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f-STABILITY OF SPACELIKE HYPERSURFACES IN WEIGHTED SPACETIMES

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Abstract. We establish the notions of f-stability and strong f-stability concerning closed spacelike hypersurfaces immersed with constant f-mean curvature in a conformally stationary spacetime endowed with a conformal timelike vector field V and a weight function f. When V is closed, with the aid of the f-Laplacian of a suitable support function, we characterize f-stable closed spacelike hypersurfaces through the analysis of the first eigenvalue of the Jacobi operator associated to the corresponding variational problem. Furthermore, we obtain sufficient conditions which assure that a strongly f-stable closed spacelike hypersurface must be either f-maximal or isometric to a leaf orthogonal to V.

1. Introduction

The notion of stability concerning hypersurfaces of constant mean curvature of Riemannian ambient spaces was first studied by Barbosa and do Carmo in [4], and by Barbosa, do Carmo and Eschenburg in [5], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. Afterwards, working in the Lorentzian context, Barbosa and Oliker [6] obtained an analogous result proving that

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constant mean curvature spacelike hypersurfaces in Lorentzian manifolds are also critical points of the area functional for variations that keep the volume constant. Later on, Barros, Brasil and Caminha [7] studied the problem of strong stability (that is, stability with respect to not necessarily volume-preserving variations) for spacelike hypersurfaces with constant mean curvature in a generalized Robertson–Walker (GRW) spacetime of constant sectional curvature, giving a characterization for the maximal spacelike hypersurfaces and spacelike slices of such an ambient space.

On the other hand, the study of variational questions associated to the area functional in (Riemannian or Lorentzian) manifolds with density, also called weighted manifolds, has been a focus of attention in the last years. We recall that a weighted manifold $\overline{M}_f^{n+1}$ is a (Riemannian or Lorentzian) manifold $\left(\overline{M}^{n+1}, \langle\, ,\rangle\right)$ endowed with a weighted volume form $d\mu = e^{-f}\, d\overline{M}$, where the weight f is a real-valued smooth function on $\overline{M}_f^{n+1}$ and $d\overline{M}$ is the volume element induced by the metric $\langle\, ,\rangle$. For a comprehensive introduction to weighted manifolds we refer the reader, for instance, to Ch. 3 of Bayle's thesis [9] or to Ch. 18 of Morgan's book [18].

In this direction, Rosales, Cañete, Bayle and Morgan [20] investigated the isoperimetric problem for Euclidean space endowed with a continuous density, showing that, for a radial log-convex density, balls about the origin are isoperimetric regions. Afterwards, Cañete and Rosales [10] studied smooth Euclidean solid cones endowed with a smooth homogeneous weight function. In this context, they proved that the unique compact, orientable, second order minima of the weighted area under variations preserving the weighted volume and with free boundary in the boundary of the cone are intersections with the cone of round spheres centered at the vertex. In [15], Impera and Rimoldi established stability properties concerning f-minimal hypersurfaces (that is, with identically zero f-mean curvature) isometrically immersed in a weighted manifold with non-negative Bakry–Émery Ricci curvature under volume growth conditions. Meanwhile, Castro and Rosales [12] obtained variational characterizations of critical points and second order minima of the weighted area with or without a volume constraint in weighted Riemannian manifolds with boundary.

Also in the branch of manifolds with density, Batista, Cavalcante and Pyo [8] showed some general inequalities involving the weighted mean curvature of compact submanifolds immersed in weighted Riemannian manifolds. As application, they obtained an isoperimetric inequality for such submanifolds. Moreover, they also proved an extrinsic upper bound to the first nonzero eigenvalue of the f-Laplacian on closed submanifolds of weighted Riemannian manifolds. Concerning the weighted product space $\mathbb{G}^n \times \mathbb{R}$, where $\mathbb{G}^n$ stands for the so-called Gaussian space which is nothing but the

Euclidian space $\mathbb{R}^n$ endowed with the Gaussian probability density

$$e^{-f(x)} = (2\pi)^{-\frac{n+1}{2}} e^{-\frac{|x|^2}{2}}, \quad x \in \mathbb{R}^n,$$

Hieu and Nam [14] extended the classical Bernstein's theorem showing that the only weighted minimal graphs $\Sigma(u)$ of smooth functions $u(x) = t$ over $\mathbb{G}^n$ are the affine hyperplanes $t = \text{const}$. Afterwards, McGonagle and Ross [16] showed that the hyperplane is the only stable, smooth solution to the isoperimetric problem in the $\mathbb{G}^{n+1}$. Meanwhile, in the works [1,13] it was applied suitable generalized maximum principles in order to obtain new Calabi–Bernstein type results concerning complete hypersurfaces immersed in certain weighted generalized Robertson–Walker spacetimes.

Motivated by the works described above, in this article we deal with closed spacelike hypersurfaces immersed with constant f-mean curvature in a *weighted conformally stationary spacetime* endowed with a conformal vector field V and a weight function f (for more details see Sections 2 and 4). For these hypersurfaces, first we extend the ideas of [6,7] introducing the notions of f-stability and strong f-stability (see Definitions 1 and 2, respectively). When V is closed, with the aid of the f-Laplacian of an appropriate support function (cf. Proposition 3) we establish a suitable relation between f-stability and the spectrum of the f-Laplacian of a closed spacelike hypersurface having constant f-mean curvature (cf. Theorem 1). Furthermore, we also improve the main result of [7] obtaining sufficient conditions to guarantee that a strong f-stable hypersurface be either f-maximal (that is, with identically zero f-mean curvature) or isometric to a leaf orthogonal to V (cf. Theorem 2 and Corollary 2).

2. Preliminaries

This section is devoted to recall some basic facts concerning spacelike hypersurfaces immersed in a weighted Lorentzian space.

Let $\left(\overline{M}^{n+1}, \langle\,,\rangle, \overline{\nabla}, d\overline{\mu}\right)$ be a *weighted time-oriented Lorentzian manifold*, that is, a time-oriented Lorentzian manifold $\overline{M}^{n+1}$ with metric tensor $\langle\,,\rangle$, Levi-Civita conection $\overline{\nabla}$ and endowed with a weighted volume form $d\overline{\mu} = e^{-f}\, d\overline{M}$, where f is a real-valued smooth function on $\overline{M}^{n+1}$, called *weight function*, and $d\overline{M}$ is the volume element induced by the metric $\langle\,,\rangle$. In order to shorten our notation, we denote $\left(\overline{M}^{n+1}, \langle\,,\rangle, \overline{\nabla}, d\overline{\mu}\right)$ simply by $\overline{M}_f^{n+1}$. We mean by $C^\infty(\overline{M})$ the ring of real functions of class C^∞ on $\overline{M}^{n+1}$ and by $\mathfrak{X}(\overline{M})$ the $C^\infty(\overline{M})$-module of vector fields of class C^∞ on $\overline{M}^{n+1}$. For $\overline{M}_f^{n+1}$, the Bakry–Émery–Ricci tensor $\overline{\text{Ric}}_f$ is defined by

$$\overline{\text{Ric}}_f = \overline{\text{Ric}} + \overline{\text{Hess}} f, \tag{2.1}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ are the Ricci tensor and the Hessian operator in $\overline{M}^{n+1}_f$, respectively.

In this context, we consider spacelike hypersurfaces $x\colon \Sigma^n \hookrightarrow \overline{M}^{n+1}_f$, namely, isometric immersions from a connected, n-dimensional orientable Riemannian manifold Σ^n into $\overline{M}^{n+1}_f$. We let ∇ denote the Levi-Civita connection of Σ^n.

Since we are supposing that $\overline{M}^{n+1}_f$ is time-orientable and $x\colon \Sigma^n \hookrightarrow \overline{M}^{n+1}_f$ is a spacelike hypersurface, then Σ^n is orientable (cf. [19]) and one can choose a globally defined unit normal vector field N on Σ^n having the same time-orientation of $\overline{M}^{n+1}_f$, which is called the *future-pointing Gauss map* of Σ^n. In this setting, let A denote the shape operator of Σ^n with respect to N, so that at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map

$$A_p\colon T_p\Sigma \longrightarrow T_p\Sigma, \quad v \mapsto A_p v = -\overline{\nabla}_v N.$$

The *f-mean curvature* of Σ^n is the function H_f given by

$$nH_f = nH - \langle \overline{\nabla} f, N\rangle, \tag{2.2}$$

where $H = -\frac{1}{n}\,\mathrm{tr}(A)$ denotes the standard mean curvature of Σ^n with respect to its future-pointing Gauss map N. The *f-divergence* on Σ^n is defined by

$$\mathrm{div}_f\colon \mathfrak{X}(\Sigma) \longrightarrow C^\infty(\Sigma), \quad X \mapsto \mathrm{div}_f X = \mathrm{div}\, X - \langle \nabla f, X\rangle, \tag{2.3}$$

where $\mathrm{div}(\cdot)$ denotes the standard divergence of Σ^n. A direct calculation assures us that

$$\mathrm{div}_f(\varphi X) = \varphi\, \mathrm{div}_f X + \langle \nabla\varphi, X\rangle$$

for all $X \in \mathfrak{X}(\Sigma)$ and any $\varphi \in C^\infty(\Sigma)$. We define the *f-Laplacian* of Σ^n by

$$\Delta_f\colon C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma), \quad u \mapsto \Delta_f u = \mathrm{div}_f \nabla u = \Delta u - \langle \nabla f, \nabla u\rangle, \tag{2.4}$$

where Δ is the standard Laplacian of Σ^n.

3. Description of the variational problem

Now, let us consider immersions $x\colon \Sigma^n \hookrightarrow \overline{M}^{n+1}_f$ of compact spacelike hypersurfaces Σ^n with boundary $\partial\Sigma$ (possibly empty). A *variation* of $x\colon \Sigma^n \hookrightarrow \overline{M}^{n+1}_f$ is a smooth mapping

$$X\colon \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}_f$$

satisfying the following two conditions:

(1) For $t \in (-\varepsilon, \varepsilon)$, the map $X_t \colon \Sigma^n \hookrightarrow \overline{M}^{n+1}_f$ given by $X_t(p) = X(t,p)$ is a spacelike immersion such that $X_0 = x$.

(2) $X_t\big|_{\partial\Sigma} = x\big|_{\partial\Sigma}$, for all $t \in (-\varepsilon, \varepsilon)$.

In all that follows, we let dM_t denote the volume element of the metric induced on Σ^n by X_t and N_t the unit normal vector field along X_t. Moreover, we also consider in Σ^n the weighted volume form given by $d\mu_t = e^{-f} dM_t$. When $t = 0$ all these objects coincide with the ones defined in Σ^n, respectively.

The *variational field* associated to the variation X is the vector field $\frac{\partial X}{\partial t}\big|_{t=0}$. Letting

$$u_t = -\Big\langle \frac{\partial X}{\partial t}, N_t \Big\rangle, \tag{3.1}$$

we get

$$\frac{\partial X}{\partial t}\Big|_{t=0} = u_0 N + \Big(\frac{\partial X}{\partial t}\Big|_{t=0}\Big)^{\top},$$

where $(\cdot)^{\top}$ stands for tangential components.

The *weighted volume* of the variation X is the functional

$$\mathcal{V}_f \colon (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad t \mapsto \mathcal{V}_f(t) = \int_{M \times [0,t]} X^*(d\overline{\mu})$$

and we say that X is *volume-preserving* if $\mathcal{V}_f(t) = \mathcal{V}_f(0)$, for all $t \in (-\varepsilon, \varepsilon)$.

The following result is well known and, in the context of weighted manifolds, it can be found in [12] or [20].

Lemma 1. *Let $\overline{M}^{n+1}_f$ be a weighted time-oriented Lorentzian manifold and let $x \colon \Sigma^n \hookrightarrow \overline{M}^{n+1}_f$ be a closed spacelike hypersurface. If $X \colon \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}_f$ is a variation of x, then*

$$\frac{d\mathcal{V}_f}{dt} = \int_M u_t \, d\mu_t,$$

where u_t is given in (3.1)*. In particular, X is volume-preserving if and only if $\int_M u_t \, d\mu_t = 0$ for all $t \in (-\varepsilon, \varepsilon)$.*

Remark 1. We observe that is not difficult to verify that [5, Lemma 2.2] still remains valid for the context of weighted time-oriented Lorentzian manifolds, that is, if $u \in C^\infty(\Sigma)$ is such that $u\big|_{\partial\Sigma} = 0$ and $\int_\Sigma u d\mu = 0$, then there exists a volume-preserving variation of Σ^n whose variational field is uN.

The *weighted area functional* associated to the variation X is given by

$$\mathcal{A}_f\colon(-\varepsilon,\varepsilon)\to\mathbb{R},\quad t\mapsto\mathcal{A}_f(t)=\int_\Sigma d\mu_t.$$

Following the same steps of the proof of Lemma 3.2 of [12], it is not difficult to see that we get the following

LEMMA 2. *Let $\overline{M}^{n+1}_f$ be a weighted time-oriented Lorentzian manifold and $x\colon\Sigma^n\hookrightarrow\overline{M}^{n+1}_f$ a closed spacelike hypersurface. If $X\colon\Sigma^n\times(-\varepsilon,\varepsilon)\to\overline{M}^{n+1}_f$ is a variation of x, then*

$$\mathcal{A}'_f(t)=n\int_\Sigma (H_f)_t\,u_t\,d\mu_t, \tag{3.2}$$

where u_t is given in (3.1) *and $(H_f)_t=H_f(t,\cdot)$ denotes the f-mean curvature of Σ^n with respect to the metric induced by X_t.*

In order to characterize hypersurfaces with constant f-mean curvature, we consider the variational problem of maximizing the functional $\mathcal{A}_f$ for all variations $X\colon\Sigma^n\times(-\varepsilon,\varepsilon)\to\overline{M}^{n+1}_f$ of $x\colon\Sigma^n\hookrightarrow\overline{M}^{n+1}_f$ that preserve the weighted volume $\mathcal{V}_f$. The *Jacobi functional* associated to this problem is given by

$$\mathcal{J}_f\colon(-\varepsilon,\varepsilon)\to\mathbb{R},\quad t\mapsto\mathcal{J}_f(t)=\mathcal{A}_f(t)-\varrho\mathcal{V}_f(t), \tag{3.3}$$

where ϱ is a constant to be determined. As an immediate consequence of Lemmas 2 and 1 we get

$$\mathcal{J}'_f(t)=\int_M\{n(H_f)_t-\varrho\}u_t\,d\mu_t. \tag{3.4}$$

In order to make an appropriated choice of ϱ, let

$$\overline{\mathcal{H}}=\frac{1}{\mathcal{A}_f(0)}\int_M (H_f)_0\,d\mu$$

be an integral mean of the f-mean curvature of Σ^n. We call the attention to the fact that, in case $(H_f)_0$ is constant, we have

$$\overline{\mathcal{H}}=(H_f)_0=H_f, \tag{3.5}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\varrho=n\overline{\mathcal{H}}$, from (3.4) we arrive at

$$\mathcal{J}'_f(t)=n\int_M\big\{(H_f)_t-\overline{\mathcal{H}}\big\}\,u_t\,d\mu_t. \tag{3.6}$$

Reasoning as in the proof of [4, Proposition 2.7], from (3.6) we get

PROPOSITION 1. *Let $\overline{M}_f^{n+1}$ be a weighted time-oriented Lorentzian manifold and let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a closed spacelike hypersurface. The following statements are equivalent*:

(a) $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ *has constant f-mean curvature H_f*;

(b) *for all variations $X\colon \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}_f^{n+1}$ of x that preserve the volume, we have that $\mathcal{J}_f'(0) = 0$*;

(c) *for all variations $X\colon \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}_f^{n+1}$ of x, we have that $\mathcal{J}_f'(0) = 0$.*

In particular, Proposition 1 guarantees that a spacelike hypersurface $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is a critical point of the variational problem described above if and only if its f-mean curvature H_f is constant. Motivated by this fact, we establish the following

DEFINITION 1. Let $\overline{M}_f^{n+1}$ be a weighted time-oriented Lorentzian manifold and let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a closed spacelike hypersurface having constant f-mean curvature H_f. We say that $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is *f-stable* if $\mathcal{A}_f''(0) \leq 0$ for all volume-preserving variation $X\colon \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}_f^{n+1}$ of x.

REMARK 2. Let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a closed spacelike hypersurface as described in Definition 1. We consider the set

$$\mathcal{G} = \left\{ u \in C^\infty(\Sigma) : \int_\Sigma u \, d\mu = 0 \right\}. \tag{3.7}$$

Just as in [4], we can establish the following criterion of f-stability: a spacelike hypersurface $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is f-stable if and only if $\mathcal{J}_f''(0)(u) \leq 0$, for all $u \in \mathcal{G}$.

On the other hand, if we change our object of study, considering closed spacelike hyperpurfaces $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ which maximize the functional Jacobi $\mathcal{J}_f$ for any variation $X\colon \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}_f^{n+1}$ of x, we obtain again from Proposition 1 that $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is a critical point of $\mathcal{J}_f$ if and only if its f-mean curvature H_f is constant. This, in turn, motivates the following

DEFINITION 2. Let $\overline{M}_f^{n+1}$ be a weighted time-oriented Lorentzian manifold and let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a closed spacelike hypersurface whose f-mean curvature H_f is constant. We say that $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is strongly f-stable if $\mathcal{J}_f''(0)(u) \leq 0$, for any $u \in C^\infty(\Sigma)$.

The sought formula for the second variation of Jacobi functional $\mathcal{J}_f$ is given in the following

PROPOSITION 2. *Let $\overline{M}_f^{n+1}$ be a weighted time-oriented Lorentzian manifold and let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a closed spacelike hypersurface whose f-mean curvature H_f is constant. If $X\colon \Sigma^n \times (-\varepsilon,\varepsilon) \to \overline{M}_f^{n+1}$ is a variation of x, then $\mathcal{J}_f''(0)$ is given by*

$$\mathcal{J}_f''(0)(u) = \int_\Sigma \big\{ \Delta_f(u) - \big\{ \overline{\mathrm{Ric}}_f(N,N) + |A|^2 \big\} u \big\} u \, d\mu, \tag{3.8}$$

for all $u \in C^\infty(\Sigma)$.

PROOF. Since H_f is constant, from (3.6) and (3.5) we have that

$$\mathcal{J}_f''(0) = \int_\Sigma n\Big(\frac{\partial H_f}{\partial t}\Big|_{t=0}\Big) u_0 \, d\mu + \int_\Sigma n\Big(\underbrace{H_f - \overline{\mathcal{H}}}_{0}\Big)\frac{\partial}{\partial t}(u_t\, d\mu_t)\Big|_{t=0}. \tag{3.9}$$

On the other hand, reasoning as in the proof of equation (3.5) of [12], we obtain

$$n\frac{\partial H_f}{\partial t}\Big|_{t=0} = \Delta_f(u_0) - \big\{ \overline{\mathrm{Ric}}_f(N,N) + |A^2| \big\} u_0.$$

Hence,

$$\mathcal{J}_f''(0) = \int_\Sigma \big\{ \Delta_f(u_0) - \big\{ \overline{\mathrm{Ric}}_f(N,N) + |A|^2 \big\} u_0 \big\} u_0 \, d\mu.$$

To finish the proof, we observe that the above expression depends only on the hypersurface $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ and on the function $u_0 \in C^\infty(\Sigma)$. □

4. The *f*-Laplacian of a support function

Proceeding with the context of the previous section, let $\overline{M}_f^{n+1}$ be a weighted Lorentzian manifold. A vector field V on $\overline{M}_f^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V\langle\,,\rangle = 2\psi\langle\,,\rangle$$

for some function $\psi \in C^\infty(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Lorentzian metric of $\overline{M}_f^{n+1}$. The function ψ is called the *conformal factor* of V. So, extending the terminology established in [2], a weighted Lorentzian manifold $\overline{M}_f^{n+1}$ endowed with a globally defined timelike conformal vector field will be called a *weighted conformally stationary spacetime.*

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathcal{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathcal{X}(\overline{M})$ is conformal if and only if

$$\langle \overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\psi_V \langle X, Y\rangle, \tag{4.1}$$

for all $X, Y \in \mathcal{X}(\overline{M})$. In particular, V is a Killing vector field if and only if $\psi_V \equiv 0$.

Let us suppose, in addition, that the conformal timelike vector field V is closed, that is,

$$\overline{\nabla}_X V = \psi_V X \tag{4.2}$$

for all $X \in \mathfrak{X}(\overline{M})$. Also assuming that V has no singularities on an open set $\mathcal{U} \subset \overline{M}_f^{n+1}$, the distribution $V^\perp$ on $\mathcal{U}$ of vector fields orthogonal to V is integrable, for if $X, Y \in V^\perp$, then

$$\langle [X, Y], V\rangle = \langle \overline{\nabla}_X Y - \overline{\nabla}_Y X, V\rangle = -\langle Y, \overline{\nabla}_X V\rangle + \langle X, \overline{\nabla}_Y V\rangle = 0.$$

We let Ξ^n be a leaf of $V^\perp$ furnished with the induced metric. From (4.2) we get

$$\overline{\nabla}\langle V, V\rangle = 2\psi_V V, \tag{4.3}$$

so that $\langle V, V\rangle$ is constant on connected leaves of $V^\perp$. Computing covariant derivatives in (4.3), we have

$$\overline{\mathrm{Hess}}\langle V, V\rangle(X, Y) = 2X(\psi_V)\langle V, Y\rangle + 2\psi_V^2\langle X, Y\rangle.$$

Since both $\overline{\mathrm{Hess}}$ and the metric $\langle\,,\rangle$ are symmetric tensors, we get

$$X(\psi_V)\langle V, Y\rangle = Y(\psi_V)\langle V, X\rangle$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. Taking $Y = V$ we then arrive at

$$\overline{\nabla}\psi_V = \frac{V(\psi_V)}{\langle V, V\rangle} V = -\nu(\psi_V)\nu, \tag{4.4}$$

where $\nu_V = \frac{V}{\sqrt{-\langle V,V\rangle}}$. Hence, ψ_V is also constant on connected leaves of $V^\perp$. If Ξ^n is such a leaf and A_Ξ denotes its shape operator with respect to ν_V, we get

$$A_\Xi(X) = -\overline{\nabla}_X \nu_V = \psi_V X$$

for any $X \in \mathfrak{X}(\Xi)$ and, hence, Ξ^n is a totally umbilical spacelike hypersurface with constant mean curvature $\mathcal{H}$ given by

$$\mathcal{H} = \frac{\psi_V}{\sqrt{-\langle V, V\rangle}}\,. \tag{4.5}$$

Under the additional hypothesis that the weight function f of $\overline{M}^{n+1}_f$ does not depend on the parameter of the flow associated to the unit timelike vector field ν_V (which means that $\langle\overline{\nabla}f,\nu_V\rangle=0$ on $\overline{M}^{n+1}_f$), from (2.2) and (4.5) we obtain that the f-mean curvature of a leaf Ξ of $V^\top$ with respect to ν is given by

$$\mathcal{H}_f=\frac{\psi_V}{\sqrt{-\langle V,V\rangle}}\,. \tag{4.6}$$

According to the terminology established in [3], a particular class of conformally stationary spacetimes is constituted by the so-called generalized Robertson–Walker (GRW) spacetimes, namely, Lorentzian warped product spaces of the type $-I\times_\phi F^n$, where $I\subset\mathbb{R}$ is an open interval with the metric $-dt^2$, F^n is an n-dimensional Riemannian manifold and $\phi\colon I\to\mathbb{R}$ is positive and smooth. A GRW spacetime $-I\times_\phi F^n$ endowed with a weight function f is called a *weighted GRW spacetime* and it will be denoted by $(-I\times_\phi F^n)_f$. For such a space, if π_I is the canonical projection onto I, then the vector field

$$V=(\phi\circ\pi_I)\partial_s \tag{4.7}$$

is a conformal, timelike and closed, with conformal factor $\psi_V=\phi'\circ\pi_I$, where the prime denotes differentiation with respect to the parameter s. Moreover, it follows from Proposition 1 of [17] that each spacelike leaf $F^n_s=\{s\}\times F^n$ is totally umbilical and, supposing that f does not depend on the parameter s, the f-mean curvature of F^n_s with respect to ∂_s is equal to $\frac{\phi'(s)}{\phi(s)}$.

Conversely, let $\overline{M}^{n+1}_f$ be a weighted conformally stationary Lorentzian manifold endowed with a closed conformal vector field V. If $p\in\overline{M}^{n+1}_f$ and Ξ^n_p is the leaf of $V^\perp$ passing through p, then we can find a neighborhood $\mathcal{U}_p$ of p in Ξ^n_p and an open interval $I\subset\mathbb{R}$ containing 0 such that the flow $\mathcal{F}$ of V is defined on $\mathcal{U}_p$ for every $s\in I$. Besides, if $\overline{M}^{n+1}_f$ is timelike geodesically complete, which means that any timelike geodesic of $\overline{M}^{n+1}_f$ is defined for all values of the parameter $s\in\mathbb{R}$, from [17, Section 3] we have that

$$\mathbb{R}\times\Xi^n_p\longrightarrow\overline{M}^{n+1}_f,\quad (s,q)\mapsto\mathcal{F}(s,q) \tag{4.8}$$

is a global parametrization on $\overline{M}^{n+1}_f$, such that $\overline{M}^{n+1}_f$ is isometric to the weighted GRW spacetime

$$(-\mathbb{R}\times_\phi\Xi^n_p)_f, \tag{4.9}$$

where

$$\phi(s) = \sqrt{-\big\langle V(\mathcal{F}(s,q)), V(\mathcal{F}(s,q))\big\rangle}, \quad s \in \mathbb{R},$$

and $q \in \Xi_p^n$ is an arbitrary point.

On the other hand, we observe that it follows from Theorem 1.2 of [11] that if $\overline{M}_f^{n+1}$ is a weighted timelike geodesic complete conformally stationary spacetime, with closed conformal timelike vector field V, endowed with a weight function $f \in C^\infty(\overline{M})$ which is bounded and such that its Bakry–Émery–Ricci tensor $\overline{\mathrm{Ric}}_f$ satisfies $\overline{\mathrm{Ric}}_f(X,X) \geq 0$ for all timelike vector fieds X, then f must be constant along the timelike line contained em $\overline{M}_f^{n+1}$, given via isometry by (4.8). So, motivated by this result, along this work we will consider weighted conformally stationary spacetimes $\overline{M}_f^{n+1}$ endowed with a closed conformal timelike vector field V whose weight function f does not depend on the parameter of the flow associated with the unit timelike vector field $\nu_V = \frac{V}{\sqrt{-\langle V,V\rangle}}$, that is, $\langle \overline{\nabla} f, \nu_V\rangle = 0$. This condition has already been used in (4.6) for calculating the f-mean curvature of the leaves $V^\top$. In particular, when the ambient space is a weighted GRW spacetime, we will make explicit this condition simply writing

$$-I \times_\phi F_f^n. \tag{4.10}$$

In the scenario described above, if $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is a spacelike hypersurface, then the smooth function

$$\eta_V\colon \Sigma^n \to \mathbb{R}, \quad p \mapsto \eta_V(p) = \langle V(p), N(p)\rangle \tag{4.11}$$

is negative and, after some standard calculations, we get

$$\nabla \eta_V = -A(V^\top), \tag{4.12}$$

where A is the shape operator of Σ^n.

The following proposition gives a suitable formula for the f-Laplacian of the support function η, which will be crucial to obtain our criteria of f-stability along the next section.

Proposition 3. *Let $\overline{M}_f^{n+1}$ be a weighted conformally stationary spacetime with a closed conformal timelike vector field V and whose weight function f does not depend on the parameter of the flow associated to ν_V. Let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a spacelike hypersurface with future-pointing Gauss map N and support function $\eta_V = \langle V, N\rangle$. Then*

$$\Delta_f \eta_V = \big\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\big\}\eta_V + nV^\top(H_f) + n\big\{\psi H_f - N(\psi_V)\big\}, \tag{4.13}$$

where ψ_V is the conformal factor of V, $\overline{\mathrm{Ric}}_f$ denotes the Bakry–Émery–Ricci tensor of $\overline{M}_f^{n+1}$, A is the shape operator of Σ^n with respect to N, H_f is the f-mean curvature of Σ^n and ∇H_f stands for the gradient of H_f in the induced metric of Σ^n.

PROOF. According to our previous digression, from (4.9) and (4.10) we have that (up to isometry) $\overline{M}_f^{n+1}$ can be locally regarded as a weighted GRW spacetime of the type $-\mathbb{R}\times_\phi \left(\Xi_p^n\right)_f$. In this setting, from (4.7) we have that $V=\phi\,\partial_t$, $\psi_V=\phi'$, $\nu_V=\partial_t$, $\sqrt{-\langle V,V\rangle}=\phi$ and, consequently, $\langle\overline{\nabla} f,\partial_t\rangle=0$.

Note that, from (2.2) we get

$$n\langle\partial_t,\nabla H\rangle=n\langle\partial_t^\top,\nabla H\rangle=n\langle\partial_t^\top,\nabla H_f\rangle+\partial_t^\top\langle\overline{\nabla} f,N\rangle. \tag{4.14}$$

On the other hand,

$$\begin{aligned}\partial_t^\top\big\langle\overline{\nabla} f,N\big\rangle&=\big\langle\overline{\nabla}_{\partial_t^\top}\overline{\nabla} f,N\big\rangle+\big\langle\overline{\nabla} f,\overline{\nabla}_{\partial_t^\top}N\big\rangle\\&=\big\langle\overline{\nabla}_{\partial_t+\Theta N}\overline{\nabla} f,N\big\rangle-\big\langle\overline{\nabla} f,A(\partial_t^\top)\big\rangle\\&=\big\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f,N\big\rangle+\Theta\,\overline{\mathrm{Hess}}_f(N,N)-\big\langle\overline{\nabla} f,A(\partial_t^\top)\big\rangle,\end{aligned} \tag{4.15}$$

where $\partial_t^\top=\partial_t+\Theta N$ and $\Theta=\langle N,\partial_t\rangle$.

Now, taking into account that $\langle\overline{\nabla} f,\partial_t\rangle=0$ and denoting by $\widetilde{\nabla}$ the Levi-Civita connection on Ξ_p^n, we have $\overline{\nabla} f=\frac{1}{\phi^2}\widetilde{\nabla} f$. Then,

$$\big\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f,N\big\rangle=\big\langle\overline{\nabla}_{\partial_t}(\phi^{-2}\widetilde{\nabla} f),N\big\rangle=\big\langle-2\phi^{-3}\phi'\widetilde{\nabla} f+\phi^{-2}\overline{\nabla}_{\partial_t}\widetilde{\nabla} f,N\big\rangle.$$

Hence, with aid of Proposition 7.35 of [19], from (4) we obtain

$$\begin{aligned}\big\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f,N\big\rangle&=\big\langle-2\phi^{-3}\phi'\widetilde{\nabla} f+\phi^{-2}\phi^{-1}\phi'\widetilde{\nabla} f,N\big\rangle\\&=-\phi'\phi^{-3}\big\langle\widetilde{\nabla} f,N\big\rangle=-\phi'\phi^{-1}\big\langle\overline{\nabla} f,N\big\rangle.\end{aligned} \tag{4.16}$$

Substituting (4.16) in equation (4.15) we get that

$$\partial_t^\top\langle\overline{\nabla} f,N\rangle=-\langle\overline{\nabla} f,N\rangle\phi^{-1}\phi'+\Theta\,\overline{\mathrm{Hess}}_f(N,N)-\langle\overline{\nabla} f,A(\partial_t^\top)\rangle. \tag{4.17}$$

From equation (4.14) and (4.17) we conclude that

$$\begin{aligned}n\phi\langle\partial_t,\nabla H\rangle&=n\phi\langle\partial_t^\top,\nabla H_f\rangle-\phi'\langle\overline{\nabla} f,N\rangle\\&\quad+\phi\,\Theta\,\overline{\mathrm{Hess}}_f(N,N)-\phi\langle\overline{\nabla} f,A(\partial_t^\top)\rangle.\end{aligned} \tag{4.18}$$

On the other hand, from Proposition 3.1 of [7] we have that

$$\Delta\langle N,\phi\,\partial_t\rangle=n\langle\phi\,\partial_t,\nabla H\rangle+\phi\;\Theta\big\{\overline{\mathrm{Ric}}(N,N)+|A|^2\big\}+n\big\{\phi' H-N(\phi')\big\}. \tag{4.19}$$

So, taking into account (2.1), substituting (4.18) into (4.19) we obtain that

$$\Delta\langle N,\phi\,\partial_t\rangle = n\langle\phi\,\partial_t,\nabla H_f\rangle + \phi\,\Theta\big\{\,\overline{\mathrm{Ric}}_f(N,N)+|A|^2\big\} + n\big\{\,\phi' H_f - N(\phi')\big\} - \phi\langle\overline{\nabla} f, A(\partial_t^\top)\rangle. \tag{4.20}$$

Moreover, from (4.12) we can verify that

$$\nabla\langle N,\phi\,\partial_t\rangle = -\phi\, A(\partial_t^\top). \tag{4.21}$$

Therefore, inserting equations (4.20) and (4.21) into (2.4) we reach at (4.13). □

In particular, from Proposition 3 we obtain the following

COROLLARY 1. *Let $\overline{M}_f^{n+1}$ be a weighted conformally stationary spacetime with a Killing timelike vector field W and whose weight function f does not depend on the parameter of the flow associated to unit vector field ν_W. Let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a spacelike hypersurface with future-pointing Gauss map N and support function $\eta_W = \langle W,N\rangle$. Then*

$$\Delta_f \eta_W = \big\{\,\overline{\mathrm{Ric}}_f(N,N)+|A|^2\big\}\,\eta_W + nW^\top(H_f).$$

In addition, if Σ^n is closed and H_f and $\lambda = \overline{\mathrm{Ric}}_f(N,N)+|A|^2$ are constant, then λ is an eigenvalue of the operator Δ_f in Σ^n with eigenfunction η_W.

5. Main results

We can now present our first f-stability criterion concerning closed spacelike hypersurfaces immersed in a weighted conformally stationary spacetime.

THEOREM 1. *Let $\overline{M}_f^{n+1}$ be a weighted conformally stationary spacetime with a closed conformal timelike vector field V and whose weight function f does not depend on the parameter of the flow associated to ν_V. Suppose that $\overline{M}_f^{n+1}$ is also equipped with a Killing timelike vector field W and that f does not depend on the parameter of the flow associated to a unit vector field ν_W. Let $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a closed spacelike hypersurface with constant f-mean curvature H_f and such that $\lambda = \overline{\mathrm{Ric}}_f(N,N)+|A|^2$ is constant. Then $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is f-stable if and only if λ is the first eigenvalue of f-Laplacian Δ_f on Σ^n.*

PROOF. Since that λ is constant and W is a Killing timelike vector field on $\overline{M}_f^{n+1}$, Corollary 1 guarantees that λ is in the spectrum of Δ_f.

Let λ_1 be the first eigenvalue of Δ_f on Σ^n. If $\lambda = \lambda_1$, then the variational characterization of λ_1 (see, for instance, Section 1 of [8]) gives

$$\lambda = \min_{u \in \mathcal{G} \setminus \{0\}} \frac{-\int_\Sigma u \Delta_f u \, d\mu}{\int_\Sigma u^2 \, d\mu},$$

where $\mathcal{G}$ is defined in (3.7). It follows that, for any $u \in \mathcal{G}$,

$$\mathcal{J}_f''(0)(u) = \int_\Sigma \left\{ u\Delta_f u + \lambda u^2 \right\} d\mu \le (-\lambda + \lambda) \int_\Sigma u^2 \, d\mu = 0,$$

and, according to Remark 2, $x \colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is f-stable.

Now suppose that $x \colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ is f-stable, so that $\mathcal{J}_f''(0)(u) \le 0$ for all $u \in \mathcal{G}$. Let u be an eigenfunction associated to the first eigenvalue λ_1 of Δ_f. From Remark 1 we obtain that there exists a volume-preserving variation of Σ^n whose variational field is uN. Consequently, by (3.8) we get

$$0 \ \ge\ \mathcal{J}_f''(0)(u) \ =\ (-\lambda_1 + \lambda) \int_\Sigma u^2 \, d\mu.$$

Therefore, since $\lambda_1 \le \lambda$, we must have $\lambda_1 = \lambda$. $\square$

Proceeding, we establish the following rigidity result

THEOREM 2. *Let $\overline{M}_f^{n+1}$ be a timelike geodesically complete weighted conformally stationary spacetime endowed with a closed conformal timelike vector field V and whose weight function f does not depend on the parameter $s \in \mathbb{R}$ of the flow associated to ν_V. Let $x \colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$ be a strongly f-stable closed spacelike hypersurface. Suppose that the conformal factor ψ_V of V satisfies the condition*

$$\frac{\partial \psi_V}{\partial s} \ge \max\{\psi_V H_f, 0\}. \tag{5.1}$$

If the set where $\psi_V = 0$ has empty interior in Σ^n, then Σ^n is either f-maximal or isometric to a leaf of the foliation $V^\perp$.

PROOF. Let us consider in $\overline{M}_f^{n+1}$ the global parametrization (4.8). Since Σ^n is strongly f-stable, it follows from (3.8) that

$$\int_\Sigma \left\{ \Delta_f u - \left\{ \overline{\mathrm{Ric}}_f(N,N) + |A|^2 \right\} u \right\} u \, d\mu \le 0, \quad \text{for all } u \in C^\infty(\Sigma). \tag{5.2}$$

In particular, since H_f is constant on Σ^n, taking the smooth function η_V defined in (4.11) we get from (4.13) that

$$\Delta_f \eta_V - \left\{ \overline{\mathrm{Ric}}_f(N,N) + |A|^2 \right\} \eta_V = n\{\psi H_f - N(\psi_V)\}.$$

Thus, from (5.2) we have that

$$\int_M \{\psi H_f - N(\psi_V)\}\, \eta_V \, d\mu \le 0. \tag{5.3}$$

On the other hand, it follows from (4.4) that

$$N(\psi_V) = \langle N, \overline{\nabla}\psi_V\rangle = -\nu_V(\psi_V)\langle N, \nu_V\rangle = \frac{\partial \psi_V}{\partial s}\cosh\theta,$$

where θ is the hyperbolic angle between N and ν. Substituting the above into (5.3), we finally arrive at

$$\int_M \Big(\frac{\partial \psi_V}{\partial s}\cosh\theta - \psi_V H_f\Big)\sqrt{-\langle V, V\rangle}\cosh\theta \, d\mu \le 0.$$

Now, from (5.1) we obtain

$$\begin{aligned} 0 &\ge \int_M \Big\{\frac{\partial \psi_V}{\partial s}\cosh\theta - \psi_V H_f\Big\}\sqrt{-\langle V, V\rangle}\cosh\theta \, d\mu \\ &\ge \int_M (\cosh\theta - 1)\frac{\partial \psi_V}{\partial s}\sqrt{-\langle V, V\rangle}\cosh\theta \, d\mu \ge 0. \end{aligned}$$

Hence,

$$\frac{\partial \psi_V}{\partial s}(\cosh\theta - 1) = 0 \quad \text{and} \quad \frac{\partial \psi_V}{\partial s} = \psi_V H_f$$

on Σ^n. But, since H_f is constant on Σ^n, Σ^n is either f-maximal or $H_f \neq 0$ on Σ^n. If this last case occurs, the condition on the zero set of ψ_V on Σ^n together with the above give $\frac{\partial \psi_V}{\partial s} \neq 0$ on a dense subset of Σ^n and hence $\cosh\theta = 1$ on this set. By continuity, $\cosh\theta = 1$ on Σ^n. Therefore in this case Σ^n must be a leaf of the foliation $V^\perp$. $\square$

We close our paper observing that, when the ambient space is a weighted GRW spacetime, we can apply Theorem 2 to obtain the following extension of [7, Theorem 1.1]

Corollary 2. *Let $x\colon \Sigma^n \hookrightarrow -\mathbb{R} \times_\phi F^n_f$ be a closed, strongly f-stable spacelike hypersurface. Suppose that $\phi'' \ge \max\{\phi' H_f, 0\}$. If the set where $\phi' = 0$ has empty interior in Σ^n, then Σ^n is either f-maximal or isometric to a slice $\{s_0\} \times F^n$, for some $s_0 \in \mathbb{R}$.*

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