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HYPERGEOMETRIC CAUCHY NUMBERS AND POLYNOMIALS

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Abstract. For positive integers N and M, the general *hypergeometric Cauchy polynomials* $c_{M,N,n}(z)$ ($M, N \ge 1$; $n \ge 0$) are defined by

$$\frac{1}{(1+t)^z}\frac{1}{{}_2F_1(M,N;N+1;-t)} = \sum_{n=0}^\infty c_{M,N,n}(z)\,\frac{t^n}{n!}\,,$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function. When $M = N = 1$, $c_n = c_{1,1,n}$ are the classical Cauchy numbers. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers. In the aspect of determinant expressions, hypergeometric Cauchy numbers are the natural extension of the classical Cauchy numbers, though many kinds of generalizations of the Cauchy numbers have been considered by many authors. In this paper, we show some interesting expressions of generalized hypergeometric Cauchy numbers. We also give a convolution identity for generalized hypergeometric Cauchy polynomials.

1. Introduction

Let

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}$$

be the Gauss hypergeometric function with the rising factorial $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ ($n \ge 1$) and $(x)^{(0)} = 1$. For $N \ge 1$, define the hy-

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pergeometric Cauchy numbers $c_{N,n}$ ([20]) by

$$\frac{1}{{}_2F_1(1,N;N+1;-t)}=\frac{(-1)^{N-1}t^N/N}{\log(1+t)-\sum_{n=1}^{N-1}(-1)^{n-1}t^n/n}=\sum_{n=0}^{\infty}c_{N,n}\frac{t^n}{n!}\,. \tag{1}$$

When $N=1$, $c_n=c_{1,n}$ are the classical Cauchy numbers ([5,33]) defined by

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}\,.$$

Notice that $b_n=c_n/n!$ are sometimes called the Bernoulli numbers of the sec ond kind. In [20], the general *hypergeometric Cauchy polynomials* $c_{M,N,n}(z)$ $(M,N\ge 1;\ n\ge 0)$ are defined by

$$\frac{1}{(1+t)^z}\frac{1}{{}_2F_1(M,N;N+1;-t)}=\sum_{n=0}^{\infty}c_{M,N,n}(z)\frac{t^n}{n!}\,, \tag{2}$$

so that $c_{N,n}=c_{1,N,n}(0)$.

Similar hypergeometric numbers are hypergeometric Bernoulli numbers $B_{N,n}$ and hypergeometric Euler numbers. For $N\ge 1$, define hypergeometric Bernoulli numbers $B_{N,n}$ ([10–13,15,34]) by

$$\frac{1}{{}_1F_1(1;N+1;t)}=\frac{t^N/N!}{e^t-\sum_{n=0}^{N-1}t^n/n!}=\sum_{n=0}^{\infty}B_{N,n}\frac{t^n}{n!}\,, \tag{3}$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a;b;z)=\sum_{n=0}^{\infty}\frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!}\,.$$

When $N=1$, $B_{1,n}=B_n$ are classical Bernoulli numbers, defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\,.$$

The hypergeometric Euler numbers $E_{N,n}$ ([24,30]) are defined by

$$\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)}=\sum_{n=0}^{\infty}E_{N,n}\frac{t^n}{n!}\,,$$

where

$${}_1F_2(a;b,c;z)=\sum_{n=0}^{\infty}\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}$$

is the hypergeometric function. When $N=0$, then $E_n=E_{0,n}$ are classical Euler numbers defined by

$$\frac{1}{\cosh t}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}\,.$$

Several kinds of generalizations of the Cauchy numbers (or the Bernoulli numbers of the second kind) have been considered by many authors. For example, poly-Cauchy number [18], multiple Cauchy numbers, shifted Cauchy numbers [28], generalized Cauchy numbers [25], incomplete Cauchy numbers [21,23,26], various types of q-Cauchy numbers [3,19,22,29], Cauchy Carlitz numbers [16,17]. The situations are similar and even more so for Bernoulli numbers and Euler numbers. One of the advantages of hypergeometric numbers is the natural extension of determinant expressions of the numbers. In [24,30], the hypergeometric Euler numbers $E_{N,2n}$ can be expressed as

$$E_{N,2n}=(-1)^n(2n)!\begin{vmatrix}\frac{(2N)!}{(2N+2)!} & 1 & & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!}\end{vmatrix}\quad (N\ge 0,\ n\ge 1)\,.$$

When $N=0$, this is reduced to a famous determinant expression of Euler numbers (cf. cite[p. 52]Glaisher):

$$E_{2n}=(-1)^n(2n)!\begin{vmatrix}\frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}\,. \tag{4}$$

In addition, when $N=1$, $E_{1,n}$ can be expressed by Bernoulli numbers as $E_{1,n}=-(n-1)B_n$ ([30]).

In [1], the hypergeometric Bernoulli numbers $B_{N,n}$ can be expressed as

$$B_{N,n}=(-1)^n n!\begin{vmatrix}\frac{N!}{(N+1)!} & 1 & & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!}\end{vmatrix}\quad (N\ge 1,\ n\ge 1)\,.$$

When $N=1$, we have a determinant expression of Bernoulli numbers ([7, p. 53]):

$$B_n=(-1)^n n!\begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}. \tag{5}$$

In addition, relations between $B_{N,n}$ and $B_{N-1,n}$ are shown in [1].

In this paper, we shall give a similar determinant expression of generalized hypergeometric Cauchy numbers and their generalizations. This allows us to find a more different expression of generalized hypergeometric Cauchy numbers as well as a converted expression. We also study the sums of products of generalized hypergeometric Cauchy polynomials.

2. Basic properties of generalized hypergeometric Cauchy numbers

Since

$${}_2F_1(M,N;N+1;-t)=\sum_{n=0}^\infty \frac{(M)^{(n)}N}{N+n}\,\frac{(-t)^n}{n!}\,,$$

from the definition (2), we have

$$\begin{aligned} 1&=\left(\sum_{l=0}^\infty \frac{(M)^{(l)}N}{N+l}\frac{(-t)^l}{l!}\right)\left(\sum_{m=0}^\infty c_{M,N,m}\frac{t^m}{m!}\right)\\ &=\sum_{n=0}^\infty\sum_{m=0}^n\binom{n}{m}\frac{(M)^{(n-m)}N(-1)^{n-m}}{N+n-m}c_{M,N,m}\frac{t^n}{n!}\,. \end{aligned}$$

Thus, we get

$$\sum_{m=0}^n\binom{n}{m}\frac{(-1)^{n-m}(M)^{(n-m)}}{N+n-m}c_{M,N,m}=0 \quad (n\ge 1) \tag{6}$$

with $c_{M,N,0}=1$. By (6), we have

$$c_{M,N,n}=\sum_{m=0}^{n-1}\binom{n}{m}\frac{(-1)^{n-m-1}(M)^{(n-m)}N}{N+n-m}c_{M,N,m}\,.$$

Then, *generalized hypergeometric Bernoulli numbers of the second kind* $b_{M,N,n} := c_{M,N,n}/n!$ satisfy the relation
(7)

$$b_{M,N,n} = \sum_{m=0}^{n-1} (-1)^{n-m-1} \frac{(M+n-m-1)!}{(n-m)!} \frac{N}{N+n-m} b_{M,N,m} \quad (n \ge 1)\,.$$

By using this expression, the first few values of $c_{M,N,n}$ are given by the following:

$$c_{M,N,0} = 1\,, \quad c_{M,N,1} = \frac{M \cdot N}{N+1}\,, \quad c_{M,N,2} = \frac{2M^2N^2}{(N+1)^2} - \frac{M(M+1)N}{N+2}\,,$$

$$c_{M,N,3} = \frac{6M^3N^3}{(N+1)^3} - \frac{6M^2(M+1)N^2}{(N+1)(N+2)} + \frac{M(M+1)(M+2)N}{N+3}\,,$$

$$c_{M,N,4} = \frac{24M^4N^4}{(N+1)^4} - \frac{36M^3(M+1)N^3}{(N+1)^2(N+2)} + \frac{8M^2(M+1)(M+2)N^2}{(N+1)(N+3)} + \frac{6M^2(M+1)^2N^2}{(N+2)^2} - \frac{M(M+1)(M+2)(M+3)N}{N+4}\,.$$

An explicit expression of hypergeometric Cauchy numbers is given as follows.

THEOREM 1. *For $N, n \ge 1$, we have*

$$c_{M,N,n} = n! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\dots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{(M)^{(i_1)} \cdots (M)^{(i_k)} N^k}{i_1! \cdots i_r! \, (N+i_1) \cdots (N+i_k)}\,.$$

Such values of $c_{M,N,n}$ can be expressed in terms of the determinant in the later section.

PROOF. Proof of Theorem 1 This is a special case of Theorem 2 in the next section. □

3. Multiple hypergeometric Cauchy numbers

In [14], the higher-order hypergeometric Bernoulli numbers and polynomials are defined and studied. For positive integers M, N and r, define the *higher-order generalized hypergeometric Cauchy numbers* $c_{M,N,n}^{(r)}$ by the generating function

(8)
$$\frac{1}{\bigl({}_2F_1(M,N;N+1;-t)\bigr)^r} = \sum_{n=0}^{\infty} c_{M,N,n}^{(r)} \frac{t^n}{n!}\,.$$

When $r=1$, $c_{M,N,n}=c_{M,N,n}^{(1)}$ are the generalized hypergeometric Cauchy numbers.

From the definition (8),

$$1=\left(\sum_{i=0}^\infty\frac{(M)^{(i)}(N)^{(i)}}{(N+i)^{(i)}}\frac{(-t)^i}{i!}\right)^r\left(\sum_{n=0}^\infty c_{M,N,n}^{(r)}\frac{t^n}{n!}\right)$$

$$=\left(\sum_{l=0}^\infty\sum_{\substack{i_1+\dots+i_r=l\\ i_1,\dots,i_r\ge 0}}\frac{(-1)^l l!\,(M)^{(i_1)}\cdots(M)^{(i_r)}(N)^{(i_1)}\cdots(N)^{(i_r)}}{i_1!\cdots i_r!\,(N+1)^{(i_1)}\cdots(N+1)^{(i_r)}}\frac{t^l}{l!}\right)$$

$$\times\left(\sum_{n=0}^\infty c_{M,N,n}^{(r)}\frac{t^n}{n!}\right)=\sum_{n=0}^\infty\sum_{m=0}^n\binom{n}{m}(-1)^{n-m}(n-m)!$$

$$\times\sum_{\substack{i_1+\dots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{(M)^{(i_1)}\cdots(M)^{(i_r)}(N)^{(i_1)}\cdots(N)^{(i_r)}}{i_1!\cdots i_r!\,(N+1)^{(i_1)}\cdots(N+1)^{(i_r)}}c_{M,N,n}^{(r)}\frac{t^n}{n!}\,.$$

Hence, for $n\ge 1$, we have the following.

PROPOSITION 1.

$$\sum_{m=0}^n\sum_{\substack{i_1+\dots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{(-1)^{n-m}(M)^{(i_1)}\cdots(M)^{(i_r)}}{m!\,i_1!\cdots i_r!\,(N+i_1)\cdots(N+i_r)}c_{M,N,m}^{(r)}=0\,.$$

By using Proposition 1 or

$$c_{M,N,n}^{(r)}=-n!\,N^r\sum_{m=0}^{n-1}\sum_{\substack{i_1+\dots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{(-1)^{n-m}(M)^{(i_1)}\cdots(M)^{(i_r)}c_{M,N,m}^{(r)}}{m!\,i_1!\cdots i_r!\,(N+i_1)\cdots(N+i_r)}\tag{9}$$

with $c_{M,N,0}^{(r)}=1$ $(N\ge 1)$, some values of $c_{M,N,n}^{(r)}$ $(0\le n\le 4)$ are explicitly given by the following.

$$c_{M,N,0}^{(r)}=1\,,\quad c_{M,N,1}^{(r)}=\frac{rMN}{N+1}\,,\quad c_{M,N,2}^{(r)}=\frac{r(r+1)M^2N^2}{(N+1)^2}\frac{rM(M+1)N}{N+2}\,,$$

$$c_{M,N,3}^{(r)}=\frac{r(r+1)(r+2)M^3N^3}{(N+1)^3}$$

$$-\frac{3r(r+1)M^2(M+1)N^2}{(N+1)(N+2)}+\frac{rM(M+1)(M+2)N}{N+3}\,,$$

$$c_{M,N,4}^{(r)}=\frac{r(r+1)(r+2)(r+3)M^4N^4}{(N+1)^4}$$
$$-\frac{6r(r+1)(r+2)M^3(M+1)N^3}{(N+1)^2(N+2)}+\frac{4r(r+1)M^2(M+1)(M+2)N^2}{(N+1)(N+3)}$$
$$+\frac{3r(r+1)M^2(M+1)^2N^2}{(N+2)^2}-\frac{rM(M+1)(M+2)(M+3)N}{N+4}.$$

We have an explicit expression for $c_{M,N,n}^{(r)}$.

THEOREM 2. *For $M,N,n\ge 1$, we have*

$$c_{M,N,n}^{(r)}=n!\sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{e_1+\dots+e_k=n\\ e_1,\dots,e_k\ge 1}}D_r(e_1)\cdots D_r(e_k),$$

where

$$D_r(e)=\sum_{\substack{i_1+\dots+i_r=e\\ i_1,\dots,i_r\ge 0}}\frac{(M)^{(i_1)}\cdots(M)^{(i_r)}N^r}{i_1!\cdots i_r!\,(N+i_1)\cdots(N+i_r)}. \tag{10}$$

The first few values of $D_r(e)$ are given by the following.

$$D_r(1)=\frac{rMN}{N+1},\quad D_r(2)=\frac{rM(M+1)N}{2(N+2)}+\frac{r(r-1)M^2N^2}{2(N+1)^2},$$
$$D_r(3)=\frac{rM(M+1)(M+2)N}{6(N+3)}+\frac{r(r-1)M^2(M+1)N^2}{2(N+1)(N+2)}+\binom{r}{3}\frac{M^3N^3}{(N+1)^3},$$
$$D_r(4)=\frac{rM(M+1)(M+2)(M+3)N}{24(N+4)}$$
$$+\frac{r(r-1)M^2(M+1)(M+2)N^2}{6(N+1)(N+3)}+\binom{r}{2}\frac{M^2(M+1)^2N^2}{4(N+2)^2}$$
$$+r\binom{r-1}{2}\frac{M^3(M+1)N^3}{2(N+1)^2(N+2)}+\binom{r}{4}\frac{M^4N^4}{(N+1)^4}.$$

We shall introduce the Hasse–Teichmüller derivative in order to prove Theorem 2 easily. Let $\mathbb{F}$ be a field of any characteristic, $\mathbb{F}[[z]]$ the ring of formal power series in one variable z, and $\mathbb{F}((z))$ the field of Laurent series in z. Let n be a nonnegative integer. We define the Hasse–Teichmüller derivative $H^{(n)}$ of order n by

$$H^{(n)}\biggl(\sum_{m=R}^{\infty} c_m z^m\biggr) = \sum_{m=R}^{\infty} c_m \binom{m}{n} z^{m-n}$$

for $\sum_{m=R}^{\infty} c_m z^m \in \mathbb{F}((z))$, where R is an integer and $c_m \in \mathbb{F}$ for any $m \ge R$. Note that $\binom{m}{n} = 0$ if $m < n$.

The Hasse–Teichmüller derivatives satisfy the product rule [36], the quo tient rule [8] and the chain rule [9]. One of the product rules can be described as follows.

LEMMA 1. *For $f_i \in \mathbb{F}[[z]]$ $(i = 1, \ldots, k)$ with $k \ge 2$ and for $n \ge 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\ge 0}} H^{(i_1)}(f_1)\cdots H^{(i_k)}(f_k)\,.$$

The quotient rules can be described as follows.

LEMMA 2. *For $f \in \mathbb{F}[[z]]\backslash\{0\}$ and $n \ge 1$, we have*

$$H^{(n)}\Bigl(\frac{1}{f}\Bigr) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\ge 1}} H^{(i_1)}(f)\cdots H^{(i_k)}(f) \tag{11}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\ge 0}} H^{(i_1)}(f)\cdots H^{(i_k)}(f)\,. \tag{12}$$

PROOF OF THEOREM 2. Put $h(t) = \bigl(f(t)\bigr)^r$, where

$$f(t) = \sum_{j=0}^{\infty} \frac{(M)^{(j)}(N)^{(j)}}{(N+1)^{(j)}}\,\frac{(-t)^j}{j!}\,.$$

Since

$$H^{(i)}(f)\big|_{t=0} = \sum_{j=i}^{\infty} \frac{(-1)^j (M)^{(j)}(N)^{(j)}}{(N+1)^{(j)}} \binom{j}{i} \frac{t^{j-i}}{j!}\Big|_{t=0} = \frac{(-1)^i (M)^{(i)} N}{i!\,(N+i)}$$

by the product rule of the Hasse–Teichmüller derivative in Lemma 1, we get

$$H^{(e)}(h)\big|_{x=0} = \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\ldots,i_r\ge 0}} H^{(i_1)}(f)\Big|_{x=0} \cdots H^{(i_r)}(f)\Big|_{x=0}$$

$$= \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}} \frac{(-1)^{i_1}(M)^{(i_1)}N}{i_1!\,(N+i_1)} \cdots \frac{(-1)^{i_r}(M)^{(i_r)}N}{i_r!\,(N+i_r)}$$

$$= (-1)^e \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}} \frac{(M)^{(i_1)}\cdots(M)^{(i_r)}N^r}{i_1!\,\cdots i_r!\,(N+i_1)\cdots(N+i_r)} := (-1)^e D_r(e)\,.$$

Hence, by the quotient rule of the Hasse–Teichmüller derivative in Lemma 2(11), we have

$$\frac{c_{M,N,n}^{(r)}}{n!} = \sum_{k=1}^n \frac{(-1)^k}{h^{k+1}}\Big|_{x=0} \sum_{\substack{e_1+\cdots+e_k=n\\ e_1,\dots,e_k\ge 1}} H^{(e_1)}(h)\Big|_{x=0} \cdots H^{(e_k)}(h)\Big|_{x=0}$$

$$= \sum_{k=1}^n (-1)^k \sum_{\substack{e_1+\cdots+e_k=n\\ e_1,\dots,e_k\ge 1}} (-1)^n D_r(e_1)\cdots D_r(e_k)\,. \qquad \square$$

4. Determinant expressions of generalized hypergeometric Cauchy numbers

THEOREM 3. *For $n \ge 1$, we have*

$$c_{M,N,n} = n! \begin{vmatrix} \frac{M\cdot N}{N+1} & 1 & & & \\ \frac{(M)^{(2)}N}{2!\,(N+2)} & \frac{M\cdot N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(M)^{(n-1)}N}{(n-1)!\,(N+n-1)} & \frac{(M)^{(n-2)}N}{(n-2)!\,(N+n-2)} & \cdots & \frac{M\cdot N}{N+1} & 1 \\ \frac{(M)^{(n)}N}{n!\,(N+n)} & \frac{(M)^{(n-1)}N}{(n-1)!\,(N+n-1)} & \cdots & \frac{(M)^{(2)}N}{2!\,(N+2)} & \frac{M\cdot N}{N+1} \end{vmatrix}.$$

REMARK. When $M = 1$, Theorem 3 is reduced to a determinant expression of hypergeometric Cauchy numbers [2]:

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix} \quad (N, n \ge 1)\,.$$

When $N=1$, we have a determinant expression of Cauchy numbers ([7, p. 50]):

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix} . \tag{13}$$

The values of this determinant, that is, $b_n = c_n/n!$ are called *Bernoulli numbers of the second kind*.

Proof of Theorem 3. This is a special case with $r=1$ of the next theorem. □

Now, we can also show a determinant expression of $c_{M,N,n}^{(r)}$.

Theorem 4. *For integers $M, N, n \ge 1$, we have*

$$c_{M,N,n}^{(r)} = n! \begin{vmatrix} D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-2) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-1) & \cdots & D_r(2) & D_r(1) \end{vmatrix} .$$

where $D_r(e)$ are given in (10).

Remark. When $r=1$ in Theorem 4, we have the result in Theorem 3.

Proof of Theorem 4. For simplicity, put $b_{M,N,n}^{(r)} = c_{M,N,n}^{(r)}/n!$. Then, we shall prove that for any $n \ge 1$

$$b_{M,N,n}^{(r)} = \begin{vmatrix} D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-2) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-1) & \cdots & D_r(2) & D_r(1) \end{vmatrix} . \tag{14}$$

When $n=1$, (14) is valid because

$$D_r(1) = \frac{rN^r}{N^{r-1}(N+1)} = \frac{rMN}{N+1} = b_{M,N,1}^{(r)} .$$

Assume that (14) is valid up to $n-1$. Notice that by (9), we have

$$b_{M,N,n}^{(r)}=\sum_{l=1}^{n}(-1)^{l-1}b_{M,N,n-l}^{(r)}D_r(l)\,.$$

Thus, by expanding the first row of the right-hand side (14), it is equal to

$$D_r(1)b_{M,N,n-1}^{(r)}-\begin{vmatrix} D_r(2) & 1 & & & \\ D_r(3) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-3) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-2) & \cdots & D_r(2) & D_r(1) \end{vmatrix}$$

$$=D_r(1)b_{M,N,n-1}^{(r)}-D_r(2)b_{M,N,n-2}^{(r)}+\begin{vmatrix} D_r(3) & 1 & & & \\ D_r(4) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-4) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-3) & \cdots & D_r(2) & D_r(1) \end{vmatrix}$$

$$=D_r(1)b_{M,N,n-1}^{(r)}-D_r(2)b_{M,N,n-2}^{(r)}+\cdots+(-1)^{n-2}\begin{vmatrix} D_r(n-1) & 1 \\ D_r(n) & D_r(1) \end{vmatrix}$$

$$=\sum_{l=1}^{n}(-1)^{l-1}D_r(l)b_{M,N,n-l}^{(r)}=b_{M,N,n}^{(r)}\,.$$

Note that $b_{M,N,1}^{(r)}=D_r(1)$ and $b_{M,N,0}^{(r)}=1$. □

We shall use the Trudi's formula to obtain a different explicit expression for the numbers $c_{M,N,n}^{(r)}$ in Theorem 4.

LEMMA 3 (Trudi's formula [27,31]). *For a positive integer m, we have*

$$\begin{vmatrix} a_1 & a_2 & \cdots & & a_m \\ a_0 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & a_0 & a_1 \end{vmatrix}$$

$$=\sum_{t_1+2t_2+\cdots+mt_m=m}\binom{t_1+\cdots+t_m}{t_1,\dots,t_m}(-a_0)^{m-t_1-\cdots-t_m}a_1^{t_1}a_2^{t_2}\cdots a_m^{t_m},$$

where $\binom{t_1+\cdots+t_m}{t_1,\dots,t_m}=\frac{(t_1+\cdots+t_m)!}{t_1!\cdots t_m!}$ *are the multinomial coefficients.*

In [32], Merca gave the following interesting relation (see also [27]).

LEMMA 4. *If* $\{\alpha_n\}_{n\ge 0}$ *is a sequence defined by* $\alpha_0=1$ *and*

$$\alpha_n=\begin{vmatrix} R(1) & 1 & & \\ R(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ R(n) & \cdots & R(2) & R(1) \end{vmatrix},$$

then

$$R(n)=\begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

Moreover, if

$$A=\begin{pmatrix} 1 & & & \\ \alpha_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_n & \cdots & \alpha_1 & 1 \end{pmatrix},$$

then

$$A^{-1}=\begin{pmatrix} 1 & & & \\ R(1) & 1 & & \\ \vdots & \ddots & \ddots & \\ R(n) & \cdots & R(1) & 1 \end{pmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}R(1)^{t_1}R(2)^{t_2}\cdots R(n)^{t_n}.$$

By applying these lemmata to Theorem 4 we obtain an explicit expression for the generalized hypergeometric Cauchy numbers.

THEOREM 5. *For* $n\ge 1$

$$c_{M,N,n}^{(r)}=n!\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}$$
$$\times(-1)^{n-t_1-\cdots-t_n}D_r(1)^{t_1}D_r(2)^{t_2}\cdots D_r(n)^{t_n}.$$

Moreover,

$$D_r(n)=\begin{vmatrix}\frac{c_{M,N,1}^{(r)}}{1!} & 1 & & \\ \frac{c_{M,N,2}^{(r)}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c_{M,N,n}^{(r)}}{n!} & \cdots & \frac{c_{M,N,2}^{(r)}}{2!} & \frac{c_{M,N,1}^{(r)}}{1!}\end{vmatrix},$$

and

$$\begin{pmatrix}1 & & & & \\ \frac{c_{M,N,1}^{(r)}}{1!} & 1 & & & \\ \frac{c_{M,N,2}^{(r)}}{2!} & \frac{c_{M,N,1}^{(r)}}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{c_{M,N,n}^{(r)}}{n!} & \cdots & \frac{c_{M,N,2}^{(r)}}{2!} & \frac{c_{M,N,1}^{(r)}}{1!} & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & & & & \\ D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & 1 & & \\ \vdots & & \ddots & & \\ D_r(n) & \cdots & D_r(2) & D_r(1) & 1\end{pmatrix}.$$

If $M=r=1$ in Theorem 5, we have a different expression for hypergeometric Cauchy numbers $c_{N,n}=c_{1,N,n}$.

COROLLARY 1. *For* $n\ge 1$

$$c_{N,n}=n!\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}$$
$$\times\left(\frac{N}{N+1}\right)^{t_1}\left(\frac{N}{N+2}\right)^{t_2}\cdots\left(\frac{N}{N+n}\right)^{t_n}.$$

Moreover,

$$\frac{N}{N+n}=\begin{vmatrix}\frac{c_{N,1}}{1!} & 1 & & \\ \frac{c_{N,2}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c_{N,n}}{n!} & \cdots & \frac{c_{N,2}}{2!} & \frac{c_{N,1}}{1!}\end{vmatrix}.$$

If $M=N=r=1$ in Theorem 5, we have a different expression for the original Cauchy numbers $c_n=c_{1,1,n}$.

COROLLARY 2. *For* $n\ge 1$

$$c_n=n!\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}\left(\frac{1}{2}\right)^{t_1}\left(\frac{1}{3}\right)^{t_2}\cdots\left(\frac{1}{n+1}\right)^{t_n}.$$

Moreover,

$$\frac{1}{n+1}=\begin{vmatrix} \frac{c_1}{1!} & 1 & & \\ \frac{c_2}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c_n}{n!} & \cdots & \frac{c_2}{2!} & \frac{c_1}{1!} \end{vmatrix}.$$

5. Sums of products of generalized hypergeometric Cauchy polynomials

As an application, we shall give a convolution identity for the generalized hypergeometric Cauchy polynomials. Similar sums of products have been studied for Bernoulli numbers, hypergeometric Bernoulli numbers, poly-Bernoulli numbers, and Cauchy numbers (see, e.g. [6,15,20,35,37]).

The general *hypergeometric Cauchy polynomials* $c_{M,N,n}(z)$ $(M, N \ge 1;\ n \ge 0)$ are defined in (2). For convenience, put

$$f(t,z) := \frac{1}{(1+t)^z}\,\frac{1}{F}$$

with $F := {}_2F_1(M,N;N+1;-t)$. Since

$$\frac{d}{dt}F = -\sum_{n=0}^{\infty}\frac{(M)^{(n+1)}(N)^{(n+1)}}{(N+1)^{(n+1)}}\,\frac{(-t)^n}{n!}\,,$$

we have

$$\frac{d}{dt}\,\frac{1}{F} = \frac{1}{F^2}\sum_{n=0}^{\infty}\frac{(M)^{(n+1)}(N)^{(n+1)}}{(N+1)^{(n+1)}}\,\frac{(-t)^n}{n!}\,.$$

Thus,

$$\begin{aligned} &N\cdot F - tF^2\frac{d}{dt}\frac{1}{F} \\ &= N\sum_{n=0}^{\infty}\frac{(M)^{(n)}(N)^{(n)}}{(N+1)^{(n)}}\,\frac{(-t)^n}{n!} - t\sum_{n=0}^{\infty}\frac{(M)^{(n+1)}(N)^{(n+1)}}{(N+1)^{(n+1)}}\,\frac{(-t)^n}{n!} \\ &= \sum_{n=0}^{\infty}(N+n)\frac{(M)^{(n)}(N)^{(n)}}{(N+1)^{(n)}}\,\frac{(-t)^n}{n!} = N\sum_{n=0}^{\infty}(M)^{(n)}\,\frac{(-t)^n}{n!} = \frac{N}{(1+t)^M}\,. \end{aligned}$$

Hence, we get

$$\frac{1}{F^2} = \frac{(1+t)^M}{F} - \frac{(1+t)^M}{N}\, t\frac{d}{dt}\,\frac{1}{F}\,. \tag{15}$$

Since

$$\frac{d}{dt}f(t,z) = -\frac{z}{1+t}f(t,z) + \frac{1}{(1+t)^z}\frac{d}{dt}\frac{1}{F}\,,$$

by dividing both sides of (15) by $(1+t)^z$, we have

$$\frac{1}{(1+t)^z}\frac{1}{F^2} = \frac{(1+t)^M}{(1+t)^z}\frac{1}{F} - \frac{(1+t)^M}{N}\frac{t}{(1+t)^z}\frac{d}{dt}\frac{1}{F} \tag{16}$$

$$= (1+t)^M f(t,z) - \frac{(1+t)^M}{N}t\frac{d}{dt}f(t,z) - \frac{(1+t)^M}{N}t\frac{z}{1+t}f(t,z)\,.$$

The first term on the right-hand side of (16) is equal to

$$(1+t)^M f(t,z) = \sum_{j=0}^{M}\binom{M}{j}t^j\sum_{n=0}^{\infty}c_{M,N,n}(z)\,\frac{t^n}{n!}$$

$$= \sum_{j=0}^{M}\binom{M}{j}\sum_{n=0}^{\infty}\frac{(n+j)!}{n!}c_{M,N,n}(z)\,\frac{t^{n+j}}{(n+j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{M}\binom{M}{j}\frac{n!}{(n-j)!}c_{M,N,n-j}(z)\frac{t^n}{n!}\,.$$

The second term on the right-hand side of (16) is equal to

$$-\frac{(1+t)^M}{N}t\frac{d}{dt}f(t,z) = -\frac{1}{N}\sum_{j=0}^{M}\binom{M}{j}t^{j+1}\sum_{n=0}^{\infty}c_{M,N,n+1}(z)\frac{t^n}{n!}$$

$$= -\frac{1}{N}\sum_{j=0}^{M}\binom{M}{j}\sum_{n=0}^{\infty}\frac{(n+j+1)!}{n!}c_{M,N,n+1}(z)\frac{t^{n+j+1}}{(n+j+1)!}$$

$$= -\frac{1}{N}\sum_{n=0}^{\infty}\sum_{j=0}^{M}\binom{M}{j}\frac{n!}{(n-j-1)!}c_{M,N,n-j}(z)\frac{t^n}{n!}\,.$$

The third term on the right-hand side of (16) is equal to

$$-\frac{z(1+t)^{M-1}t}{N}f(t,z)=-\frac{z}{N}\sum_{j=0}^{M-1}\binom{M-1}{j}t^{j+1}\sum_{n=0}^{\infty}c_{M,N,n}(z)\,\frac{t^n}{n!}$$

$$=-\frac{z}{N}\sum_{j=0}^{M-1}\binom{M-1}{j}\sum_{n=0}^{\infty}\frac{(n+j+1)!}{n!}c_{M,N,n}(z)\,\frac{t^{n+j+1}}{(n+j+1)!}$$

$$=-\frac{z}{N}\sum_{n=0}^{\infty}\sum_{j=0}^{M-1}\binom{M-1}{j}\frac{n!}{(n-j-1)!}c_{M,N,n-j-1}(z)\frac{t^n}{n!}\,.$$

On the other hand, taking $z=x+y$,

$$\frac{1}{(1+t)^z}\frac{1}{F^2}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}c_{M,N,k}(x)c_{M,N,n-k}(y)\frac{t^n}{n!}\,.$$

By comparing the coefficients on both sides, we have

$$\sum_{k=0}^{n}\binom{n}{k}c_{M,N,k}(x)c_{M,N,n-k}(y)=\sum_{j=0}^{M}\binom{M}{j}\frac{n!}{(n-j)!}c_{M,N,n-j}(x+y)$$

$$-\frac{1}{N}\sum_{j=0}^{M}\binom{M}{j}\frac{n!}{(n-j-1)!}c_{M,N,n-j}(x+y)$$

$$-\frac{z}{N}\sum_{j=0}^{M-1}\binom{M-1}{j}\frac{n!}{(n-j-1)!}c_{M,N,n-j-1}(x+y)$$

$$=\frac{n!}{N}\sum_{j=0}^{M}\left(\binom{M}{j}\frac{N-n+j}{(n-j)!}-z\binom{M-1}{j-1}\frac{1}{(n-j)!}\right)c_{M,N,n-j}(x+y)\,.$$

Therefore, we obtain a convolution identity for the sums of products of two hypergeometric Cauchy polynomials. For convenience, take $\binom{n}{-1}=0$ for $n\ge 0$.

THEOREM 6. *For $M,N\ge 1$ and $n\ge M$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}c_{M,N,k}(x)c_{M,N,n-k}(y)$$

$$= \frac{n!}{N}\sum_{j=0}^M\left(\binom{M}{j}\frac{N-n+j}{(n-j)!}-(x+y)\binom{M-1}{j-1}\frac{1}{(n-j)!}\right)c_{M,N,n-j}(x+y)\,.$$

Remark. When $M=1$ in Theorem 6, we have a convolution identity for the sums of products of two original hypergeometric Cauchy polynomials $c_{N,n}(x)=c_{1,N,n}(x)$.

$$\sum_{k=0}^n\binom{n}{k}c_{N,k}(x)c_{N,n-k}(y)$$

$$=\frac{N-n}{N}c_{N,n}(x+y)+\frac{n}{N}\big(-(x+y)+N-n+1\big)\,c_{N,n-1}(x+y)\,.$$

When $M=1$ and $z=0$ in Theorem 6, we have a convolution identity for the sums of products of two original hypergeometric Cauchy numbers $c_{N,n}$ ([20, Theorem 2]).

$$\sum_{k=0}^n\binom{n}{k}c_{N,k}c_{N,n-k}=\frac{N-n}{N}c_{N,n}+\frac{n}{N}(N-n+1)c_{N,n-1}\,.$$

When $M=N=1$ in Theorem 6, we have a convolution identity for the sums of products of two classical Cauchy polynomials $c_n(x)$ (see e.g. [4]).

$$\sum_{k=0}^n\binom{n}{k}c_k(x)c_{n-k}(y)=-(n-1)c_n(x+y)-n(x+y+n-2)c_{n-1}(x+y)\,.$$

When $M=N=1$ and $z=0$ in Theorem 6, we have a convolution identity for the sums of products of two classical Cauchy numbers c_n. This is also a special case of the result by Zhao for $m=2$ (see [37]).

$$\sum_{k=0}^n\binom{n}{k}c_kc_{n-k}=-(n-1)c_n-n(n-2)c_{n-1}\,.$$

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