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UPPER AND LOWER BOUNDS OF LARGE DEVIATIONS FOR SOME DEPENDENT SEQUENCES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $S_n = \sum_{i=1}^n X_i$ and $M_n = \max\{X_1, X_2, \dots, X_n\}$. Under some suitable conditions, we establish the upper bound of large deviations for S_n and M_n based on some dependent sequences including acceptable random variables, widely acceptable random variables and a class of random variables that satisfies the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality. In addition, the lower bound of large deviations for some dependent sequences is also obtained. The results obtained in the paper generalize and improve some corresponding ones for independent random variables and negatively associated random variables.

1. Introduction

It is well known that the large deviation techniques are very useful tools in many areas of probability theory, statistics, statistical physics, insurance mathematics, and other related fields. For independent and identically distributed (i.i.d.) non-negative random variables $\{X, X_n, n \geq 1\}$ with $S_n = \sum_{i=1}^n X_i$ for each $n \geq 1$, Gantert [2] studied the logarithmic asymptotic behaviors for the partial sum S_n and obtained the following result.

Theorem A. *Let $\gamma > 1$. Then the following holds.*

(i) *If $EX^\gamma = \infty$ and $P(X > t) \geq L(t)/t^\gamma$ for some slowly varying function L, then for every $m > 0$ and $x \geq 1$,*

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(S_n \geq n^x m) \geq 1 - \gamma x. \tag{1.1}$$

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(ii) *If* $EX^p < \infty$ *for each* $p < \gamma$, *then for every* $m > EX$ *and* $x \geq 1$,

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(S_n \geq n^x m) \leq 1 - \gamma x. \tag{1.2}$$

However, the conditions of Theorem A look not very perfect, since the condition "$P(X > t) \geq L(t)/t^\tau$ for some slowly varying function L" is needed. To remove this condition, Hu and Nyrhinen [4] further investigated the logarithmic asymptotic behaviors for the partial sum S_n, and gave the exact limit inferiors and limit superiors by introducing two parameters, namely

$$\bar{\gamma} \doteq -\limsup_{t\to\infty} \frac{1}{t} \log P(\log X > t) \in [0, \infty] \tag{1.3}$$

and

$$\underline{\gamma} \doteq -\liminf_{t\to\infty} \frac{1}{t} \log P(\log X > t) \in [0, \infty]. \tag{1.4}$$

Write $\bar{x} = \max\{1, 1/\bar{\gamma}\}$ if $\bar{\gamma} \in (0, \infty]$ and $\underline{x} = \max\{1, 1/\underline{\gamma}\}$ if $\underline{\gamma} \in (0, \infty]$. For convention, we assume that $1/\infty = 0$.

Based on the notations above, Hu and Nyrhinen [4] established the large deviations for the partial sums of non-negative i.i.d. random variables as follows.

THEOREM B. *Assume that* $\{X, X_n, n \geq 1\}$ *are non-negative i.i.d. random variables. Let* $\bar{\gamma} \in (0, \infty)$. *Then for every* $x > \bar{x}$,

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(S_n > n^x) = 1 - \bar{\gamma} x.$$

If in addition $\underline{\gamma} < \infty$, *then for every* $x > \bar{x}$,

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(S_n > n^x) = 1 - \underline{\gamma} x.$$

Recently, Miao et al. [7] extended the result of Theorem B for non-negative i.i.d. random variables to the cases of stationary m-dependent sequence and stationary negatively associated sequence. However, another two parameters are needed for this purpose, namely,

$$\bar{\alpha} \doteq -\limsup_{t\to\infty} \frac{1}{t} \log P(\log |X| > t) \in [0, \infty] \tag{1.5}$$

and

$$\underline{\alpha} \doteq -\liminf_{t\to\infty} \frac{1}{t} \log P(\log |X| > t) \in [0, \infty]. \tag{1.6}$$

Remark 1.1. It is easily checked that $\bar{\alpha} \le \underline{\alpha}$, and the two parameters $\bar{\alpha}$ and $\underline{\alpha}$ are finite and equal to the common value α if and only if

$$\frac{1}{t^{\alpha+\varepsilon}} \le P(|X| > t) \le \frac{1}{t^{\alpha-\varepsilon}} \tag{1.7}$$

for every $\varepsilon > 0$, and for large t. A further useful fact is that

$$\bar{\alpha} = \sup\left\{ \lambda \ge 0 : E|X|^{\lambda} < \infty \right\}. \tag{1.8}$$

One may refer to Rolski et al. [8] for the details of the proof. The representation (1.8) shows that if $\bar{\alpha}$ is finite, then X is heavy tailed, namely, $E\exp(\lambda X) = \infty$ for every $\lambda > 0$.

Write $\bar{x} = \max\{1, 1/\bar{\alpha}\}$ if $\bar{\alpha} \in (0, \infty]$ and $\underline{x} = \max\{1, 1/\underline{\alpha}\}$ if $\underline{\alpha} \in (0, \infty]$. For convention, we assume that $1/\infty = 0$.

Based on the notations above, Miao et al. [7] established the large deviations for the partial sums of stationary m-dependent sequence and stationary negatively associated sequence. In addition, they obtained the upper bound of large deviations for general stationary sequence as follows.

Theorem C. *Let $\{X, X_n, n \ge 1\}$ be a stationary sequence of random variables. Suppose that the parameters $\bar{\alpha}$ and $\underline{\alpha}$ are finite and equal to the common value α. If $0 < \alpha < 1$, then for every $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\left(|S_n| > n^x\right) \le 1 - \alpha x, \tag{1.9}$$

and if $\alpha \ge 1$, then for every $x > \bar{x}$,

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\left(|S_n| > n^x\right) \le \alpha - \alpha x. \tag{1.10}$$

Inspired by the ideas of Gantert [2], Hu and Nyrhinen [4] and Miao et al. [7], we aim to generalize and improve the result of Theorem C for stationary sequence of random variables to a class of random variables. The condition of stationarity will be weakened by stochastic domination, and the upper bound of the large deviations will be improved.

The concept of stochastic domination below will play an important role throughout the paper.

Definition 1.1. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \le CP(|X| > x)$$

for all $x \ge 0$ and $n \ge 1$.

The paper is organized as follows. Some important lemmas are provided in Section 2. The upper bounds of the large deviations for acceptable random variables, widely acceptable random variables and a class of random variables are presented in Section 3, Section 4 and Section 5, respectively. In Section 6, we discuss the upper bounds of the large deviations for widely orthant dependent random variables. The condition of stationarity is not needed, while the condition of stochastic domination is needed.

Throughout the paper, C denotes a positive constant not depending on n, which may be different in various places. Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. Set $S_n = \sum_{i=1}^n X_i$ and $M_n = \max\{X_1, X_2, \dots, X_n\}$ for each $n \geq 1$. Denote $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$.

2. Some important lemmas

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for symmetric random variables.

LEMMA 2.1 (Ledoux and Talagrand [5]). *Let $\{X_n, n \geq 1\}$ be a sequence of symmetric random variables. For $n \geq 1$, set $S_n = \sum_{k=1}^n X_k$. Then for any $x > 0$,*

$$P\Big(\max_{1 \leq j \leq n} |X_j| \geq x\Big) \leq 2P(|S_n| \geq x). \tag{2.1}$$

The next one is a basic property for the two parameters $\bar{\alpha}$ and $\underline{\alpha}$.

LEMMA 2.2. *Let $\bar{\alpha}$ and $\underline{\alpha}$ be defined by* (1.5) *and* (1.6), *respectively. Then for any $x > 0$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(\log |X| > x \log n) = -\bar{\alpha} x \tag{2.2}$$

and

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(\log |X| > x \log n) = -\underline{\alpha} x. \tag{2.3}$$

In addition, for any constant C, we have

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(\log |X| > x \log n - C) = -\bar{\alpha} x \tag{2.4}$$

and

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(\log |X| > x \log n - C) = -\underline{\alpha} x. \tag{2.5}$$

PROOF. (2.2) and (2.3) can be obtained by the definitions of $\bar{\alpha}$ and $\underline{\alpha}$ immediately. We only need to show (2.4) and (2.5).

Noting that for any fixed $x > 0$ and constant C,

$$\lim_{n\to\infty} \frac{x\log n - C}{\log n} = x,$$

we have by the definitions of $\bar{\alpha}$ and $\underline{\alpha}$ that

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{\log n} \log P(\log|X| > x\log n - C)\\ &= \limsup_{n\to\infty} \frac{x\log n - C}{\log n} \cdot \frac{1}{x\log n - C} \log P(\log|X| > x\log n - C) = -\bar{\alpha}x\end{aligned}$$

and

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{\log n} \log P(\log|X| > x\log n - C)\\ &= \liminf_{n\to\infty} \frac{x\log n - C}{\log n} \cdot \frac{1}{x\log n - C} \log P(\log|X| > x\log n - C) = -\underline{\alpha}x. \qquad \square\end{aligned}$$

REMARK 2.1. Under the conditions of Lemma 2.2, we have that for any $x > 0$ and for any constant C,

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(\log|X| \geq x\log n) = -\bar{\alpha}x, \tag{2.6}$$

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(\log|X| \geq x\log n) = -\underline{\alpha}x, \tag{2.7}$$

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(\log|X| \geq x\log n - C) = -\bar{\alpha}x, \tag{2.8}$$

and

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(\log|X| \geq x\log n - C) = -\underline{\alpha}x. \tag{2.9}$$

Their proofs are similar to those of (2.2)–(2.5) respectively. So the details are omitted.

The last one is a basic property for stochastic domination.

LEMMA 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$, we have*

$$E|X_n|^\alpha \leq CE|X|^\alpha, \tag{2.10}$$

where C is a positive constant.

PROOF. It follows from the definition of stochastic domination that

$$E|X_n|^\alpha = \alpha \int_0^\infty x^{\alpha-1} P(|X_n| > x)\, dx$$
$$\le C\alpha \int_0^\infty x^{\alpha-1} P(|X| > x)\, dx = CE|X|^\alpha,$$

which implies (2.10). □

3. Upper bound of large deviations for acceptable random variables

In this section, we present an upper bound of large deviations for acceptable random variables. The concept of acceptable random variables was introduced by Giuliano et al. [3] as follows.

DEFINITION 3.1. We say that a finite collection of random variables X_1, X_2, …, X_n is acceptable if for any real number λ,

$$E \exp\left(\lambda \sum_{i=1}^n X_i\right) \le \prod_{i=1}^n E \exp(\lambda X_i). \tag{3.1}$$

An infinite sequence of random variables $\{X_n, n \ge 1\}$ is acceptable if every finite subcollection is acceptable.

It is easily seen that independent random variables are acceptable. As is mentioned in Giuliano et al. [3], a sequence of negatively orthant dependent random variables with a finite Laplace transform or finite moment generating function near zero (and hence a sequence of negatively associated random variables with finite Laplace transform, too) provides us an example of acceptable random variables.

Another interesting example of a sequence $\{Z_n, n \ge 1\}$ of acceptable random variables can be constructed in the following way. Feller [1, Problem III.1] (cf. also Romano and Siegel [9, Section 4.30]) provides an example of two random variables X and Y such that the density of their sum is the convolution of their densities, yet they are not independent. It is easy to see that X and Y are not negatively dependent either. Since they are bounded, their Laplace transforms $E \exp(\lambda X)$ and $E \exp(\lambda Y)$ are finite for any λ. Next, since the density of their sum is the convolution of their densities, we have

$$E \exp(\lambda(X + Y)) = E \exp(\lambda X) E \exp(\lambda Y).$$

The announced sequence of acceptable random variables $\{Z_n, n \ge 1\}$ can be now constructed in the following way. Let (X_k, Y_k) be independent copies of the random vector $(X,Y), k \geq 1$. For any $n \geq 1$, set $Z_n = X_k$ if $n = 2k+1$ and $Z_n = Y_k$ if $n = 2k$. Hence, the model of acceptable random variables that we consider in this paper (Definition 3.1) is more general than models considered in the previous literature. Studying the limiting behavior of acceptable random variables is of interest.

For more details about the limiting behavior of acceptable random variables, one can refer to Shen et al. [11], Shen and Wu [12] and Sung et al. [13] among others.

The aim of this section is to present the upper bound of the large deviations for acceptable random variables, while the following assumption is needed.

(H_1) Let $\{X_n, n \geq 1\}$ be a sequence of acceptable random variables. For any fixed $C > 0$, denote

$$\tilde{X}_n = -CI\left(X_n < -C\right) + X_n I\left(|X_n| \leq C\right) + CI\left(X_n > C\right), \ n \geq 1.$$

Then $\{\tilde{X}_n, n \geq 1\}$, $\{\tilde{X}_n^+, n \geq 1\}$ and $\{\tilde{X}_n^-, n \geq 1\}$ are all sequences of acceptable random variables.

REMARK 3.1. We point out that there are many sequences of random variables satisfying the definition of acceptability and assumption (H_1), such as independent sequence, negatively associated (NA, for short) sequence, negatively orthant dependent (NOD, for short) sequence, and so on.

Our results are as follows. The first one is the upper bound of large deviations for S_n.

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of acceptable random variables, which is stochastically dominated by a random variable X. Assume that (H_1) is satisfied. If $\bar{\alpha} \in (0,\infty)$, then for any $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(|S_n| > n^x) \leq 1 - \bar{\alpha}x. \tag{3.2}$$

Furthermore, if $\underline{\alpha} < \infty$, then for any $x > \bar{x}$,

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P(|S_n| > n^x) \leq 1 - \underline{\alpha}x. \tag{3.3}$$

The next one is the upper bound of large deviations for M_n.

THEOREM 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of symmetric acceptable random variables, which is stochastically dominated by a random variable X. Assume that (H_1) is satisfied. If $\bar{\alpha} \in (0,\infty)$, then for any $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\left(|M_n| > n^x\right) \leq 1 - \bar{\alpha}x. \tag{3.4}$$

Furthermore, if $\underline{\alpha} < \infty$, *then for any* $x > \bar{x}$,

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P\left(|M_n| > n^x\right) \le 1 - \underline{\alpha} x. \tag{3.5}$$

To prove Theorems 3.1 and 3.2, we need the following important probability inequality for the partial sums of acceptable random variables. The method used to prove the lemma is inspired by the ideas of Hu and Nyrhinen [4].

LEMMA 3.1. *Let* $\{X_n, n \ge 1\}$ *be a sequence of acceptable random variables, which is stochastically dominated by a random variable* X *with* $E|X|^\alpha < \infty$ *for some* $\alpha \in (0,\infty)$. *Denote* $\beta = \min(1,\alpha)$ *and* $\mu = E|X|^\beta$. *Assume that* (H_1) *is satisfied. Then there exist two positive constants* C_1 *and* C_2 *such that for any* $v > 0$, $t > 0$ *and* $n \ge 1$,

$$P(|S_n| > t^{1/\beta}) \le C_1 n P\Big(|X| > \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + 2C_2^{\frac{1}{2}v^{1/\beta}} e^v \Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}}, \tag{3.6}$$

and

$$P(|S_n| \ge t^{1/\beta}) \le C_1 n P\Big(|X| \ge \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + 2C_2^{\frac{1}{2}v^{1/\beta}} e^v \Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}}. \tag{3.7}$$

PROOF. For fixed $n \ge 1$, denote

$$\begin{aligned}\tilde{X}_n = -\Big(\frac{t}{v}\Big)^{1/\beta} I\Big(X_n < -\Big(\frac{t}{v}\Big)^{1/\beta}\Big) + X_n I\Big(|X_n| \le \Big(\frac{t}{v}\Big)^{1/\beta}\Big)\\ + \Big(\frac{t}{v}\Big)^{1/\beta} I\Big(X_n > \Big(\frac{t}{v}\Big)^{1/\beta}\Big)\end{aligned}$$

and $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$. It is easily checked that

$$\begin{aligned}P\big(|S_n| > t^{1/\beta}\big) &\le P\Big(\bigcup_{i=1}^n \{X_i \ne \tilde{X}_i\}\Big) + P\big(|\tilde{S}_n| > t^{1/\beta}\big)\\ &\le \sum_{i=1}^n P\big(X_i \ne \tilde{X}_i\big) + P\big(|\tilde{S}_n| > t^{1/\beta}\big)\\ &\le \sum_{i=1}^n P\Big(|X_i| > \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + P\big(|\tilde{S}_n| > t^{1/\beta}\big),\end{aligned}$$

which together with the definition of stochastic domination yields that

$$P\big(|S_n| > t^{1/\beta}\big) \le C_1 n P\Big(|X| > \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + P\big(|\tilde{S}_n| > t^{1/\beta}\big). \tag{3.8}$$

In the following, we take $P(|\tilde{S}_n| > t^{1/\beta})$ into account. For any positive constant h, we have for $1 \le i \le n$ that

$$Ee^{h\tilde{X}_i^+} = E\big(e^{h\tilde{X}_i^+} I(\tilde{X}_i^+ > 0)\big) + P\big(\tilde{X}_i^+ = 0\big)$$
$$= E\bigg(\frac{e^{h\tilde{X}_i^+} - 1}{|\tilde{X}_i|^\beta} \cdot |\tilde{X}_i|^\beta I(\tilde{X}_i^+ > 0)\bigg) + 1$$
$$\le E\bigg(\frac{e^{h|\tilde{X}_i|} - 1}{|\tilde{X}_i|^\beta} \cdot |\tilde{X}_i|^\beta I(\tilde{X}_i^+ > 0)\bigg) + 1.$$

Since $0 < \beta \le 1$, we can see that $\frac{e^{hs}-1}{s^\beta}$ is nondecreasing for $s > 0$. Note that $|\tilde{X}_i| \le \min\big(|X_i|, (\frac{t}{v})^{1/\beta}\big)$ for $1 \le i \le n$. We have

$$Ee^{h\tilde{X}_i^+} \le \frac{\exp(h(\frac{t}{v})^{1/\beta}) - 1}{\frac{t}{v}} \cdot E|X_i|^\beta + 1 \tag{3.9}$$
$$\le \exp\bigg\{\frac{\exp(h(\frac{t}{v})^{1/\beta}) - 1}{\frac{t}{v}} \cdot E|X_i|^\beta\bigg\} \le \exp\bigg\{C_2\mu \cdot \frac{\exp(h(\frac{t}{v})^{1/\beta}) - 1}{\frac{t}{v}}\bigg\},$$

where the last inequality above follows from Lemma 2.3 and C_2 is a positive constant. By the assumption (H_1), we can see that $\{\tilde{X}_1^+, \tilde{X}_2^+, \cdots, \tilde{X}_n^+\}$ are still acceptable random variables. Hence, we have by Markov's inequality, Definition 3.1 and (3.9) that

$$P\bigg(\sum_{i=1}^n \tilde{X}_i^+ > \frac{1}{2}t^{1/\beta}\bigg) \le e^{-\frac{1}{2}ht^{1/\beta}} E\exp\bigg(h\sum_{i=1}^n \tilde{X}_i^+\bigg) \tag{3.10}$$
$$\le e^{-\frac{1}{2}ht^{1/\beta}} \prod_{i=1}^n Ee^{h\tilde{X}_i^+} \le \exp\bigg\{C_2\mu n \cdot \frac{\exp(h(\frac{t}{v})^{1/\beta}) - 1}{\frac{t}{v}} - \frac{1}{2}ht^{1/\beta}\bigg\}.$$

Taking $h = (\frac{v}{t})^{1/\beta}\log(\frac{t}{C_2\mu n} + 1)$ into the right hand side of (3.10), we can get

$$P\bigg(\sum_{i=1}^n \tilde{X}_i^+ > \frac{1}{2}t^{1/\beta}\bigg) \le \exp\bigg\{v - \frac{1}{2}\,v^{1/\beta}\log\Big(\frac{t}{C_2\mu n} + 1\Big)\bigg\} \tag{3.11}$$
$$= e^v\Big(\frac{t}{C_2\mu n} + 1\Big)^{-\frac{1}{2}v^{1/\beta}} \le C_2^{\frac{1}{2}v^{1/\beta}} e^v\Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}}.$$

Note that $\{\tilde{X}_1^-, \tilde{X}_2^-, \dots, \tilde{X}_n^-\}$ are still acceptable random variables by the assumption (H_1). Hence, similarly to the proof of (3.11), we have

$$P\Big(\sum_{i=1}^n \tilde{X}_i^- > \frac{1}{2}t^{1/\beta}\Big) \le C_2^{\frac{1}{2}v^{1/\beta}} e^v \Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}}. \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$P\big(|\tilde{S}_n| > t^{1/\beta}\big) \le P\Big(\sum_{i=1}^n |\tilde{X}_i| > t^{1/\beta}\Big) \tag{3.13}$$

$$\le P\Big(\sum_{i=1}^n \tilde{X}_i^+ > \frac{1}{2}t^{1/\beta}\Big) + P\Big(\sum_{i=1}^n \tilde{X}_i^- > \frac{1}{2}t^{1/\beta}\Big) \le 2C_2^{\frac{1}{2}v^{1/\beta}} e^v \Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}},$$

which together with (3.8) yields the desired result (3.6).

Similarly to the proof of (3.6), we can get (3.7) immediately, provided that $\tilde{X}_n$ is replaced by

$$\tilde{X}_n' = -\Big(\frac{t}{v}\Big)^{1/\beta} I\Big(X_n \le -\Big(\frac{t}{v}\Big)^{1/\beta}\Big) + X_n I\Big(|X_n| < \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + \Big(\frac{t}{v}\Big)^{1/\beta} I\Big(X_n \ge \Big(\frac{t}{v}\Big)^{1/\beta}\Big). \quad \square$$

Proof of Theorem 3.1. Applying Lemma 3.1 with $t = n^{\beta x}$, we can get that

$$P(|S_n| > n^x) \le C_1 n P\Big(|X| > \Big(\frac{n^{\beta x}}{v}\Big)^{1/\beta}\Big) + 2C_2^{\frac{1}{2}v^{1/\beta}} e^v \Big(\frac{\mu n}{n^{\beta x}}\Big)^{\frac{1}{2}v^{1/\beta}} \tag{3.14}$$

$$= C_1 n P\Big(|X| > \frac{n^x}{v^{1/\beta}}\Big) + 2C_2^{\frac{1}{2}v^{1/\beta}} e^v \cdot \mu^{\frac{1}{2}v^{1/\beta}} \cdot n^{\frac{(1-\beta x)v^{1/\beta}}{2}},$$

which implies that

$$P(|S_n| > n^x) \le \max\Big\{2C_1 n P\Big(|X| > \frac{n^x}{v^{1/\beta}}\Big), 4C_2^{\frac{1}{2}v^{1/\beta}} e^v \cdot \mu^{\frac{1}{2}v^{1/\beta}} \cdot n^{\frac{(1-\beta x)v^{1/\beta}}{2}}\Big\}.$$

Hence,

$$\frac{1}{\log n} \log P(|S_n| > n^x)$$

$$\le \frac{1}{\log n} \max\Big\{ \log 2C_1 + \log n + \log P\big(\log|X| > x\log n - \log v^{1/\beta}\big),$$

$$\log\big(4C_2^{\frac{1}{2}v^{1/\beta}} e^v \cdot \mu^{\frac{1}{2}v^{1/\beta}}\big) + \frac{(1-\beta x)v^{1/\beta}}{2}\log n\Big\},$$

which together with Lemma 2.2 yields that

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(|S_n| > n^x) \le \max\left\{1 - \bar{\alpha}x, \frac{(1-\beta x)v^{1/\beta}}{2}\right\}. \tag{3.15}$$

Note that $x > \bar{x} = \max\{1, 1/\bar{\alpha}\}$, we can choose a suitable $\alpha > 0$ such that $1 - \alpha x < 0$, which together with $\beta = \min(1, \alpha)$ yields that $1 - \beta x < 0$. In view of the arbitrariness of v, the desired result follows from (3.15) immediately.

If we further assume that $\underline{\alpha} < \infty$, then similarly to the proof of (3.2), we can see that (3.3) holds for any $x > \bar{x}$. □

PROOF OF THEOREM 3.2. Noting that

$$|M_n| = \Big|\max_{1\le j\le n} X_j\Big| \le \max_{1\le j\le n} |X_j|,$$

we have by Lemma 2.1 that for any $x > 0$,

$$P(|M_n| \ge x) \le P\Big(\max_{1\le j\le n} |X_j| \ge x\Big) \le 2P(|S_n| \ge x),$$

which together with Lemma 3.1 yields that

$$P(|M_n| > n^x) \le 2C_1 nP\Big(|X| \ge \Big(\frac{n^{\beta x}}{v}\Big)^{1/\beta}\Big) + 4C_2^{\frac{1}{2}v^{1/\beta}} e^v \cdot \mu^{\frac{1}{2}v^{1/\beta}} \cdot n^{\frac{(1-\beta x)v^{1/\beta}}{2}}. \tag{3.16}$$

The rest of the proof is similar to that of Theorem 3.1, provided that Lemma 2.2 is replaced by Remark 2.1. □

4. Upper bound of large deviations for widely acceptable random variables

In this section, we consider a more general dependence structure than acceptable random variables, which is inspired by Definition 3.1.

DEFINITION 4.1. We say that a finite collection of random variables X_1, X_2, …, X_n is widely acceptable if there exists a finite real number $g(n)$ such that for any real number λ,

$$E\exp\left(\lambda\sum_{i=1}^n X_i\right) \le g(n)\prod_{i=1}^n E\exp(\lambda X_i). \tag{4.1}$$

An infinite sequence of random variables $\{X_n, n \ge 1\}$ is widely acceptable if every finite subcollection is widely acceptable, and $g(n)$, $n \ge 1$ are called dominating coefficients.

It is easily seen that $g(n) \geq 1$. If $g(n) \equiv M$ for each $n \geq 1$, where $M \geq 1$ is some positive constant, then we say $\{X_n, n \geq 1\}$ is extended acceptable. In this case, we can see that the extended negatively dependence (END, for short) structure satisfies the condition (4.1) when $g(n) \equiv M$. One can refer to Liu [6] and Shen [10] for instance. If $g(n) \equiv 1$ for each $n \geq 1$, then the concept of widely acceptable random variables is reduced to acceptable random variables. In addition, we point out that the widely orthant dependence (WOD, for short) structure also satisfies condition (4.1) with the dominating coefficients $g(n)$, $n \geq 1$. One can refer to Wang et al. [14] or Wang et al. [15] for instance. Hence, the concept of widely acceptable random variables includes independent sequence, NA sequence, NOD sequence, END sequence and WOD sequence as special cases. For examples of widely acceptable random variables, one can refer to Wang et al. [14] for instance.

The aim of this section is to present the upper bound of large deviations for widely acceptable random variables, while the following assumption is needed.

(H_2) Let $\{X_n, n \geq 1\}$ be a sequence of widely acceptable random variables with the dominating coefficients $g(n)$, $n \geq 1$. For any fixed $C > 0$, denote

$$\tilde{X}_n = -CI(X_n < -C) + X_n I(|X_n| \leq C) + CI(X_n > C), \quad n \geq 1.$$

Then $\{\tilde{X}_n, n \geq 1\}$, $\{\tilde{X}_n^+, n \geq 1\}$ and $\{\tilde{X}_n^-, n \geq 1\}$ are all sequences of widely acceptable random variables with the dominating coefficients $g(n)$, $n \geq 1$.

REMARK 4.1. We point out that there are many sequences of random variables satisfying the assumption (H_2), such as independent sequence, NA sequence, NOD sequence, END sequence, WOD sequence, and so on.

The upper bounds of large deviations for widely acceptable random variables are as follows.

THEOREM 4.1. *Let $\{X_n, n \geq 1\}$ be a sequence of widely acceptable random variables, which is stochastically dominated by a random variable X. Assume that* (H_2) *is satisfied and $\log g(n) = o(\log n)$. If $\bar{\alpha} \in (0, \infty)$, then for any $x > \bar{x}$,* (3.2) *holds. Furthermore, if $\underline{\alpha} < \infty$, then for any $x > \bar{x}$,* (3.3) *holds.*

THEOREM 4.2. *Let $\{X_n, n \geq 1\}$ be a sequence of symmetric widely acceptable random variables, which is stochastically dominated by a random variable X. Assume that* (H_2) *is satisfied and $\log g(n) = o(\log n)$. If $\bar{\alpha} \in (0, \infty)$, then for any $x > \bar{x}$,* (3.4) *holds. Furthermore, if $\underline{\alpha} < \infty$, then for any $x > \bar{x}$,* (3.5) *holds.*

The proofs of Theorems 4.1 and 4.2 are similar to those of Theorems 3.1 and 3.2, while Lemma 3.1 is replaced by the following lemma. The proof is similar to that of Lemma 3.1, so the details of the proof are omitted.

LEMMA 4.1. *Let $\{X_n, n \ge 1\}$ be a sequence of widely acceptable random variables, which is stochastically dominated by a random variable X with $E|X|^\alpha < \infty$ for some $\alpha \in (0, \infty)$. Denote $\beta = \min(1, \alpha)$ and $\mu = E|X|^\beta$. Assume that* (H_2) *is satisfied. Then there exist two positive constants C_1 and C_2 such that for any $v > 0$, $t > 0$ and $n \ge 1$,*

$$(4.2)\quad P(|S_n| > t^{1/\beta}) \le C_1 n P\Big(|X| > \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + 2C_2^{\frac{1}{2}v^{1/\beta}} e^v g(n) \Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}},$$

and

$$(4.3)\quad P(|S_n| \ge t^{1/\beta}) \le C_1 n P\Big(|X| \ge \Big(\frac{t}{v}\Big)^{1/\beta}\Big) + 2C_2^{\frac{1}{2}v^{1/\beta}} e^v g(n) \Big(\frac{\mu n}{t}\Big)^{\frac{1}{2}v^{1/\beta}}.$$

REMARK 4.2. We have pointed out that if $g(n) \equiv 1$ for each $n \ge 1$, then the concept of widely acceptable random variables is reduced to acceptable random variables. Hence, Theorems 4.1 and 4.2 are generalizations of Theorems 3.1 and 3.2, respectively.

5. Upper bound of large deviations for a class of random variables

It is well known that the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality play important roles in various proofs of limit theorems. In particular, they provide a measure of convergence rate for the strong law of large numbers. The aim of the this section is to provide an upper bound of large deviations for a class of random variables that satisfies the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality.

Let $p > 1$ and $\{X_n, n \ge 1\}$ be a sequence of random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for each $i \ge 1$.

(i) Marcinkiewicz–Zygmund type inequality:

$$E\Big|\sum_{i=1}^n X_i\Big|^p \le C_1 \sum_{i=1}^n E|X_i|^p \quad \text{for } 1 < p \le 2;$$

(ii) Rosenthal type inequality:

$$E\Big|\sum_{i=1}^n X_i\Big|^p \le C_2 \Big[\sum_{i=1}^n E|X_i|^p + \Big(\sum_{i=1}^n EX_i^2\Big)^{p/2}\Big] \quad \text{for } p > 2,$$

where C_1 and C_2 are two positive constants depending only on p.

To establish the main results of this section, we need the following assumption.

(H_3) Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality. For any fixed $x > 0$, denote

$$Y_n = -n^x I(X_n < -n^x) + X_n I(|X_n| \leq n^x) + CI(X_n > n^x), \quad n \geq 1.$$

Then $\{Y_n, n \geq 1\}$ is still a sequence of random variables satisfying the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality.

REMARK 5.1. We point out that there are many sequences of random variables satisfying the assumption (H_3), such as independent sequence, NA sequence, NOD sequence, END sequence, ρ-mixing sequence, φ-mixing sequence, $\tilde{\rho}$-mixing sequence, and so on.

The upper bound of large deviations for the partial sum of a class of random variables that satisfies the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality is as follows.

THEOREM 5.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. Assume that* (H_3) *is satisfied.*

(i) *If $\bar{\alpha} \in (0, 2]$, then for any $x > \bar{x}$,* (3.2) *holds.*

(ii) *If $\bar{\alpha} \in (2, \infty)$, then for any $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\big(|S_n| > n^x\big) \leq \bar{\alpha}/2 - \bar{\alpha}x. \tag{5.1}$$

PROOF. For fixed $n \geq 1$ and $x > \bar{x}$, set

$$Y_n = -n^x I(X_n < -n^x) + X_n I(|X_n| \leq n^x) + n^x I(X_n > n^x),$$

and $T_n = \sum_{i=1}^n Y_i$. It is easily checked that

$$\begin{aligned} P(|S_n| > n^x) &\leq \sum_{i=1}^n P(|X_i| > n^x) + P(|T_n| > n^x) \\ &\leq CnP(|X| > n^x) + P(|T_n| > n^x). \end{aligned} \tag{5.2}$$

(i) If $\bar{\alpha} \in (0, 2]$, then we can choose any small ε such that $0 < \varepsilon < \bar{\alpha}$. Noting that $|Y_i| \leq \min(|X_i|, n^x)$, we have by Markov's inequality, C_r-inequality, Marcinkiewicz–Zygmund type inequality and Lemma 2.3 that for any $x > \bar{x}$,

$$\begin{aligned} P(|T_n| > n^x) &\leq Cn^{-\bar{\alpha}x} E|T_n|^{\bar{\alpha}} \leq Cn^{-\bar{\alpha}x} \sum_{i=1}^n E|Y_i|^{\bar{\alpha}} \\ &\leq Cn^{\varepsilon x - \bar{\alpha}x} \sum_{i=1}^n E|X_i|^{\bar{\alpha}-\varepsilon} \leq Cn^{1+\varepsilon x - \bar{\alpha}x} E|X|^{\bar{\alpha}-\varepsilon}, \end{aligned}$$

which together with (5.2) yields that

$$P(|S_n| > n^x) \le CnP(|X| > n^x) + Cn^{1+\varepsilon x-\bar{\alpha}x}E|X|^{\bar{\alpha}-\varepsilon} \le \max\left\{2CnP(|X| > n^x), 2Cn^{1+\varepsilon x-\bar{\alpha}x}E|X|^{\bar{\alpha}-\varepsilon}\right\}. \tag{5.3}$$

Applying Lemma 2.2, we have by (5.3) that

$$\limsup_{n\to\infty}\frac{1}{\log n}\log P(|S_n| > n^x) \le \max\{1-\bar{\alpha}x, 1+\varepsilon x-\bar{\alpha}x\} = 1+\varepsilon x-\bar{\alpha}x. \tag{5.4}$$

Noting that $\varepsilon > 0$ is arbitrary, the desired result follows from (5.4) immediately.

(ii) If $\bar{\alpha} \in (2, \infty)$, then we can choose any small ε such that $0 < \varepsilon < \bar{\alpha}$. Noting that $|Y_i| \le \min(|X_i|, n^x)$ and $EX^2 < \infty$, we have by Markov's inequality, Rosenthal type inequality and Lemma 2.3 that for any $x > \bar{x}$,

$$\begin{aligned} P(|T_n| > n^x) &\le Cn^{-\bar{\alpha}x}E|T_n|^{\bar{\alpha}} \\ &\le Cn^{-\bar{\alpha}x}\left[\sum_{i=1}^n E|Y_i|^{\bar{\alpha}} + \left(\sum_{i=1}^n EY_i^2\right)^{\bar{\alpha}/2}\right]s \\ &\le Cn^{1+\varepsilon x-\bar{\alpha}x}E|X|^{\bar{\alpha}-\varepsilon} + Cn^{-\bar{\alpha}x}\left(\sum_{i=1}^n EX_i^2\right)^{\bar{\alpha}/2} \\ &\le Cn^{1+\varepsilon x-\bar{\alpha}x}E|X|^{\bar{\alpha}-\varepsilon} + Cn^{\bar{\alpha}/2-\bar{\alpha}x}EX^2, \end{aligned} \tag{5.5}$$

which together with (5.2) yields that

$$\begin{aligned} P(|S_n| > n^x) &\le CnP(|X| > n^x) + Cn^{1+\varepsilon x-\bar{\alpha}x}E|X|^{\bar{\alpha}-\varepsilon} + Cn^{\bar{\alpha}/2-\bar{\alpha}x}EX^2 \\ &\le \max\left\{2CnP\left(|X| > n^x\right), 2Cn^{1+\varepsilon x-\bar{\alpha}x}E|X|^{\bar{\alpha}-\varepsilon}, 2Cn^{\bar{\alpha}/2-\bar{\alpha}x}EX^2\right\}. \end{aligned} \tag{5.6}$$

Applying Lemma 2.2, we have by (5.6) that

$$\begin{aligned} &\limsup_{n\to\infty}\frac{1}{\log n}\log P(|S_n| > n^x) \\ &\le \max\left\{1-\bar{\alpha}x, 1+\varepsilon x-\bar{\alpha}x, \bar{\alpha}/2-\bar{\alpha}x\right\} = \max\left\{1+\varepsilon x-\bar{\alpha}x, \bar{\alpha}/2-\bar{\alpha}x\right\}. \end{aligned} \tag{5.7}$$

Noting that $\varepsilon > 0$ is arbitrary and $\bar{\alpha}/2 > 1$, (5.1) follows from (5.7) immediately. □

Remark 5.2. Comparing Theorem 5.1 with Theorem C, we have the following improvements:

(i) The condition of stationarity in Theorem C is weakened by stochastic domination in Theorem 5.1. The condition of identical distribution is not needed.

(ii) For $\bar{\alpha} \in (1, 2]$, the upper bound $1 - \bar{\alpha}x$ in Theorem 5.1 is sharper than $\bar{\alpha} - \bar{\alpha}x$ in Theorem C.

(iii) For $\bar{\alpha} \in (2, \infty)$, the upper bound $\bar{\alpha}/2 - \bar{\alpha}x$ in Theorem 5.1 is sharper than $\bar{\alpha} - \bar{\alpha}x$ in Theorem C.

If the partial sum S_n is replaced by the maximum M_n, then we can get the following upper bound of large deviations.

Theorem 5.2. *Let $\{X_n, n \geq 1\}$ be a sequence of symmetric random variables, which is stochastically dominated by a random variable X. Assume that* (H_3) *is satisfied.*

(i) *If $\bar{\alpha} \in (0, 2]$, then for any $x > \bar{x}$,* (3.4) *holds.*

(ii) *If $\bar{\alpha} \in (2, \infty)$, then for any $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\big(|M_n| > n^x\big) \leq \bar{\alpha}/2 - \bar{\alpha}x.$$

The proof of Theorem 5.2 is the same as those in Theorem 3.2 and Theorem 5.1, so the details of the proof are omitted.

Remark 5.3. We point out that there exist some sequences of random variables that don't satisfy the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality exactly, such as WOD random variables. Wang et al. [15] established the following moment inequalities for WOD random variables with the dominating coefficients $g(n)$, $n \geq 1$.

Lemma 5.1. *Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exist positive constants $C_1(p)$ and $C_2(p)$ depending only on p such that*

$$E\Big|\sum_{i=1}^n X_i\Big|^p \leq \big[C_1(p) + C_2(p)g(n)\big] \sum_{i=1}^n E|X_i|^p, \quad \textit{for } 1 \leq p \leq 2$$

and

$$E\Big|\sum_{i=1}^n X_i\Big|^p \leq C_1(p)\sum_{i=1}^n E|X_i|^p + C_2(p)g(n)\bigg(\sum_{i=1}^n E|X_i|^2\bigg)^{p/2}, \quad \textit{for } p > 2.$$

Note that the sequence of WOD random variables satisfies the following property. One can refer to Wang et al. [15] for instance.

Property 5.1. *Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with the dominating coefficients $g(n)$, $n \geq 1$. If $\{f_n(\cdot), n \geq 1\}$ are all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of WOD random variables with the dominating coefficients $g(n)$, $n \geq 1$.*

With Lemma 5.1 and Property 5.1 accounted for, and similarly to the proofs of Theorems 5.1 and 5.2, we can establish the upper bound of large deviations for WOD random variables with the dominating coefficients $g(n)$ as follows.

Theorem 5.3. *Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables, which is stochastically dominated by a random variable X. Assume that $\log g(n) = o(\log n)$.*

(i) *If $\bar{\alpha} \in (0, 2]$, then for any $x > \bar{x}$, (3.2) holds.*

(ii) *If $\bar{\alpha} \in (2, \infty)$, then for any $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(|S_n| > n^x) \leq \bar{\alpha}/2 - \bar{\alpha}x.$$

Theorem 5.4. *Let $\{X_n, n \geq 1\}$ be a sequence of symmetric WOD random variables, which is stochastically dominated by a random variable X. Assume that $\log g(n) = o(\log n)$.*

(i) *If $\bar{\alpha} \in (0, 2]$, then for any $x > \bar{x}$, (3.4) holds.*

(ii) *If $\bar{\alpha} \in (2, \infty)$, then for any $x > \bar{x}$,*

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P(|M_n| > n^x) \leq \bar{\alpha}/2 - \bar{\alpha}x.$$

6. Further discussion

In the previous sections, we established upper bounds of large deviations for the partial sum S_n and maximum M_n based on acceptable random variables, widely acceptable random variables and a class of random variables that satisfies the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality. However, the lower bounds of large deviations for the partial sum S_n and maximum M_n are not established. For this purpose, we introduce the following assumption for the sequence $\{X_n, n \geq 1\}$:

(H_4) For any $n \geq 1$ and any real numbers $x_1, x_2, \ldots, x_n$,

$$\begin{aligned} &P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \\ &\leq P(X_1 \leq x_1)P(X_2 \leq x_2) \cdots P(X_n \leq x_n). \end{aligned}$$

Remark 6.1. We point out that there are many sequences of random variables satisfying the assumption (H_4), such as independent sequence, NA sequence, NOD sequence, and so on.

Based on the assumption (H_4), we can get the following lower bounds of large deviations for the partial sum S_n and maximum M_n. The proof is similar to those of Theorems 2.1 and 3.2 in Miao et al. [7], so the details are omitted.

THEOREM 6.1. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of non-negative and identically distributed random variables satisfying* (H_4). *If* $\bar{\alpha} \in (0, \infty)$, *then for any* $x > \bar{x}$,

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\big(|S_n| > n^x\big) \geq 1 - \bar{\alpha}x$$

and

$$\limsup_{n\to\infty} \frac{1}{\log n} \log P\big(|M_n| > n^x\big) \geq 1 - \bar{\alpha}x.$$

Furthermore, if $\underline{\alpha} < \infty$, *then for any* $x > \bar{x}$,

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P\big(|S_n| > n^x\big) \geq 1 - \underline{\alpha}x$$

and

$$\liminf_{n\to\infty} \frac{1}{\log n} \log P\big(|M_n| > n^x\big) \geq 1 - \underline{\alpha}x.$$

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