. We check that $\mathcal{U}_a$ is Nash–Williams: Let $\mathcal{G} \subseteq \mathcal{AR}_\alpha^{(a)}$ be Nash–Williams and $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$. For $i = 0, 1$, let $\mathcal{G}_i' = \{a \cup y : y \in \mathcal{G}_i\}$, and similarly for $\mathcal{G}'$. We obtain a Nash–Williams subset $\mathcal{G}' \subseteq \mathcal{AR}_\alpha$ and a partition $\mathcal{G}' = \mathcal{G}_0' \sqcup \mathcal{G}_1'$. Since $\mathcal{U}$ is Nash–Williams, there exists $[X] \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{G}_i' | X = \emptyset$. So $[X/a] \in \mathcal{U}_a$ and $\mathcal{G}_i | (X/a) = \emptyset$.

So by Theorem 4.26 there exist $[X_a] \in \mathcal{U}_a$ and $p' \le p$ such that $[\emptyset, X_a] \times [p'] \subseteq \mathcal{B}_a$ or $\mathcal{B}_a^c$. Thus $[a, a \cup X_a] \times [p'] \subseteq \mathcal{B}$ or $[a, a \cup X_a] \times [p'] \cap \mathcal{B} = \emptyset$. Hence by Lemma 4.18 we can find a $\mathcal{U}$-tree T' with stem a such that $[T'] \times [p'] \subseteq \mathcal{B}$ or $\mathcal{B}^c$. $\square$

Then by a standard procedure (see [22, §2.2]), we have the following.

Theorem 4.28. *The field of perfectly $\mathcal{U}$-Ramsey subsets of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is closed under the Souslin operation.*

Corollary 4.29. *Every Souslin-measurable subset of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is perfectly $\mathcal{U}$-Ramsey.*

Restating the above corollary, we have the main theorem of this subsection, Theorem 1.3.

5. Preservation under countable-support side-by-side Sacks forcing

Let κ be a cardinal. Recall that the countable-support side-by-side Sacks forcing is the set $\mathcal{P}_\kappa$ consisting of conditions $p = (p^i)_{i<\kappa}$, where, for all $i < \kappa$, p^i is a perfect tree and, for all but countably many $i < \kappa$, p^i is the full binary tree $2^{<\omega}$. The partial order is given by $p \le q$ if $p^i \subseteq q^i$ for every $i < \kappa$. Recall also that $\mathrm{supp}(p) = \{i < \kappa : p^i \ne 2^{<\omega}\}$.

Let $\mathcal{U}$ be a Nash–Williams ultrafilter in $\mathcal{R}_\alpha$. We show that the upward closure of $\mathcal{U}$, after forcing by $\mathcal{P}_\kappa$, is still Nash–Williams. First we see it is selective.

Lemma 5.1 [1, Lemma 1.9]. *Let $p \in \mathcal{P}_\kappa$, $n \in \omega$ and $F \subseteq \kappa$ be finite. If $q \le p$ then there exists $\sigma \in l(F, n, p)$ such that q and $p|\sigma$ are compatible.*

Corollary 5.2 [1, Corollary 1.10]. *Suppose $p \in \mathcal{P}_\kappa$, $n \in \omega$, and $F \subseteq \kappa$ finite. If $p \Vdash (\tau \in V)$ then there exists $q \le^{F,n} p$ such that for every $\sigma \in l(F, n, q)$ there exists $a_\sigma \in V$ such that $q|\sigma \Vdash (\tau = a_\sigma)$.*

If q and τ are as in Corollary 5.2 then we say q *determines* τ *with respect to* (F, n). We say q *determines* τ if there exist $F \in [\omega]^{<\omega}$ and $n \in \omega$ such that q determines τ with respect to (F, n). For $q \Vdash (\tau : [\mathbb{T}_\alpha] \to V)$, q *determines* τ if q determines $\tau(x)$ for every $x \in [\mathbb{T}_\alpha]$.

As $[\mathbb{T}_\alpha]$ is countable, we can write it as $[\mathbb{T}_\alpha] = \{x_n : n \in \omega\}$, and refine p step by step, considering x_n at step n, and achieve the following as in [22].

LEMMA 5.3. *If $p \in \mathcal{P}_\kappa$ and $p \Vdash (\tau : [\mathbb{T}_\alpha] \to 2)$, then $\exists q \leq p$ such that q determines τ.*

Exactly the same as in [22], using Theorem 1.3, we have the following theorem saying that the upward closure of $\mathcal{U}$ is still "ultra" in the extension.

THEOREM 5.4 [22, Theorem 3.4]. *If $p \in \mathcal{P}_\kappa$ and $p \Vdash \tau \subseteq [\mathbb{T}_\alpha]$, then there exist $q \leq p$ and $[B] \in \mathcal{U}$ such that $q \Vdash ([B] \subseteq \tau)$ or $q \Vdash ([B] \cap \tau = \emptyset)$.*

It is then straightforward to show that the upward closure is selective in the extension. Now we check that it is also Nash–Williams in the extension.

THEOREM 5.5. *Suppose $p \in \mathcal{P}_\kappa$ and $p \Vdash ((\mathcal{G} \subseteq \mathcal{AR}_\alpha$ is Nash–Williams$) \wedge (\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1))$. Then there exists $q \leq p$, $i < 2$ and $[X] \in \mathcal{U}$ such that $q \Vdash (\mathcal{G}_i | X = \emptyset)$.*

PROOF. Let $p \in \mathcal{P}_\kappa$ as in the statement of the theorem. We may consider $\mathcal{G} \subseteq \mathcal{AR}_\alpha$ with $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ as a function $g : \mathcal{AR}_\alpha \to 3$ given by the following formula. For $a \in \mathcal{AR}_\alpha$,

$$g(a) = \begin{cases} 0 & \text{if } a \in \mathcal{G}_0; \\ 1 & \text{if } a \in \mathcal{G}_1; \\ 2 & \text{if } a \in \mathcal{AR}_\alpha \setminus \mathcal{G}. \end{cases}$$

Since $\mathcal{AR}_\alpha$ is countable, we enumerate it as $\mathcal{AR}_\alpha = \{a_k : k \in \omega\}$.

We construct $(p_k)_k$ recursively as follows, starting with $p_{-1} = p$. Suppose we have $p_k \leq p$. By Corollary 5.2, there exists $p_{k+1} \leq^{k+1,k+1} p_k$ such that for every $\sigma \in l(k+1, k+1, p_k)$ there is $i_\sigma \in 3$ such that $p_{k+1}|\sigma \Vdash (g(a_{k+1}) = i_\sigma)$. Let $p_\infty = \big(\bigcap_{k \in \omega} p_k^i\big)_{i<\omega}$. Then $p_\infty \in \mathcal{P}_\kappa$ and for each $k < \omega$,

$$\forall \sigma \in l(k, k, p_\infty)\, \exists i_\sigma \in 3 \quad p_\infty|\sigma \Vdash (g(a_k) = i_\sigma).$$

Let

$$\mathcal{F} = \big\{ (a_k, \varepsilon) : \exists \sigma \in l(k, k, p_\infty)((\sigma \sqsubseteq^* \varepsilon) \wedge (p_\infty|\sigma \Vdash g(a_k) = 0 \text{ or } 2))\big\}$$
$$\subseteq \mathcal{AR}_\alpha \times [p_\infty],$$
$$\mathcal{X} = \big\{ (X, \varepsilon) : (\forall a_k \sqsubseteq X)((a_k, \varepsilon) \in \mathcal{F})\big\} \subseteq \mathcal{R}_\alpha \times [p_\infty].$$

Note that $\mathcal{X}^c = \{(X, \varepsilon) : (\exists a_k \sqsubseteq X)((a_k, \varepsilon) \notin \mathcal{F})\}$ is an open subset of $\mathcal{R}_\alpha \times [p_\infty]$. Then by Theorem 1.3, there exists $[X] \in \mathcal{U}$ and $q \leq p_\infty$ such that $[\emptyset, X] \times [q] \subseteq \mathcal{X}$ or $[\emptyset, X] \times [q] \cap \mathcal{X} = \emptyset$.

Suppose first $[\emptyset, X] \times [q] \subseteq \mathcal{X}$. We show that $a_k \leq X$ implies $q \Vdash (g(a_k) = 0 \text{ or } 2)$, so $q \Vdash (\mathcal{G}_1 | X = \emptyset)$: Assume $r \leq q$ is such that $r \Vdash (g(a_k) = 1)$. We aim for a contradiction. By Lemma 5.1, there is $\sigma \in l(k, k, p_\infty)$ such that r is compatible with $q|\sigma$ and $q|\sigma \neq \emptyset$. Let $\varepsilon \in [q]$ be such that $\sigma \sqsubseteq^* \varepsilon$, and let $Y \leq X$ be such that $a_k \sqsubseteq Y$. As $(Y, \varepsilon) \in [\emptyset, X] \times [q] \subseteq \mathcal{X}$, $(a_k, \varepsilon) \in \mathcal{F}$. Hence $q|\sigma \leq p_\infty|\sigma \Vdash (g(a_k) = 0 \text{ or } 2)$, contradicting that $r \Vdash g(a_k) = 1$ and r is compatible with $q|\sigma$.

Now suppose $[\emptyset, X] \times [q] \subseteq \mathcal{X}^c$. We check that $a_k \leq X$ implies

$$q \Vdash (\exists a_l \leq X)(((a_l \sqsubseteq a_k) \vee (a_l \sqsupseteq a_k)) \wedge (g(a_l) = 1)).$$

Then, as $q \Vdash (\mathcal{G} \text{ is Nash–Williams})$, $q \Vdash (\mathcal{G}_0 | X = \emptyset)$. Assume $r \leq q$ is such that $r \Vdash ((\forall a_l \leq X)((a_l \sqsubseteq a_k \text{ or } a_l \sqsupseteq a_k) \rightarrow (g(a_l) = 0 \text{ or } 2))$. We aim for a contradiction. Let $\varepsilon \in [r]$ and $Y \leq X$ be such that $a_k \sqsubseteq Y$. Then $(Y, \varepsilon) \in \mathcal{X}^c$. So there is $a_l \sqsubseteq Y$ such that $(a_l, \varepsilon) \notin \mathcal{F}$. Let $\tau \in l(l, l, p_\infty)$ be such that $\tau \sqsubseteq^* \varepsilon$. Then $p_\infty|\tau \Vdash (g(a_l) = 1)$. This contradicts that $r|\tau \leq p_\infty|\tau$ and $r|\tau \leq r \Vdash ((\forall a_l \leq X)((a_l \sqsubseteq a_k \text{ or } a_l \sqsupseteq a_k) \rightarrow (g(a_l) = 0 \text{ or } 2))$. $\square$

Thus we have proved that every Nash–Williams ultrafilter in $\mathcal{R}_\alpha$ is preserved under countable-support side-by-side Sacks forcing (Theorem 1.1).

6. Necessity of Nash–Williams

In this section, we show that the ultrafilter being Nash–Williams is a necessary condition for the Local Parametrized $\mathcal{R}_\alpha$ Theorem 1.3 to hold.

For a subset $\mathcal{X} \subseteq \mathcal{R}_\alpha$, we say $\mathcal{X}$ is *weakly $\mathcal{U}$-Ramsey* if for every $[X] \in \mathcal{U}$ there exists $[Y] \in \mathcal{U}$ with $Y \leq X$ such that either $[\emptyset, Y] \subseteq \mathcal{X}$ or $[\emptyset, Y] \cap \mathcal{X} = \emptyset$. Note that, if $\mathcal{U}$ is selective, then by Lemma 4.18, every $\mathcal{U}$-Ramsey set is weakly $\mathcal{U}$-Ramsey.

LEMMA 6.1. *If every open subset of $\mathcal{R}_\alpha$ is weakly $\mathcal{U}$-Ramsey, then $\mathcal{U}$ is Nash–Williams. Namely,*

$$(\forall \mathcal{X} \subseteq \mathcal{R}_\alpha \textit{ open})(\forall [X] \in \mathcal{U})(\exists [Y] \in \mathcal{U})$$
$$((Y \leq X) \wedge ([\emptyset, Y] \subseteq \mathcal{X}) \vee ([\emptyset, Y] \cap \mathcal{X} = \emptyset))$$
$$\Rightarrow (\forall \mathcal{G} \subseteq \mathcal{AR}_\alpha \textit{ Nash–Williams})(\forall \mathcal{G}_0 \sqcup \mathcal{G}_1 = \mathcal{G})(\exists [X] \in \mathcal{U})(\exists i \in 2)(\mathcal{G}_i | X = \emptyset).$$

PROOF. Let $\mathcal{G} \subseteq \mathcal{AR}_\alpha$ be Nash–Williams, $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$. Then the set $\mathcal{X} = \bigcup_{a \in \mathcal{G}_0} [a, \mathbb{T}]$ is open in $\mathcal{R}_\alpha$. As every open set is weakly $\mathcal{U}$-Ramsey, there is $[Y] \in \mathcal{U}$ such that $Y \leq X$ and $[\emptyset, Y] \subseteq \mathcal{X}$ or $[\emptyset, Y] \cap \mathcal{X} = \emptyset$.

In the first case, $[\emptyset, Y] \subseteq \bigcup_{a\in\mathcal{F}_0}[a,\mathbb{T}]$. So for every $Z \le Y$, Z has an initial segment in $\mathcal{G}_0$. But $\mathcal{G}$ is Nash–Williams so Z cannot have initial segment in $\mathcal{G}_1$. i.e. $a \notin \mathcal{G}_1$ for all $a \le Y$, so $\mathcal{G}_1|Y = \emptyset$. In the second case, $[\emptyset, Y] \cap \big(\bigcup_{a\in\mathcal{G}_0}[a,\mathbb{T}]\big) = \emptyset$, so $a \notin \mathcal{G}_0$ for all $a \le Y$, i.e. $\mathcal{G}_0|Y = \emptyset$. $\square$

From the above lemma, we know that if $\mathcal{U}$ is selective and every open subset of $\mathcal{R}_\alpha$ is $\mathcal{U}$-Ramsey, then $\mathcal{U}$ must be Nash–Williams. Moreover, it follows from the Local Parametrized $\mathcal{R}_\alpha$ Theorem 1.3 that every open subset of $\mathcal{R}_\alpha$ is weakly $\mathcal{U}$-Ramsey. So for Theorem 1.3 to hold for a selective ultrafilter, it is necessary that the ultrafilter is Nash–Williams.

Thus, we see that Nash–Williams ultrafilters in $\mathcal{R}_\alpha$ are analogous to selective ultrafilters in the Ellentuck space: They are preserved under countable-support side-by-side Sacks forcing. In particular, the ultrafilters $\mathcal{U}_\alpha$ are preserved under such forcings.

We showed in Section 4 that *selective* and *weakly selective* coincide in $\mathcal{R}_\alpha$. In [21], Trujillo constructed an ultrafilter that is weakly selective but not Ramsey in $\mathcal{R}_1$, answering a question of Dobrinen. So the ultrafilter is selective but not Ramsey. We also showed that every Nash–Williams ultrafilter is selective. It remains to be seen the exact relation among the notions of *weakly selective*, *selective*, *Nash–Williams* and *Ramsey* in $\mathcal{R}_\alpha$. It is not difficult to see that these four notions coincide in the Milliken space (see [22]) and the Ellentuck space. It would also be interesting to see how the relation among the notions and their preservation under Sacks forcing depend on the structure of topological Ramsey spaces.

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