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STRONG BOUNDEDNESS, STRONG CONVERGENCE AND GENERALIZED VARIATION

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To the memory of Naza Tanović-Miller

Abstract. A trigonometric series strongly bounded at two points and with coefficients forming a log-quasidecreasing sequence is necessarily the Fourier series of a function belonging to all L^p spaces, $1 \le p < \infty$. We obtain new results on strong convergence of Fourier series for functions of generalized bounded variation.

1. Introduction

Extending Hardy–Littlewood's concept of strong $(C, 1)$ summability to Cesàro methods (C, α) of order $\alpha \ge 0$, Hyslop [8] arrived at his notion of strong convergence. Subsequently, this was successfully applied to the study of trigonometric series in several papers written by N. Tanović-Miller and her co-workers [13,14,16–18]. Strong convergence of trigonometric series attracts attention because of its position between ordinary and absolute convergence [4,16,17].

Interesting results about the global behaviour of a series deduced from its behaviour at one or two points were initially related to absolute convergence and obtained by O. Szász [15] and R. Pippert [10]. The assumption on the coefficients of a series was that their magnitudes form an almost decreasing sequence. The analogues are valid in the case of strong convergence [3].

We introduce new notions of strong boundedness (in Hyslop's sense) and logarithmic quasimonotonicity. We prove that if one requests only strong boundedness of a trigonometric series at two points but imposes logarithmic

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quasimonotonicity on the magnitudes of its coefficients, then the respective trigonometric series is the Fourier series of a function belonging to all L^p spaces, $1 \le p < \infty$.

In the area of strong convergence, our attention is turned to Fourier series of regulated functions, i.e., functions belonging to various classes of generalized bounded variation.

2. Banach spaces of strongly bounded sequences

DEFINITION 2.A. A sequence of numbers $\{d_n\}$ is strongly $(C,1)$ summable to a limit d with index $\lambda > 0$ (λ-strongly $(C,1)$ summable to d), and we write $d_n \to d\ [C_1]_\lambda$, if

$$\sum_{k=1}^{n} |d_k - d|^\lambda = o(n) \quad \text{as } n \to \infty.$$

DEFINITION 2.B. A sequence of numbers $\{d_n\}$ is strongly convergent to a limit d with index $\lambda > 0$ (λ-strongly convergent to d), and we write $d_n \to d$ $[I]_\lambda$, if

1) $d_n \to d$ as $n \to \infty$,
2) $\sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda = o(n)$ as $n \to \infty$, i.e., $k(d_k - d_{k-1}) \to 0\ [C_1]_\lambda$.

If $\lambda = 1$, we simply denote it by $[I]$.

DEFINITION 2.1. A sequence of numbers $\{d_n\}$ is said to be strongly bounded with index $\lambda > 0$ (λ-strongly bounded), if

1) $d_n = O(1)$ as $n \to \infty$,
2) $\sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda = O(n)$ as $n \to \infty$.

If $\lambda = 1$, we say that sequence $\{d_n\}$ is strongly bounded.

The set of λ-strongly bounded sequences is denoted by $\mathscr{B}^\lambda$.

It is obvious that every λ-strongly convergent sequence is λ-strongly bounded. The converse does not hold as illustrated by the following examples.

EXAMPLE 2.2. Let $d_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$ for $n \in \mathbb{N}$. This sequence is obviously convergent. Therefore, $d_n = O(1)$ as $n \to \infty$. For any $\lambda > 0$, we have

$$\sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda = \sum_{k=1}^{n} k^\lambda \left| \frac{(-1)^{k-1}}{k} \right|^\lambda = \sum_{k=1}^{n} 1 = n.$$

Hence, $\{d_n\}$ is λ-strongly bounded but not λ-strongly convergent.

Example 2.3. Consider the sequence

$$d_n = \begin{cases} 1, & \text{if } n = 2^k,\ k \in \mathbb{N}, \\ 0, & \text{if } n \neq 2^k,\ k \in \mathbb{N}. \end{cases}$$

This sequence is bounded but it is not convergent since it has two partial limits, 0 and 1. Therefore, it is not λ-strongly convergent. Now, we have $|d_n - d_{n-1}| = 1$ if $n = 2^k$ or $n = 2^k + 1$. Otherwise, $d_n - d_{n-1} = 0$. Let $0 < \lambda \le 1$. Then

$$\sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda = \sum_{j=0}^{\lfloor \log_2 n \rfloor} [2^{j\lambda} + (2^j + 1)^\lambda] = O(n^\lambda).$$

Therefore, this sequence is λ-strongly bounded for $0 < \lambda \le 1$.

Every $\mathscr{B}^\lambda$ is a linear space. The next theorem introduces a norm in $\mathscr{B}^\lambda$ that turns $\mathscr{B}^\lambda$ into a Banach space.

Theorem 2.4. i) *$\mathscr{B}^\mu \supseteq \mathscr{B}^\lambda$ for $0 < \mu < \lambda$.*
ii) *For $d = \{d_n\}_{n=1}^\infty \in \mathscr{B}^\lambda$, $\lambda \ge 1$, let*

$$\|d\|_{\mathscr{B}^\lambda} = \sup_n |d_n| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda \right)^{\frac{1}{\lambda}}.$$

Then $\|\cdot\|_{\mathscr{B}^\lambda}$ is a norm on $\mathscr{B}^\lambda$, $\lambda \ge 1$.
iii) *$\mathscr{B}^\lambda$, $\lambda \ge 1$, is a Banach space under the norm given in* ii).

Proof. i) This is an immediate consequence of Hölder's inequality

$$\sum_{k=1}^{n} k^\mu |d_k - d_{k-1}|^\mu \le \left(\sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda \right)^{\frac{\mu}{\lambda}} \left(\sum_{k=1}^{n} 1 \right)^{1-\frac{\mu}{\lambda}} = O(n)$$

for $0 < \mu < \lambda$.

ii) It is obvious that $\|d\|_{\mathscr{B}^\lambda} \ge 0$ and that the equality holds if and only if $d = \{0\}_{n=1}^\infty$. If α is an arbitrary complex number and $\alpha d := \{\alpha d_n\}_{n=1}^\infty$, then

$$\|\alpha d\|_{\mathscr{B}^\lambda} = \sup_n |\alpha d_n| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} k^\lambda |\alpha d_k - \alpha d_{k-1}|^\lambda \right)^{\frac{1}{\lambda}}$$

$$= |\alpha| \left(\sup_n |d_k| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} k^\lambda |d_k - d_{k-1}|^\lambda \right)^{\frac{1}{\lambda}} \right) = |\alpha| \|d\|_{\mathscr{B}^\lambda}.$$

If $d^{(1)} = \{d_n^{(1)}\}_{n=1}^{\infty}$ and $d^{(2)} = \{d_n^{(2)}\}_{n=1}^{\infty}$ are two λ-strongly bounded sequences, $\lambda \geq 1$, and $d^{(1)} + d^{(2)} := \left\{ d_n^{(1)} + d_n^{(2)} \right\}_{n=1}^{\infty}$, then by Minkowski's inequality we get

$$\left| d_n^{(1)} + d_n^{(2)} \right| + \left(\frac{1}{n} \sum_{k=1}^{n} k^{\lambda} \left| \left(d_k^{(1)} + d_k^{(2)} \right) - \left(d_{k-1}^{(1)} + d_{k-1}^{(2)} \right) \right|^{\lambda} \right)^{\frac{1}{\lambda}}$$

$$= |d_n^{(1)} + d_n^{(2)}| + \left(\frac{1}{n} \sum_{k=1}^{n} k^{\lambda} \left| (d_k^{(1)} - d_{k-1}^{(1)}) + (d_k^{(2)} - d_{k-1}^{(2)}) \right|^{\lambda} \right)^{\frac{1}{\lambda}}$$

$$\leq |d_n^{(1)}| + |d_n^{(2)}| + \left(\frac{1}{n} \sum_{k=1}^{n} k^{\lambda} |d_k^{(1)} - d_{k-1}^{(1)}|^{\lambda} \right)^{\frac{1}{\lambda}} + \left(\frac{1}{n} \sum_{k=1}^{n} k^{\lambda} |d_k^{(2)} - d_{k-1}^{(2)}|^{\lambda} \right)^{\frac{1}{\lambda}}.$$

Hence,

$$\|d^{(1)} + d^{(2)}\|_{\mathscr{B}^{\lambda}} \leq \sup_n |d_n^{(1)}| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} k^{\lambda} |d_k^{(1)} - d_{k-1}^{(1)}|^{\lambda} \right)^{\frac{1}{\lambda}}$$

$$+ \sup_n |d_n^{(2)}| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} k^{\lambda} |d_k^{(2)} - d_{k-1}^{(2)}|^{\lambda} \right)^{\frac{1}{\lambda}} = \|d^{(1)}\|_{\mathscr{B}^{\lambda}} + \|d^{(2)}\|_{\mathscr{B}^{\lambda}}.$$

Thus, $\mathscr{B}^{\lambda}$, $\lambda \geq 1$, is a normed linear space.

iii) It remains to check that $\mathscr{B}^{\lambda}$, $\lambda \geq 1$, is complete. Let $d^{(1)}$, $d^{(2)}$, … be a Cauchy sequence in $\mathscr{B}^{\lambda}$. Now,

$$(\forall \varepsilon > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall m > n \geq n_0)\ \left\| d^{(m)} - d^{(n)} \right\|_{\mathscr{B}^{\lambda}} < \frac{\varepsilon}{3}.$$

Note that

$$\|d^{(m)} - d^{(n)}\|_{l^{\infty}} \leq \|d^{(m)} - d^{(n)}\|_{\mathscr{B}^{\lambda}}.$$

Thus, $d^{(n)}$ is a Cauchy sequence in l^{∞}. Since l^{∞} is a Banach space, there exists $d = \{d_i\}_{i=1}^{\infty} \in l^{\infty}$ such that $\|d^{(n)} - d\|_{l^{\infty}} \to 0$ $(n \to \infty)$. Hence, for $\varepsilon > 0$ chosen above,

$$(\exists n_0^* \in \mathbb{N})\ (\forall n \geq n_0^*)\ \|d^{(n)} - d\|_{l^{\infty}} < \frac{\varepsilon}{3}. \tag{2.1}$$

Moreover,

$$(\forall k \in \mathbb{N})\ (\exists n_k \in \mathbb{N})\ (\forall n \geq n_k)\ |d_k^{(n)} - d_k| < \frac{\varepsilon}{6(k+1)2^{\frac{k+1}{\lambda}}}. \tag{2.2}$$

Let us show that $\{d^{(n)}\}_{n=1}^{\infty}$ converges to $d=\{d_i\}_{i=1}^{\infty}$ in $\mathscr{B}^{\lambda}$. Take an arbitrary $i\in\mathbb{N}$ and fix it. Put $n_i^*=\max\{n_0^*,n_1,n_2,\dots,n_i\}\in\mathbb{N}$. To simplify notation, let us put

$$\sigma(i,d)=\left(\frac{1}{i}\sum_{k=1}^{i}k^{\lambda}|d_k-d_{k-1}|^{\lambda}\right)^{\frac{1}{\lambda}}.$$

Minkowski's inequality and (2.2) yield

$$\sigma(i,d^{(n_i^*)}-d)=\left(\frac{1}{i}\sum_{k=1}^{i}k^{\lambda}|(d_k^{(n_i^*)}-d_k)+(d_{k-1}-d_{k-1}^{(n_i^*)})|^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\le\left(\frac{1}{i}\sum_{k=1}^{i}k^{\lambda}|d_k^{(n_i^*)}-d_k|^{\lambda}\right)^{\frac{1}{\lambda}}+\left(\frac{1}{i}\sum_{k=1}^{i}k^{\lambda}|d_{k-1}-d_{k-1}^{(n_i^*)}|^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\le\left(\frac{1}{i}\sum_{k=1}^{i}k^{\lambda}\frac{\varepsilon^{\lambda}}{6^{\lambda}\,(k+1)^{\lambda}\,2^{k+1}}\right)^{\frac{1}{\lambda}}+\left(\frac{1}{i}\sum_{k=1}^{i}k^{\lambda}\frac{\varepsilon^{\lambda}}{6^{\lambda}k^{\lambda}2^{k}}\right)^{\frac{1}{\lambda}}\le\frac{\varepsilon}{3}.$$

Taking into account (2.1), we get

$$\sigma(i,d^{(m)}-d)\le\sigma(i,d^{(m)}-d^{(n_i^*)})+\sigma(i,d^{(n_i^*)}-d)$$

$$\le\|d^{(m)}-d^{(n_i^*)}\|_{\mathscr{B}^{\lambda}}+\frac{\varepsilon}{3}<\frac{2\varepsilon}{3}\quad(\forall m\ge n_0^{**}=\max\{n_0,n_0^*\}\text{ and }\forall i\in\mathbb{N})$$

Therefore,

$$\|d^{(m)}-d\|_{\mathscr{B}^{\lambda}}=\|d^{(m)}-d\|_{l^{\infty}}+\sup_i\sigma(i,d^{(m)}-d)<\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}=\varepsilon$$

for all $m\ge n_0^{**}$. Finally,

$$\|d\|_{\mathscr{B}^{\lambda}}\le\|d-d^{(n_0^{**})}\|_{\mathscr{B}^{\lambda}}+\|d^{(n_0^{**})}\|_{\mathscr{B}^{\lambda}}<\infty.$$

Hence, $d\in\mathscr{B}^{\lambda}$. $\square$

3. Local to global: behaviour of trigonometric series of a special type

Definition 3.A. A sequence of positive numbers $\{d_n\}$ is said to be almost decreasing if there exists a constant M such that $d_{n+1}\le Md_n$ holds for every $n\in\mathbb{N}$. M is the index of almost monotonicity of $\{d_n\}$. The space of almost decreasing sequences with index M is denoted by $\mathcal{A}_M\mathcal{M}$. If $d_{n+1}\le Md_n$ holds true starting from some integer $n>1$, the corresponding space is denoted by $\mathcal{G}\mathcal{A}_M\mathcal{M}$.

REMARK 3.1. Note that $\mathcal{A}_1\mathcal{M} = \mathcal{M}$ is the space of decreasing sequences.

The role of almost decreasing sequences is nicely illustrated by the following theorem.

THEOREM 3.B. *Let* $\rho_n = \sqrt{a_n^2 + b_n^2}$, $n \in \mathbb{N}$, *form an almost decreasing sequence and let*

$$\left\{ \begin{aligned} &\sum A_n(x) \equiv \frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx, \\ &\sum B_n(x) \equiv \sum a_n \sin nx - b_n \cos nx. \end{aligned} \right. \tag{3.1}$$

(a) (*cf.* [10, Theorem 2]) *If one of the series* $\sum A_n(x)$ *or* $\sum B_n(x)$ *is absolutely convergent at two points* x_0 *and* x_1 *with* $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$, *then* $\sum \rho_n < \infty$.

(b) (*cf.* [3, Theorem 2.2]) *If one of the series* $\sum A_n(x)$ *or* $\sum B_n(x)$ *is* $[I]_\lambda$, $\lambda \geq 1$, *convergent at two points* x_0 *and* x_1 *with* $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$, *then* $n\rho_n \to 0$ $[C_1]_\lambda$. *If* $\lambda > 1$, *then* $\sum A_n(x)$ *is the Fourier series of a function* $f \in \bigcap_{1 \leq p < \infty} L^p$, $[I]_\lambda$ *convergent to* f *a.e., and* $\sum B_n(x)$ *is the Fourier series of its conjugate function* $\widetilde{f}$, $[I]_\lambda$ *convergent to* $\widetilde{f}$ *a.e.*

In the next theorem, we shall replace the condition of λ-strong convergence by λ-strong boundedness. A series is said to be λ-strongly bounded if the sequence of its partial sums is λ-strongly bounded. Moreover, a sequence $\{\rho_n\}$ will be taken from a larger class $\mathcal{GA}_M\mathcal{M}$, $M > 1$. (See Remark 3.6 below.)

THEOREM 3.2. *Let* $\rho_n = \sqrt{a_n^2 + b_n^2}$, $n \in \mathbb{N}$, *form a sequence from* $\mathcal{GA}_M\mathcal{M}$, $M > 1$. *If one of the series* (3.1) *is* λ*-strongly bounded,* $\lambda > 1$, *at two points* x_0 *and* x_1, $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$, *then* $\sum A_n(x)$ *and* $\sum B_n(x)$ *are the Fourier series of the functions* f, $\widetilde{f}$, *resp., belonging to* L^p *for each* $1 \leq p < \infty$.

PROOF. Let θ_n be chosen such that $\sin\theta_n = \frac{a_n}{\rho_n}$ and $\cos\theta_n = \frac{b_n}{\rho_n}$. Then, $\sum A_n(x)$ may be written in the form $\sum \rho_n \sin(nx + \theta_n)$ and $\sum B_n(x) = -\sum \rho_n \cos(nx + \theta_n)$. Assume that $\sum A_n(x)$ is λ-strongly bounded at two points:

$$\sum_{k=1}^{n} k^\lambda \rho_k^\lambda |\sin(kx_i + \theta_k)|^\lambda = O(n) \quad \text{as } n \to \infty, \text{ for } i = 0, 1. \tag{3.2}$$

Let $h = x_0 - x_1$. Then $nh = (nx_0 + \theta_n) - (nx_1 + \theta_n)$ and

$$\sin nh = \sin(nx_0 + \theta_n)\cos(nx_1 + \theta_n) - \cos(nx_0 + \theta_n)\sin(nx_1 + \theta_n).$$

Therefore,

$$|\sin nh|^{\lambda} \le 2^{\lambda}\big(\, |\sin(nx_0+\theta_n)|^{\lambda} + |\sin(nx_1+\theta_n)|^{\lambda}\big)\,.$$

The last inequality and (3.2) imply

$$\sum_{k=1}^{n} k^{\lambda}\rho_k^{\lambda}\,|\sin kh|^{\lambda} = O(n) \quad \text{as } n\to\infty. \tag{3.3}$$

Let $\{\rho_k\} \in \mathcal{GA}_M\mathcal{M}$, $M>1$. There exists $K\in\mathbb{N}$ such that

$$\rho_{k-1} \ge \frac{1}{M}\rho_k \quad \text{for } k\ge K.$$

One has

$$\begin{aligned}
&(k-1)\rho_{k-1}|\sin(k-1)h| + k\rho_k\,|\sin kh| \\
&\ge \frac{1}{M}(k-1)\rho_k\,|\sin(k-1)h| + k\rho_k\,|\sin kh| \\
&\ge \frac{k-1}{Mk}k\rho_k(|\sin(k-1)h| + |\sin kh|) \ge \frac{1}{2M}k\rho_k(|\sin(k-1)h| + |\sin kh|)
\end{aligned}$$

for $k>K$. This and

$$\begin{aligned}
|\sin(k-1)h| + |\sin kh| &\ge \sin^2(k-1)h + \sin^2 kh \\
&= 1-\cos h\cos\left(2k-1\right)h \ge 1-|\cos h|
\end{aligned}$$

yield

$$(k-1)\rho_{k-1}\,|\sin(k-1)h| + k\rho_k\,|\sin kh| \ge M_1 k\rho_k$$

for $k>K$, where $M_1=M_1(h)>0$. Hence,

$$\begin{aligned}
\sum_{k=1}^{n} k^{\lambda}\rho_k^{\lambda} &= \sum_{k=1}^{K} k^{\lambda}\rho_k^{\lambda} + \sum_{k=K+1}^{n} k^{\lambda}\rho_k^{\lambda} \\
&\le \sum_{k=1}^{K} k^{\lambda}\rho_k^{\lambda} + \frac{2^{\lambda}}{M_1^{\lambda}}\sum_{k=K+1}^{n}\big[(k-1)^{\lambda}\rho_{k-1}^{\lambda}\,|\sin(k-1)h|^{\lambda} + k^{\lambda}\rho_k^{\lambda}\,|\sin kh|^{\lambda}\big].
\end{aligned}$$

Since the first summand in the last line is a finite sum and the second one is $O(n)$ as $n\to\infty$ by (3.3), we get

$$u_{n,\lambda} := \sum_{k=1}^{n} k^{\lambda}\rho_k^{\lambda} = O(n) \quad \text{as } n\to\infty.$$

Now, let $p > \max\{\frac{\lambda}{\lambda-1}, 2\}$ and $\frac{1}{q} = 1 - \frac{1}{p}$. It is straightforward that $1 < q < \min\{\lambda, 2\}$. Notice that $\sum_{k=1}^{n} k^{\lambda}\rho_k^{\lambda} = O(n)$ implies $u_{n,q} = \sum_{k=1}^{n} k^{q}\rho_k^{q} = O(n)$.

Abel's partial summation formula gives us

(3.4)

$$\sum_{k=1}^{n} \rho_k^q = \sum_{k=1}^{n} \frac{k^q\rho_k^q}{k^q} = \sum_{k=1}^{n} \frac{u_{k,q} - u_{k-1,q}}{k^q} = \frac{u_{n,q}}{n^q} + \sum_{k=1}^{n-1} u_{k,q}\Big(\frac{1}{k^q} - \frac{1}{(k+1)^q}\Big)$$
$$= O\Big(\frac{1}{n^{q-1}}\Big) + O\bigg(\sum_{k=1}^{n-1} \frac{1}{k^q}\bigg) = O(1) \quad \text{as } n \to \infty.$$

By the Hausdorff–Young theorem [24, (2.3), (ii), p. 101], there exists $f \in L^p$ such that $\sum A_n(x)$ is the Fourier series of f. This and the uniqueness property of Fourier series yield that f belongs to all L^p spaces, $1 \le p < \infty$. Then $\sum B_n(x)$ is the Fourier series of $\widetilde{f} \in \bigcap_{1\le p<\infty} L^p$.

The proof is analogous if $\sum B_n(x)$ is λ-strongly bounded at x_0, x_1 or if $\sum A_n(x)$ is λ-strongly bounded at x_0 and $\sum B_n(x)$ at x_1. $\square$

Definition 3.C. A sequence of positive numbers $\{d_n\}$ is said to be quasi decreasing if there exists $\alpha > 0$ such that $\{d_n/n^{\alpha}\}$ is a decreasing sequence starting from some integer $n \ge 1$. α is the index of quasimonotonicity of $\{d_n\}$. The space of quasi decreasing sequences with index α is denoted by $\mathcal{Q}_{\alpha}\mathcal{M}$.

As an application of the concept introduced by Definition 3.C, we cite the next result.

Theorem 3.D ([3, Theorem 3.1]). *Let $\rho_n = \sqrt{a_n^2 + b_n^2}$, $n \in \mathbb{N}$, form a quasi decreasing sequence with index $0 < \alpha < 1$. Let a trigonometric series $\frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx$ be strongly convergent at two points x_0 and x_1 with $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$. Then this series and its conjugate are Fourier series, strongly convergent a.e.*

Having in mind that the classes of λ-strongly bounded sequences, $\lambda > 1$, are contained in the class of strongly bounded (i.e., 1-strongly bounded) sequences, we pay a closer attention to the latter case.

We shall consider a new class of logarithmic quasi decreasing sequences.

Definition 3.3. A sequence of positive numbers $\{d_n\}$ is said to be logarithmic quasi decreasing if there exists $\beta > 0$ such that $\{d_n/\log^{\beta} n\}$ is a decreasing sequence starting from some integer $n \ge 2$. β is the index of logarithmic quasimonotonicity of $\{d_n\}$. The set of logarithmic quasi decreasing sequences with index β is denoted by $\mathcal{L}_{\beta}\mathcal{QM}$.

THEOREM 3.4. *Let $\rho_n = \sqrt{a_n^2 + b_n^2}$, $n \in \mathbb{N}$, form a logarithmic quasi decreasing sequence with index $\beta > 1$ ($\rho_n \in \mathcal{L}_\beta\mathcal{QM}$). If one of the series* (3.1) *is strongly bounded at two points x_0, x_1, $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$, then $\sum \frac{n\rho_n^2}{\log^\beta n} < \infty$. $\sum A_n(x)$ and $\sum B_n(x)$ are the Fourier series of the functions f, $\widetilde{f}$ which belong to L^p for each $1 \le p < \infty$.*

PROOF. Since $\{\rho_k\} \in \mathcal{L}_\beta\mathcal{QM}$, we have

$$\rho_{k-1} \ge \frac{\log^\beta(k-1)}{\log^\beta k}\rho_k \quad \text{for } k \ge K \ge 2.$$

Reasoning as in the proof of Theorem 3.2, we get

$$u_n := \sum_{k=1}^{n} k\rho_k = O(n) \quad \text{as } n \to \infty.$$

Now, for any $\alpha > 1$, one has

$$\sum_{k=2}^{n} \frac{\rho_k}{\log^\alpha k} = \sum_{k=2}^{n} \frac{k\rho_k}{k\log^\alpha k} = \sum_{k=2}^{n} \frac{u_k - u_{k-1}}{k\log^\alpha k} \tag{3.5}$$

$$= \frac{u_n}{n\log^\alpha n} - \frac{u_1}{2\log^\alpha 2} + \sum_{k=2}^{n-1} u_k\Big(\frac{1}{k\log^\alpha k} - \frac{1}{(k+1)\log^\alpha(k+1)}\Big).$$

Obviously

$$\frac{u_n}{n\log^\alpha n} = o(1) \quad \text{as } n \to \infty. \tag{3.6}$$

Notice that

$$\frac{1}{k\log^\alpha k} - \frac{1}{(k+1)\log^\alpha(k+1)} = \frac{1}{\xi_k^2\log^\alpha \xi_k}\Big(1 + \frac{\alpha}{\log \xi_k}\Big),$$

where $\xi_k \in (k, k+1)$. From $\frac{1}{\xi_k^2\log^\alpha \xi_k}(1 + \frac{\alpha}{\log\xi_k}) < \frac{1}{k^2\log^\alpha k}(1 + \frac{\alpha}{\log 2})$ and $u_k = O(k)$, we get

$$u_k\Big(\frac{1}{k\log^\alpha k} - \frac{1}{(k+1)\log^\alpha(k+1)}\Big) = O\Big(\frac{1}{k\log^\alpha k}\Big).$$

Thus,

$$\sum_{k=2}^{n-1} u_k\Big(\frac{1}{k\log^\alpha k} - \frac{1}{(k+1)\log^\alpha(k+1)}\Big) = O\Big(\sum_{k=2}^{n-1}\frac{1}{k\log^\alpha k}\Big) = O(1) \tag{3.7}$$

as $n \to \infty$.

The relations (3.5), (3.6) and (3.7) yield

$$\sum_{k=2}^{n} \frac{\rho_k}{\log^\alpha k} = O(1) \quad \text{as } n \to \infty, \text{ for } \alpha > 1. \tag{3.8}$$

In particular, the series $\sum_{k=2}^{\infty} \frac{\rho_k}{\log^\beta k}$ is convergent. This and the fact that the sequence $\{\frac{\rho_k}{\log^\beta k}\}$ is decreasing yield $\frac{k\rho_k}{\log^\beta k} = o(1)$ as $k \to \infty$ by Olivier's theorem. Now,

$$\sum_{k=2}^{n} \frac{k\rho_k^2}{\log^\beta k} = \sum_{k=2}^{n} \frac{\rho_k}{\log^\beta k}(u_k - u_{k-1})$$

$$= \frac{u_n\rho_n}{\log^\beta n} + \sum_{k=2}^{n-1} u_k\Big(\frac{\rho_k}{\log^\beta k} - \frac{\rho_{k+1}}{\log^\beta(k+1)}\Big) - \frac{\rho_2 u_1}{\log^\beta 2}$$

$$\le o(1) + C\sum_{k=2}^{n-1} k\Big(\frac{\rho_k}{\log^\beta k} - \frac{\rho_{k+1}}{\log^\beta(k+1)}\Big)$$

$$= o(1) + C\sum_{k=2}^{n-1}\Big(\frac{k\rho_k}{\log^\beta k} - \frac{(k+1)\rho_{k+1}}{\log^\beta(k+1)}\Big) + C\sum_{k=2}^{n-1}\frac{\rho_{k+1}}{\log^\beta(k+1)}$$

$$= o(1) + C\Big(\frac{2\rho_2}{\log^\beta 2} - \frac{n\rho_n}{\log^\beta n}\Big) + C\sum_{k=2}^{n-1}\frac{\rho_{k+1}}{\log^\beta(k+1)} = O(1) \quad \text{as } n \to \infty.$$

This proves the first assertion

$$\sum_{k=2}^{\infty} \frac{k\rho_k^2}{\log^\beta k} < \infty. \tag{3.9}$$

Concerning the second assertion, (3.9) and the Riesz–Fischer theorem yield that $\sum A_n(x)$ and $\sum B_n(x)$ are Fourier series of $f, \widetilde{f} \in L^2$.

Now, let $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Obviously, $1 < q < 2$. As above,

$$\sum_{k=1}^{n} \rho_k^q = \sum_{k=1}^{n} \frac{1}{k^q} k^q \rho_k^q = \sum_{k=1}^{n-1}\Big(\Delta\frac{1}{k^q}\Big)\sum_{i=1}^{k} i^q\rho_i^q + \frac{1}{n^q}\sum_{i=1}^{n} i^q\rho_i^q \tag{3.10}$$

$$= O\bigg(\sum_{k=1}^{n-1}\frac{1}{k^{q+1}}\sum_{i=1}^{k} i^q\rho_i^q\bigg) + \frac{1}{n^q}\sum_{i=1}^{n} i^q\rho_i^q.$$

Since $\frac{k\rho_k}{\log^\beta k} = o(1)$ as $k \to \infty$, we have that $k^q \rho_k^q = o(\log^{\beta q} k)$. Therefore,

$$\sum_{i=1}^{m} i^q \rho_i^q = O(m \log^{\beta q} m) \quad \text{for } m \in \mathbb{N}.$$

The last equality, relation (3.10) and the fact that $q > 1$ yield

$$\sum_{k=1}^{n} \rho_k^q = O\biggl(\sum_{k=1}^{n-1} \frac{\log^{\beta q} k}{k^q}\biggr) + O\biggl(\frac{\log^{\beta q} n}{n^{q-1}}\biggr) = O(1) \quad \text{as } n \to \infty.$$

Thus, $f, \widetilde{f} \in \bigcap_{1 \le p < \infty} L^p$ (cf. the end of the proof of Theorem 3.2). $\square$

Remark 3.5. Pointwise convergence a.e. of the series $\sum A_n(x)$ and $\sum B_n(x)$ in Theorem 3.4 follows, of course, from the Carleson–Hunt theorem. However, the Kolmogorov–Selyverstov–Plessner theorem [6, p. 332] already serves the purpose since

$$\sum_{k=2}^{\infty} \rho_k^2 \log k < \sum_{k=2}^{\infty} \frac{k\rho_k^2}{\log^\beta k} < \infty$$

by (3.9).

The following remark concerns the relationship between various sequence spaces considered in this paper.

Remark 3.6. For $0 < M_1 < 1 < M_2$, one has

$$\mathcal{A}_{M_1}\mathcal{M} \subset \mathcal{M} \subset \bigcap_{\beta>0} \mathcal{L}_\beta \mathcal{Q}\mathcal{M} \subset \bigcup_{\beta>0} \mathcal{L}_\beta \mathcal{Q}\mathcal{M}$$
$$\subset \bigcap_{\alpha>0} \mathcal{Q}_\alpha \mathcal{M} \subset \bigcup_{\alpha>0} \mathcal{Q}_\alpha \mathcal{M} \subset \mathcal{G}\mathcal{A}_{M_2}\mathcal{M}.$$

Proof. It is obvious that $\mathcal{A}_{M_1}\mathcal{M} \subset \mathcal{M} \subset \bigcap_{\beta>0} \mathcal{L}_\beta \mathcal{Q}\mathcal{M}$ since $0 < M_1 < 1 < \frac{\log^\beta(n+1)}{\log^\beta n}$ for any $\beta > 0$ and $n \in \mathbb{N}$. The inclusions $\bigcap_{\beta>0} \mathcal{L}_\beta \mathcal{Q}\mathcal{M} \subset \bigcup_{\beta>0} \mathcal{L}_\beta \mathcal{Q}\mathcal{M}$ and $\bigcap_{\alpha>0} \mathcal{Q}_\alpha \mathcal{M} \subset \bigcup_{\alpha>0} \mathcal{Q}_\alpha \mathcal{M}$ are trivial. The inclusion $\bigcup_{\alpha>0} \mathcal{Q}_\alpha \mathcal{M} \subset \mathcal{G}\mathcal{A}_{M_2}\mathcal{M}$ follows from $\frac{(n+1)^\alpha}{n^\alpha} < M_2$ for any $\alpha > 0$, $M_2 > 1$ and sufficiently large $n \in \mathbb{N}$. Finally, to establish $\bigcup_{\beta>0} \mathcal{L}_\beta \mathcal{Q}\mathcal{M} \subset \bigcap_{\alpha>0} \mathcal{Q}_\alpha \mathcal{M}$, it is enough to check that

$$\frac{\log^\beta(n+1)}{\log^\beta n} \le \frac{(n+1)^\alpha}{n^\alpha} \tag{3.11}$$

holds true for $\beta > 0$, $\alpha > 0$ and n sufficiently large. The last inequality is equivalent to

$$\frac{\log(n+1)}{\log n} \le \Big(1+\frac{1}{n}\Big)^{\gamma}$$

where we put $\gamma = \frac{\alpha}{\beta} > 0$. Subtracting 1 from both sides, we get

$$\frac{\log(1+\frac{1}{n})}{\log n} \le \Big(1+\frac{1}{n}\Big)^{\gamma} - 1.$$

According to Taylor's formula, the left hand side is equal to

$$\frac{1}{n\log n} - \frac{1}{2n^2\log n} + O\Big(\frac{1}{n^3\log n}\Big),$$

while the right hand side is equal to $\frac{\gamma}{n} + \frac{\gamma(\gamma-1)}{2n^2} + O(\frac{1}{n^3})$. Therefore, inequality (3.11) holds true for $\beta > 0$, $\alpha > 0$ and sufficiently large n. $\square$

Remark 3.7. In Remark 3.6 we are actually dealing with equivalence classes. Namely, while proving the inclusions, we suppose that $\{d_k\}_{k\ge k_0}$ and $\{d_k\}_{k\ge k_1}$, $k_0 \ne k_1$, represent the same sequence.

Remark 3.8. We have seen in Theorem 3.2 that if $\{\rho_k\} \in \mathcal{GA}_M\mathcal{M}$, $M > 1$, then a mere λ-boundedness, $\lambda > 1$, of the series $\sum A_n(x)$ or $\sum B_n(x)$ at two distinct points is sufficient to conclude that these are Fourier series of the functions f, $\widetilde{f}$ belonging to all L^p spaces, $1 \le p < \infty$. In the case $\lambda = 1$, the same conclusion is valid under a stronger assumption $\rho_k \in \mathcal{L}_\beta\mathcal{QM}$, $\beta > 1$. For intermediate classes $\mathcal{Q}_\alpha\mathcal{M}$, $\alpha > 0$, the same techniques of the proof yield the following theorem.

Theorem 3.9. *Let* $\{\rho_k\} \in \mathcal{Q}_\alpha\mathcal{M}$, $\alpha > 0$. *If* $\sum A_n(x)$ *or* $\sum B_n(x)$ *is strongly bounded at two points* x_0 *and* x_1, $|x_0 - x_1| \not\equiv 0 \pmod{\pi}$, *then* $\sum k^{1-\alpha}\rho_k^2 < \infty$. *These series are Fourier series of the functions* f, $\widetilde{f}$ *which belong to* L^2 *if* $\alpha \in (0,1)$. *Moreover,* $f, \widetilde{f} \in L^p$, $2 < p < \frac{1}{\alpha}$, *if* $\alpha \in (0, \frac{1}{2})$.

4. Strong convergence and generalized variation

Given a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \tag{4.1}$$

let $s_n(x)$ and $\sigma_n(x)$ denote the ordinary n-th partial sum and n-th Cesàro $(C,1)$ partial sum of (4.1), respectively. If (4.1) is a Fourier series of $f \in L^1$, we shall write $s_n f$ and $\sigma_n f$ for the partial sums s_n and σ_n.

We will consider the following classes of functions

$$\mathcal{S}^{\lambda} = \left\{ f \in L^1 : s_n f \to f \ [I]_{\lambda} \text{ a.e.} \right\},$$
$$\mathscr{S}^{\lambda} = \left\{ f \in C : s_n f \to f \ [I]_{\lambda} \text{ uniformly} \right\},$$
$$\mathscr{U} = \left\{ f \in C : s_n f \to f \text{ uniformly} \right\},$$

where C is the space of 2π-periodic continuous functions.

For $\lambda \geq 1$, it is known (see [18]) that

$$\mathcal{S}^{\lambda} = \left\{ f \in L^1 : \sum_{k=1}^{n} k^{\lambda} \rho_k^{\lambda} = o(n) \right\} \text{ and } \mathscr{S}^{\lambda} = \left\{ f \in C : \sum_{k=1}^{n} k^{\lambda} \rho_k^{\lambda} = o(n) \right\}.$$

By W we denote the class of *regulated functions*, i.e. functions possessing one-sided limits at each point. Every regulated function is bounded and has at most a countable set of discontinuities. Regulated functions have a particular role in the matter of everywhere convergence of Fourier series.

Important subclasses of the class W stem from various concepts of generalized bounded variation. In the sequel, let $f(I) := f(b) - f(a)$ for arbitrary subinterval (a, b) of $(0, 2\pi)$ and the supremum in defining sums below is always taken over all finite collections of nonoverlapping subintervals I_i of $(0, 2\pi)$.

According to N. Wiener [22], a function f is of *p-bounded variation*, $p \geq 1$, on $[0, 2\pi]$ and belongs to the class V_p if

$$V_p(f) = \sup \left\{ \sum_i |f(I_i)|^p \right\}^{1/p} < \infty.$$

A function f is of *ϕ-bounded variation* (L. C. Young [23]) on $[0, 2\pi]$ and belongs to the class V_{ϕ} if

$$V_{\phi}(f) = \sup \left\{ \sum_i \phi(|f(I_i)|) \right\} < \infty.$$

Here, ϕ is a continuous function defined on $[0, \infty)$ and strictly increasing from 0 to ∞.

Notice that by taking $\phi(u) = u$ we get Jordan's class BV, while $\phi(u) = u^p$ gives Wiener's class V_p.

A function f is of *Λ-bounded variation* (D. Waterman [20]) on $[0, 2\pi]$ and belongs to the class ΛBV if

$$V_{\Lambda}(f) = \sup \left\{ \sum_i |f(I_i)|/\lambda_i \right\} < \infty,$$

where $\Lambda = \{\lambda_n\}$ is a nondecreasing sequence of positive numbers tending to infinity, such that $\sum 1/\lambda_n$ diverges.

In the case when $\Lambda = \{n\}$, the sequence of positive integers, the function f is said to be of *harmonic bounded variation* and the corresponding class is denoted by HBV.

BV is the intersection of all ΛBV spaces and W is the union of all ΛBV spaces [9].

D. Waterman also introduced the notion of continuity in Λ-variation to provide a sufficient condition for (C,α)-summability of Fourier series [21]. Let $\Lambda^m = \{\lambda_{n+m}\}$, $m = 0,1,2,\dots$. A function $f \in \Lambda BV$ is said to be *continuous in Λ-variation* (or to belong to $\Lambda_c BV$) if $V_{\Lambda^m}(f) \to 0$ as $m \to \infty$.

Clearly, $\Lambda_c BV \subseteq \Lambda BV$. Functions from $\Lambda_c BV$ admit much better estimates of their Fourier coefficients (see [12,19]).

The *modulus of variation* (Z. Chanturiya [7]) of a bounded function f is the function ν_f whose domain is the set of positive integers, given by

$$\nu_f(n) = \sup\left\{ \sum_{k=1}^{n} |f(I_k)| \right\}.$$

The modulus of variation of any bounded function is nondecreasing and concave. Given a function ν whose domain is the set of positive integers with such properties, then by $V[\nu]$ one denotes the class of functions f for which $\nu_f(n) = O(\nu(n))$ as $n \to \infty$. We note that $V_\phi \subseteq V[n\phi^{-1}(1/n)]$ and $W = \{f : \nu_f(n) = o(n)\}$ [7].

The relationship between Waterman's and Chanturiya's concepts was established in [1]. We proved the following inclusions between Wiener's, Waterman's and Chanturiya's classes of functions of generalized bounded variation.

THEOREM 4.A (cf. [2, Theorem 4.4]).

$$\{n^\alpha\}BV \subset V_{\frac{1}{1-\alpha}} \subset V[n^\alpha] \subset \{n^\beta\}BV,$$

for $0 < \alpha < \beta < 1$.

The next two theorems are related to strong convergence and strong boundedness of Fourier series of regulated functions. As always, by $\widetilde{f}$ we denote the conjugate function of a function f.

THEOREM 4.1. *Let* $\lambda \geq 1$. *Then*

i) $W \cap \mathcal{S}^\lambda = \mathscr{S}^\lambda$.

ii) *If* $f, \widetilde{f} \in W$, *then* $f, \widetilde{f} \in C$.

iii) *If* $f \in \mathcal{S}^\lambda$ *and* $\widetilde{f} \in W$, *then* $\widetilde{f} \in \mathscr{S}^\lambda$.

iv) *If* $f \in HBV$ *and* $\widetilde{f} \in W$, *then* $f, \widetilde{f} \in \mathscr{U}$.

PROOF. i) Let f be an arbitrary function in $W \cap \mathcal{S}^\lambda$. Recall that $\mathcal{S}^\lambda \subset \mathcal{S}$ [18, Theorem 1(iii)]. Thus, $\sum_{k=1}^n k\rho_k = o(n)$, as $n \to \infty$. By [6, Theorem 3, p. 183 and Corrolary 2, p. 185], f can not have discontinuities of the first kind. It follows that f is a continuous function. Its Fourier series is $(C,1)$ uniformly summable. Therefore, $f \in \mathscr{S}^\lambda$. The converse, $\mathscr{S}^\lambda \subseteq W \cap \mathcal{S}^\lambda$, is trivial.

ii) Let $f, \widetilde{f} \in W$. If there exists a point x_0 such that, e.g., $f(x_0+0) - f(x_0-0) > 0$, then by [24, Teorem 8.13, vol. I, p. 60] $\widetilde{S}_n(x_0, f) \to -\infty$. Hence, $\tilde{\sigma}_n(x_0, f) \to -\infty$, which contradicts the fact that

$$\widetilde{\sigma}_n(x_0, f) = \sigma_n(x_0, \widetilde{f}) \to \frac{1}{2}\big[\widetilde{f}(x_0+0) + \widetilde{f}(x_0-0)\big]$$

[24, Fejér's theorem 3.4, vol. I, p. 89]. Therefore, function f is continuous. Analogously, the function $\widetilde{f}$ is continuous.

iii) Let $\widetilde{f} \in W$. The conjugate series is $(C,1)$ summable to $\widetilde{f}$ a.e. [6, p. 524]. Therefore, $f \in \mathcal{S}^\lambda$ implies $\widetilde{f} \in \mathcal{S}^\lambda$. Hence, $\widetilde{f} \in W \cap \mathcal{S}^\lambda = \mathscr{S}^\lambda$ by i).

iv) By ii) above, $f, \widetilde{f} \in C$. Now, $f \in HBV \cap C$ implies uniform convergence of its Fourier series [20]. However, $\widetilde{f}$ being also continuous, its Fourier series is necessarily uniformly convergent as well, by [6, Theorem 1, p. 592]. □

THEOREM 4.2. i) $\{n^{1/2}\}BV \cap C \subset \mathscr{S}^2$.

ii) *If* $f \in \{n^{1/2}\}\, BV$ *and* $\widetilde{f} \in W$, *then* $f, \widetilde{f} \in \mathscr{S}^2$.

iii) *If* $f \in V_2$, *then sequence* $\{s_n f\}$ *is* 2*-strongly bounded.*

PROOF. i) Let $f \in \{n^{1/2}\}\, BV \cap C$. Uniform convergence of the Fourier series follows from [20]. We [2, Theorem 11.1] proved that the condition

$$\frac{1}{n}\sum_{k=1}^{n} k^2 \rho_k^2 = o(1) \quad \text{as } n \to \infty \tag{4.2}$$

is necessary and sufficient for continuity of $f \in \{n^{1/2}\}_c BV$. According to [11, Theorem 3.1] the equality $\Lambda_c BV = \Lambda BV$ holds if and only if $S_\lambda < 2$, where S_λ is the Shao–Sablin index defined by

$$S_\lambda := \limsup_{n\to\infty} \frac{\sum_{i=1}^{2n} \frac{1}{\lambda_i}}{\sum_{i=1}^{n} \frac{1}{\lambda_i}}$$

for every proper Λ-sequence $\Lambda = \{\lambda_i\}$. In case of $\Lambda = \{i^{1/2}\}$, we have

$$S_\lambda = \limsup_{n\to\infty} \frac{\sum_{i=1}^{2n} \frac{1}{\sqrt{i}}}{\sum_{i=1}^{n} \frac{1}{\sqrt{i}}} = \lim_{n\to\infty} \frac{\int_1^{2n} \frac{dx}{\sqrt{x}}}{\int_1^{n} \frac{dx}{\sqrt{x}}} = \lim_{n\to\infty} \frac{\sqrt{2n}-1}{\sqrt{n}-1} = \sqrt{2} < 2.$$

Therefore, (4.2) holds for $f \in \{n^{1/2}\}BV \cap C$. Since

$$\frac{1}{n}\sum_{k=1}^{n} k^2 |s_k f - s_{k-1} f|^2 = \frac{1}{n}\sum_{k=1}^{n} k^2 \rho_k^2 |\sin(kx+\theta_k)|^2 \le \frac{1}{n}\sum_{k=1}^{n} k^2 \rho_k^2,$$

(4.2) and uniform convergence of $\{s_n f\}$ imply that $\{s_n f\}$ is 2-strongly convergent uniformly, i.e. $f \in \mathscr{S}^2$.

ii) If $f \in \{n^{1/2}\}BV$ and $\widetilde{f} \in W$, then $f, \widetilde{f} \in C$ by Theorem 4.1 ii). Now, $f \in \mathscr{S}^2$ according to i) above. Moreover, $\widetilde{f} \in \mathscr{S}^2$ by Theorem 4.1 iii).

iii) If $f \in V_2$, then $\frac{1}{n}\sum_{k=1}^{n} k^2 \rho_k^2 = O(1)$ [5, proof of Lemma 3.1], and the sequence $\{s_n f\}$ is 2-strongly bounded. $\square$

Remark 4.3. In view of Theorem 4.A, the analogues of Theorem 4.2 i) and ii) are valid for Wiener classes V_p, $1 \le p < 2$, and Chanturiya classes $V[n^\alpha]$, $0 < \alpha < \frac{1}{2}$.

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