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ON A CERTAIN ARITHMETICAL DETERMINANT

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Abstract. Smith showed in 1875 that if $n \geqq 1$ is an integer and $G := (\gcd(i,j))_{1 \leqq i,j \leqq n}$ is the $n \times n$ matrix having $\gcd(i,j)$ as its i,j-entry for all integers i and j between 1 and n, then $\det(G) = \prod_{k=1}^n \varphi(k)$, where φ is the Euler's totient function. We show that if $n \geqq 2$ is an integer and $H := (\gcd(i,j))_{2 \leqq i,j \leqq n}$ is the $(n-1) \times (n-1)$ matrix having $\gcd(i,j)$ as its i, j-entry for all integers i and j between 2 and n, then

$$\det(H) = \left(\prod_{k=1}^n \varphi(k) \right) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^n \frac{1}{\varphi(k)}.$$

We also calculate the determinants of the matrices $(f(\gcd(x_i,x_j)))_{1 \leqq i,j \leqq n}$ and $(f(\mathrm{lcm}(x_i,x_j)))_{1 \leqq i,j \leqq n}$ having f evaluated at $\gcd(x_i,x_j)$ and $\mathrm{lcm}(x_i,x_j)$ as their (i,j)-entries, respectively, where $S = \{x_1, \dots, x_n\}$ is a set of distinct positive integers such that $x_i > 1$ for all integers i with $1 \leqq i \leqq n$ and $S \cup \{1\}$ is factor closed (that is, $S \cup \{1\}$ contains every divisor of x for any $x \in S \cup \{1\}$). Our result answers partially an open problem raised by Ligh [18].

1. Introduction

One hundred and forty years ago, Professor H.J.S. Smith at University of Oxford published [21] his famous result stating that if n is a positive integer, then the determinant of the $n \times n$ matrix $\big(\gcd(i,j)\big)_{1 \leqq i,j \leqq n}$ having the

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greatest common divisor $\gcd(i,j)$ of i and j as the i,j-entry for all integers i and j between 1 and n is equal to $\prod_{k=1}^{n}\varphi(k)$, where φ is the Euler's totient function.

Let f be an arithmetic function and $S=\{x_1,\dots,x_n\}$ a set of n distinct positive integers. Denote by

$$\big(f\big(\gcd(x_i,x_j)\big)\big)_{1\leqq i,j\leqq n} \quad \text{and} \quad \big(f\big(\mathrm{lcm}(x_i,x_j)\big)\big)_{1\leqq i,j\leqq n}$$

the $n\times n$ matrices having f evaluated at the greatest common divisor $\gcd(x_i,x_j)$ and the least common multiple $\mathrm{lcm}(x_i,x_j)$ of x_i and x_j as their (i,j)-entries, respectively. Smith [21] showed also that

$$\det\big(\mathrm{lcm}(x_i,x_j)\big)_{1\leqq i,j\leqq n}=\prod_{i=1}^{n}\varphi(x_i)\pi(x_i)$$

and

$$\det\big(f\big(\gcd(x_i,x_j)\big)\big)_{1\leqq i,j\leqq n}=\prod_{i=1}^{n}(f*\mu)(x_i)$$

if S is *factor closed* (i.e., $d\in S$ if $x\in S$ and $d\mid x$), where $f*\mu$ is the Dirichlet convolution of f and the Möbius function μ and π is the multiplicative function defined for any prime power p^r by $\pi(p^r):=-p$. After Smith's paper published, this topic has received a lot of attention from many authors and particularly became extremely active in the past decades (see, for example, [1]–[20] and [22]–[25]). Twenty years ago, Bourque and Ligh [7] showed that if S is factor closed and f is a multiplicative function such that $f(x)\neq 0$ for all $x\in S$, then

$$\det\big(f\big(\mathrm{lcm}(x_i,x_j)\big)\big)_{1\leqq i,j\leqq n}=\prod_{i=1}^{n}\big(f(x_i)\big)^2\left(\frac{1}{f}*\mu\right)(x_i),$$

where $\frac{1}{f}(x):=\frac{1}{f(x)}$ if $f(x)\neq 0$, and 0 otherwise.

Ligh [18] raised an open problem of calculating the determinant

$$\det\big(\gcd(x_i,x_j)\big)_{1\leqq i,j\leqq n}$$

with $\{x_1,\dots,x_n\}$ being an arithmetic progression. As a special case of Ligh's problem, one has the following natural interesting question:

Problem 1. *Let m and n be positive integers such that $m\leqq n$. Calculate the determinants* $\det\big(\gcd(i,j)\big)_{m\leqq i,j\leqq n}$ *and* $\det\big(\mathrm{lcm}(i,j)\big)_{m\leqq i,j\leqq n}$.

Clearly, Smith's result [21] gave an answer to Problem 1 when $m = 1$. But so far it is still kept open when $m \geqq 2$.

Our main goal in this paper is to study Ligh's question and particularly, we mainly concern with the above Problem 1. We will present explicit formulas for the determinants of the $(n-1) \times (n-1)$ matrices $\big(\gcd(i,j)\big)_{2 \leqq i,j \leqq n}$ and $\big(\mathrm{lcm}(i,j)\big)_{2 \leqq i,j \leqq n}$. Recall that a positive integer is called *squarefree* if it is divisible by no other perfect square than 1. The first ten squarefree integers are given as follows: 1, 2, 3, 5, 6, 7, 10, 11, 13, 14. We have the following result.

THEOREM 2. *Let $n \geqq 2$ be an integer. Then*

$$\det \big(\gcd(i,j)\big)_{2 \leqq i,j \leqq n} = \Big(\prod_{k=1}^{n} \varphi(k)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{n} \frac{1}{\varphi(k)}$$

and

$$\det \big(\mathrm{lcm}(i,j)\big)_{2 \leqq i,j \leqq n} = \Big(\prod_{k=1}^{n} \varphi(k)\pi(k)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{n} \frac{\mu(k)k}{\varphi(k)}.$$

Obviously, Theorem 2 gives a partial answer to Ligh's problem [18]. It also answers Problem 1 when $m = 2$. Furthermore, the following more general result holds.

THEOREM 3. *Let $n \geqq 1$ be an integer and f an arithmetic function. Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers such that $x_i > 1$ for all integers i with $1 \leqq i \leqq n$ and $S \cup \{1\}$ is factor closed. Then*

$$\det \Big(f\big(\gcd(x_i,x_j)\big)\Big)_{1 \leqq i,j \leqq n} = \prod_{l=1}^{n} (f * \mu)(x_l) + f(1) \sum_{\substack{k=1 \\ x_k \text{ is squarefree}}}^{n} \prod_{\substack{l=1 \\ l \neq k}}^{n} (f * \mu)(x_l).$$

If f is multiplicative and $f(x) \neq 0$ for all $x \in S$, then

$$\det \Big(f\big(\mathrm{lcm}(x_i,x_j)\big)\Big)_{1 \leqq i,j \leqq n}$$

$$= \Big(\prod_{k=1}^{n} \big(f(x_k)\big)^2\Big) \Big(\prod_{l=1}^{n} \Big(\frac{1}{f} * \mu\Big)(x_l) + \sum_{\substack{k=1 \\ x_k \text{ is squarefree}}}^{n} \prod_{\substack{l=1 \\ l \neq k}}^{n} \Big(\frac{1}{f} * \mu\Big)(x_l)\Big).$$

It should be pointed out that in Theorem 3, the formula about the determinant $\det \Big(f\big(\gcd(x_i,x_j)\big)\Big)_{1 \leqq i,j \leqq n}$ is the main result of this paper while the formula about the determinant $\det\big(f\big(\mathrm{lcm}(x_i,x_j)\big)\big)_{1\leqq i,j\leqq n}$ is essentially a byproduct using the well-known equality

$$g(a)g(b)=g\big(\gcd(a,b)\big)\,g\big(\mathrm{lcm}(a,b)\big)$$

with a and b being positive integers and g being a multiplicative function.

The proof of Theorem 3 presented here is similar to that of Smith [21] in character, but it is more complicated than Smith's proof.

2. Preliminary lemmas

In this section, we present two useful lemmas that are needed in the next section. In what follows, we let $\omega(x)$ denote the number of distinct prime factors of the positive integer x.

LEMMA 4. *Let $m\geqq 2$ be a given integer. Define the arithmetic function F_m for any positive integer n by*

$$F_m(n):=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)f\big(\gcd(m,d)\big).$$

Then

$$F_m(n)=\begin{cases}(f*\mu)(n), & \text{if } n\mid m,\\ 0, & \text{otherwise.}\end{cases}$$

PROOF. Evidently, Lemma 4 is true if $n=1$. In what follows, we let $n\geqq 2$. First let $n\mid m$. Then $\gcd(m,\frac{n}{d})=\frac{n}{d}$ for any positive integer d dividing n. Hence

$$F_m(n)=\sum_{d\mid n}\mu(d)f\left(\frac{n}{d}\right)=(f*\mu)(n)$$

as required.

Now let $n\nmid m$. Obviously, we have

$$(1)\qquad F_m(n)=\sum_{d\mid n}f\left(\gcd\left(m,\frac{n}{d}\right)\right)\mu(d)=\sum_{c\mid\gcd(m,n)}f(c)\sum_{\substack{d\mid n\\ \gcd\left(m,\frac{n}{d}\right)=c}}\mu(d)$$

$$=\sum_{c\mid\gcd(m,n)}f(c)\sum_{\substack{d\mid\frac{n}{c}\\ \gcd\left(\frac{m}{c},\frac{n}{cd}\right)=1}}\mu(d).$$

Since

$$\sum_{e|\gcd\left(\frac{m}{c},\frac{n}{cd}\right)} \mu(e)=1$$

holds only when $\gcd\left(\frac{m}{c},\frac{n}{cd}\right)=1$, one then derives that the inner sum in (1) is equal to

$$\sum_{d|\frac{n}{c}}\mu(d)\sum_{e|\gcd\left(\frac{m}{c},\frac{n}{cd}\right)}\mu(e)=\sum_{e|\gcd\left(\frac{m}{c},\frac{n}{c}\right)}\mu(e)\sum_{d|\frac{n}{ce}}\mu(d). \tag{2}$$

But $n\nmid m$ implies that $\frac{n}{c}\nmid\gcd(\frac{m}{c},\frac{n}{c})$. Thus the term $e=\frac{n}{c}$ does not occur on the right-hand side of (2), and so $\frac{n}{ce}>1$ which implies that the inner sum on the right-hand side of (2) is always zero. Therefore, by (1) and (2) we deduce immediately that $F_m(n)=0$ if $n\nmid m$. $\square$

Lemma 5. *Let m and n be positive integers with m dividing n and $m<n$. Then*

$$\sum_{\substack{m|d|n\\ d\geqq 2}}\mu\left(\frac{n}{d}\right)=\begin{cases}(-1)^{\omega(n)+1}, & \text{if } m=1 \text{ and } n \text{ is squarefree},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. First of all, one denotes by $\Delta_{n,m}$ the sum

$$\sum_{\substack{m|d|n\\ d\geqq 2}}\mu\left(\frac{n}{d}\right).$$

Thus

$$\Delta_{n,m}=\sum_{\substack{mk|n\\ 2\leqq mk}}\mu\left(\frac{n}{mk}\right)=\sum_{\substack{k|\frac{n}{m}\\ 2\leqq mk}}\mu\left(\frac{n}{mk}\right).$$

If $m=1$, then

$$\Delta_{n,m}=\sum_{k|n,\ 2\leqq k}\mu\left(\frac{n}{k}\right)=\sum_{k|n}\mu\left(\frac{n}{k}\right)-\mu(n)$$

$$=\begin{cases}(-1)^{\omega(n)+1}, & \text{if } n \text{ is squarefree},\\ 0, & \text{otherwise.}\end{cases}$$

If $2 \leqq m < n$, then $mk \geqq 2$ and $\frac{n}{m} \geqq 2$ since $m \mid n$ and $m < n$. So we have

$$\Delta_{n,m} = \sum_{\substack{mk|n \\ 2\leqq mk}} \mu\left(\frac{n}{mk}\right) = \sum_{k|\frac{n}{m}} \mu\left(\frac{n}{mk}\right) = \sum_{k|\frac{n}{m}} \mu(k) = 0$$

as desired. □

3. Proofs of Theorems 2 and 3

Proof of Theorem 3. Let $x_0 := 1$. Without loss of any generality, we assume that $x_1 < x_2 < \dots < x_n$. We define the $(n+1) \times (n+1)$ matrix $A = (a_{ij})$ as follows: $a_{11} := 1$, $a_{i1} := 0$ for all $2 \leqq i \leqq n+1$, and $a_{ij} := f\big(\gcd(x_{i-1}, x_{j-1})\big)$ for all $1 \leqq i \leqq n+1$ and $2 \leqq j \leqq n+1$. For each integer r with $1 \leqq r \leqq n$, we define two sets R_r and T_r of positive integers as follows:

$$R_r := \{x_d : x_d \mid x_r,\ 0 \leqq d < r\}, \quad T_r := R_r \setminus \{x_0\}.$$

Then $1 = x_0 \in R_r$ for any integer $r \geqq 1$, and $T_r = \phi$ if and only if x_r is a prime number.

First, for each integer r with $1 \leqq r \leqq n$ and every $x_d \in R_r$, we multiply by $\mu(\frac{x_r}{x_d})$ all the entries of the $(d+1)$-th row of A and then add them to the corresponding entries of the $(r+1)$-th row of A. We arrive at a new $(n+1)$ $\times$ $(n+1)$ matrix, denoted by $B := (b_{ij})$. Claim that the following is true: For all integers i and j with $1 \leqq i, j \leqq n+1$, one has

$$\text{(I)} \qquad b_{ij} = \begin{cases} \mu(x_{i-1}), & \text{if } j = 1, \\ (f * \mu)(x_{i-1}), & \text{if } j > 1 \text{ and } x_{i-1} \mid x_{j-1}, \\ 0, & \text{otherwise,} \end{cases}$$

which will be proved in what follows.

Obviously, one has $b_{11} = a_{11} = 1$ and

$$b_{1j} = a_{1j} = f\big(\gcd(x_0, x_{j-1})\big) = f(1) = (f * \mu)(1)$$

for each integer j with $2 \leqq j \leqq n+1$. For all integers i and j with $2 \leqq i$ $\leqq n+1$ and $1 \leqq j \leqq n+1$, we have

$$(3) \qquad b_{ij} = a_{ij} + \sum_{x_d \in R_{i-1}} \mu\left(\frac{x_{i-1}}{x_d}\right) a_{d+1,j} = \sum_{x_d | x_{i-1}} \mu\left(\frac{x_{i-1}}{x_d}\right) a_{d+1,j}.$$

Since $a_{11}=1$ and $a_{k1}=0$ for all integers k with $2 \leqq k \leqq n+1$, it follows that for any integer $2 \leqq i \leqq n+1$, one has

$$b_{i1}=\sum_{x_d \mid x_{i-1}} \mu\left(\frac{x_{i-1}}{x_d}\right) a_{d+1,1}=\mu(x_{i-1})a_{11}=\mu(x_{i-1})$$

as required.

Now let i and j be integers such that $2 \leqq i, j \leqq n+1$. Since $S \cup \{1\}$ is factor closed and $a_{kj}=f\big(\gcd(x_{k-1}, x_{j-1})\big)$ for all integers k with $1 \leqq k \leqq n+1$, one can derive from Lemma 4 and (3) that

$$b_{ij}=\sum_{x_d \mid x_{i-1}} \mu\left(\frac{x_{i-1}}{x_d}\right) f\big(\gcd(x_{j-1}, x_d)\big)$$

$$=\sum_{d \mid x_{i-1}} \mu\left(\frac{x_{i-1}}{d}\right) f\big(\gcd(x_{j-1}, d)\big)=\begin{cases}(f*\mu)(x_{i-1}), & \text{if } x_{i-1} \mid x_{j-1},\\ 0, & \text{otherwise.}\end{cases}$$

Therefore claim (I) is proved.

Second, for each integer r with $1 \leqq r \leqq n$ and for each $x_d \in T_r$ (if T_r is nonempty), we multiply by $\mu\big(\frac{x_r}{x_d}\big)$ all the entries of the $(d+1)$-th column of B, and then add them to the corresponding entries of the $(r+1)$-th column of B, we obtain another new $(n+1)\times(n+1)$ matrix, denoted by $C:=(c_{ij})$ with c_{ij} being the (i,j)-entry of C for all integers i and j with $1 \leqq i, j \leqq n+1$. Claim that the following holds:

$$\text{(II)} \quad c_{ij}=\begin{cases}1, & \text{if } i=j=1,\\ (-1)^{\omega(x_{i-1})}, & \text{if } i>1,\ j=1 \text{ and } x_{i-1} \text{ is squarefree},\\ (-1)^{\omega(x_{j-1})+1}f(1), & \text{if } i=1,\ j>1 \text{ and } x_{j-1} \text{ is squarefree},\\ (f*\mu)(x_{i-1}), & \text{if } 1 \leqq i=j \leqq n+1,\\ 0, & \text{otherwise.}\end{cases}$$

Now we show that claim (II) is true. Evidently, $c_{i1}=b_{i1}$ for all $1 \leqq i \leqq n+1$. So for every integer i with $1 \leqq i \leqq n+1$, one has

$$c_{i1}=\begin{cases}(-1)^{\omega(x_{i-1})}, & \text{if } x_{i-1} \text{ is squarefree},\\ 0, & \text{otherwise}\end{cases}$$

as claimed.

For all the integers i and j with $1 \leqq i \leqq n+1$ and $2 \leqq j \leqq n+1$, we have

$$c_{ij} = b_{ij} + \sum_{x_{d-1} \in T_{j-1}} \mu\left(\frac{x_{j-1}}{x_{d-1}}\right) b_{id} = \sum_{\substack{x_{d-1} | x_{j-1} \\ d \geqq 2}} \mu\left(\frac{x_{j-1}}{x_{d-1}}\right) b_{id}. \tag{4}$$

Since $b_{1k} = f(1)$ for all integers k with $2 \leqq k \leqq n+1$ and noticing that $S \cup \{1\}$ being factor closed, we deduce from (4) that for all integers j with $2 \leqq j \leqq n+1$, one has

$$c_{1j} = \sum_{\substack{x_{d-1} | x_{j-1} \\ d \geqq 2}} \mu\left(\frac{x_{j-1}}{x_{d-1}}\right) f(1) = -\mu(x_{j-1}) f(1)$$

$$= \begin{cases} (-1)^{\omega(x_{j-1})+1} f(1), & \text{if } \ x_{j-1} \text{ is squarefree and } j > 1, \\ 0, & \text{otherwise} \end{cases}$$

as required.

Now let i be an integer with $1 \leqq i \leqq n+1$. Then by (4), one has

$$c_{ii} = \sum_{\substack{x_{d-1} | x_{i-1} \\ x_{i-1} | x_{d-1},\ d \geqq 2}} \mu\left(\frac{x_{i-1}}{x_{d-1}}\right) (f * \mu)(x_{i-1}) = (f * \mu)(x_{i-1})$$

as desired.

Consequently, for all integers i and j with $2 \leqq j < i \leqq n+1$, one has $x_{i-1} \nmid x_{j-1}$ implying that $b_{ij} = 0$, and if $x_{d-1} \mid x_{j-1}$ with $d \geqq 2$, then $x_{i-1} \nmid x_{d-1}$ implying that $b_{id} = 0$. So by (4), we deduce that $c_{ij} = 0$.

Finally, we let $2 \leqq i < j \leqq n+1$. It remains to show that $c_{ij} = 0$ that will be done in what follows.

If $x_{i-1} \nmid x_{j-1}$, then by claim (I), one has that $b_{ij} = 0$, and that $b_{id} = 0$ if $x_{d-1} \mid x_{j-1}$ since we must have $x_{i-1} \nmid x_{d-1}$, otherwise, one deduces from $x_{i-1} \mid x_{d-1}$ and $x_{d-1} \mid x_{j-1}$ that $x_{i-1} \mid x_{j-1}$, a contradiction. Hence by (4), one gets that $c_{ij} = 0$.

If $x_{i-1} \mid x_{j-1}$, then it follows from claim (I) that $b_{ij} = (f * \mu)(x_{i-1})$, $b_{id} = (f * \mu)(x_{i-1})$ if $x_{i-1} \mid x_{d-1}$, and $b_{id} = 0$ if $x_{i-1} \nmid x_{d-1}$. Since $S \cup \{1\}$ is factor closed and $x_{i-1} > 1$, then by (4) and Lemma 5, one derives that

$$c_{ij} = (f * \mu)(x_{i-1}) \sum_{\substack{x_{i-1} | x_{d-1} | x_{j-1} \\ d \geqq 2}} \mu\left(\frac{x_{j-1}}{x_{d-1}}\right)$$

$$= (f * \mu)(x_{i-1}) \sum_{\substack{x_{i-1}|d|x_{j-1} \\ d \geqq 2}} \mu\Big(\frac{x_{j-1}}{d}\Big) = 0.$$

Therefore claim (II) is proved.

Now note that if $r > 1$ and x_{r-1} is not squarefree, then by claim (II) we know that $c_{rr} = (f * \mu)(x_{r-1})$ and $c_{rj} = c_{ir} = 0$ for all integers $i \neq r$ and $j \neq r$. It then follows that

$$\det(C) = \det(D) \prod_{\substack{r=2 \\ x_{r-1} \text{ is not squarefree}}}^{n+1} (f * \mu)(x_{r-1}), \tag{5}$$

where

$$D := \begin{pmatrix} 1 & (-1)^{\omega(x_{i_1})+1} f(1) & (-1)^{\omega(x_{i_2})+1} f(1) & \cdots & (-1)^{\omega(x_{i_t})+1} f(1) \\ (-1)^{\omega(x_{i_1})} & (f * \mu)(x_{i_1}) & 0 & \cdots & 0 \\ (-1)^{\omega(x_{i_2})} & 0 & (f * \mu)(x_{i_2}) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{\omega(x_{i_t})} & 0 & 0 & \cdots & (f * \mu)(x_{i_t}) \end{pmatrix}$$

with t being a positive integer and $x_{i_1}, \dots, x_{i_t}$ being all the squarefree integers in the set $\{x_1, \dots, x_n\}$. Let $x_{i_0} := 1$. Since $f(1) = (f * \mu)(1)$, one can easily check that

$$\det(D) = \sum_{l=0}^{t} \prod_{\substack{j=0 \\ j \neq l}}^{t} (f * \mu)(x_{i_j}). \tag{6}$$

But

$$\det\big(f\big(\gcd(x_i, x_j)\big)\big)_{1 \leqq i,j \leqq n} = \det(A) = \det(B) = \det(C).$$

Hence from (5) and (6) we deduce that

$$\det\big(f\big(\gcd(x_i, x_j)\big)\big)_{1 \leqq i,j \leqq n}$$

$$= \Bigg(\prod_{\substack{r=1 \\ x_r \text{ is not squarefree}}}^{n} (f * \mu)(x_r) \Bigg) \sum_{\substack{k=0 \\ x_k \text{ is squarefree}}}^{n} \prod_{\substack{j=0,\ j \neq k \\ x_j \text{ is squarefree}}}^{n} (f * \mu)(x_j)$$

$$= \sum_{\substack{i=0 \\ x_i \text{ is squarefree}}}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} (f * \mu)(x_j)$$

as required. This concludes the proof of the first part of Theorem 3.

We are now in the position to show the second part. Since f is multiplicative, one has for all integers i and j with $1 \leqq i, j \leqq n$ that

$$f\big(\gcd(x_i, x_j)\big) f\big(\mathrm{lcm}(x_i, x_j)\big) = f(x_i) f(x_j).$$

It then follows that

$$\big(f\big(\mathrm{lcm}(x_i, x_j)\big)\big)_{1 \leqq i,j \leqq n} = \mathrm{diag}\big(f(x_1), \dots, f(x_n)\big)$$

$$\cdot \left(\frac{1}{f}\big(\gcd(x_i, x_j)\big)\right)_{1 \leqq i,j \leqq n} \cdot \mathrm{diag}\big(f(x_1), \dots, f(x_n)\big),$$

where $\mathrm{diag}\big(f(x_1), \dots, f(x_n)\big)$ is the $n \times n$ diagonal matrix with $f(x_1), \dots, f(x_n)$ as its diagonal elements. So one obtains that

$$\det\big(f\big(\mathrm{lcm}(x_i, x_j)\big)\big)_{1 \leqq i,j \leqq n} = \left(\prod_{i=1}^{n} \big(f(x_i)\big)^2\right) \det\left(\frac{1}{f}\big(\gcd(x_i, x_j)\big)\right)_{1 \leqq i,j \leqq n}.$$

Since f is a nonzero multiplicative function, one has $f(1) = 1$. Thus the first part of Theorem 3 applied to $\frac{1}{f}$ gives us the desired result. □

PROOF OF THEOREM 2. The first formula is a direct consequence of that of Theorem 3 by letting $f = I$, where the arithmetic function I is defined for any positive integer x by $I(x) = x$. On the other hand, one notes that for any positive integer x,

$$\left(\frac{1}{I} * \mu\right)(x) = \frac{\pi(x)\varphi(x)}{x^2}.$$

Then the second formula follows immediately. □

REMARK 6. Although the cases $m = 1$ and $m = 2$ of Problem 1 were answered by Smith [21] and Theorem 2 of this paper, respectively, it keeps widely open when $m \geqq 3$. We will continue to explore this interesting question in the close future.

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