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APPLICATIONS OF STRONGLY CONVERGENT SEQUENCES TO FOURIER SERIES BY MEANS OF MODULUS FUNCTIONS

K. RAJ* and C. SHARMA

Department of Mathematics, Shri Mata Vaishno Devi University, Shri Mata Vaishno Devi University, Katra, J&K, India
e-mails: kuldipraj68@gmail.com, charu145.cs@gmail.com

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Abstract. Recently, Kórus [7] studied the Λ^2-strong convergence of numerical sequences. By using the idea of this paper, we introduce $[\Lambda^2, F, u, p]$-strongly convergent sequence spaces defined by a sequence of modulus functions. We also make an effort to study some inclusion relations, topological and geometric properties of these spaces. Some characterizations for strong convergent sequences are given. Finally, we study statistical convergence over these spaces and problems related to Fourier series.

1. Introduction

Let w be the set of all real or complex sequences and l_∞, c and c_0, respectively be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $\|x\| = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers. Mursaleen and Noman [9] introduced the notion of λ-convergent and λ-bounded sequences. Let $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity. A sequence $x = (x_k) \in w$ is said to be λ-convergent to the number L and called the λ-limit of x if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=0}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

* Corresponding author.

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A sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [9] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=0}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=0}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0,$$

which gives that $\lim_m \Lambda_m(x) = a$ and we say $x = (x_k)$ is λ-convergent to a. Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to zero, that is, $\lambda_{-1} = x_{-1} = 0$.
In [7] Kórus gave a new appropriate definition for the Λ^2-strong convergence by generalizing the original Λ-strong convergence concept given by Móricz [11].

Let $\Lambda = \{\lambda_k : k = 0, 1, \dots\}$ be a non-decreasing sequence of positive numbers tending to infinity. A sequence (x_k) of complex numbers converges Λ^2-strongly to a complex number x if

$$\lim_n \Lambda^2(x) - x = \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} \big| \lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x) \big| = 0,$$

with the argument $\lambda_{-1} = \lambda_{-2} = x_{-1} = x_{-2} = 0$.

A function $f\colon [0,\infty) \to [0,\infty)$ is said to be modulus if it satisfy the following:

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x + y) \le f(x) + f(y)$, for all $x, y \ge 0$,
(3) f is increasing,
(4) f is right continuous at 0.

Since $|f(x) - f(y)| \le f(|x - y|)$, it follows from condition (4) that f is continuous on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then f is bounded but if we choose $f(x) = x^p, 0 < p < 1$ then f is unbounded.

Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \ge 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [20], Theorem 10.4.2, p. 183). For more details about sequence spaces (see [1,10,12–16]) and references therein.

Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Let $\Lambda = \{\lambda_k\}$ be a non-decreasing sequence of positive numbers tending to infinity. In the present paper we define the following classes of sequences:

$$[\Lambda^2, F, u, p] = \Big\{x = (x_k) : \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k} = 0 \quad \text{as } n \to \infty\Big\},$$

$$[\Lambda^2, F, u, p]_0 = \Big\{x = (x_k) : \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big)\big]^{p_k} = 0 \quad \text{as } n \to \infty\Big\}$$

and

$$[\Lambda^2, F, u, p]_\infty = \Big\{x = (x_k) : \sup_n \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big)\big]^{p_k} < \infty\Big\}.$$

Let us consider a few special cases of the above classes of sequences. If $F(x) = x$, then the sequences $[\Lambda^2, F, u, p]$, $[\Lambda^2, F, u, p]_0$ and $[\Lambda^2, F, u, p]_\infty$ reduces to $[\Lambda^2, u, p]$, $[\Lambda^2, u, p]_0$ and $[\Lambda^2, u, p]_\infty$ as follows:

$$[\Lambda^2, u, p] = \Big\{x = (x_k) : \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k} = 0 \quad \text{as } n \to \infty\Big\},$$

$$[\Lambda^2, u, p]_0 = \Big\{x = (x_k) : \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big)\big]^{p_k} = 0 \quad \text{as } n \to \infty\Big\}$$

and

$$[\Lambda^2, u, p]_\infty = \Big\{x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big)\big]^{p_k} < \infty\Big\}.$$

If $p_k = 1$ for all $k \in \mathbb{N}$, we shall write above sequences as

$$[\Lambda^2, F, u] = \Big\{x = (x_k) : \frac{1}{\lambda_n} \sum_{k=0}^{n} u_k f_k\big(|\lambda_k (x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big) = 0 \quad \text{as } n \to \infty\Big\},$$

$$[\Lambda^2, F, u]_0 = \Big\{x = (x_k) : \frac{1}{\lambda_n} \sum_{k=0}^{n} u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big) = 0 \quad \text{as } n \to \infty\Big\}$$

and

$$[\Lambda^2, F, u]_\infty = \Big\{x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big) < \infty\Big\}.$$

The following inequality will be used throughout the paper. If $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H$, $K = \max\{1, 2^{H-1}\}$, then

$$|a_k + b_k|^{p_k} \le K\big\{|a_k|^{p_k} + |b_k|^{p_k}\big\} \tag{1.1}$$

for all $k \in \mathbb{N}$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

The main objective of this paper is to introduce the concept of strongly convergent sequences using modulus function and to construct some new sequence spaces. We also make an effort to study some topological properties and prove some inclusion relations between these sequence spaces. Finally, by using the concept of strong convergence we study statistical convergence and results related to Fourier series.

2. Main results

THEOREM 2.1. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $[\Lambda^2, F, u, p]$, $[\Lambda^2, F, u, p]_0$ and $[\Lambda^2, F, u, p]_\infty$ are linear over the complex field $\mathbb{C}$.*

PROOF. Suppose $x = (x_k)$ and $y = (y_k) \in [\Lambda^2, F, u, p]_\infty$. Then

$$\sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|\big)\big]^{p_k} < \infty$$

and

$$\sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k y_k - \lambda_{k-2} y_{k-2}| \big) \big]^{p_k} < \infty.$$

For $\alpha, \beta \in \mathbb{C}$, then there exist integers M_α and N_β such that $|\alpha| \le M_\alpha$ and $|\beta| \le N_\beta$. Using inequality (1.1) and definition of modulus function, we have

$$\sup_n \frac{1}{\lambda} \sum_{k=0}^{n} \big[u_k f_k\big(|\alpha(\lambda_k x_k - \lambda_{k-2} x_{k-2}) + \beta(\lambda_k y_k - \lambda_{k-2} y_{k-2})| \big) \big]^{p_k}$$

$$\le \sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} \Big[u_k f_k\big(|\alpha(\lambda_k x_k - \lambda_{k-2} x_{k-2})| \big) + u_k f_k\big(|\beta(\lambda_k y_k - \lambda_{k-2} y_{k-2})| \big) \Big]^{p_k}$$

$$\le K(M_\alpha)^H \sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}| \big) \big]^{p_k}$$

$$+ K(N_\beta)^H \sup_n \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k y_k - \lambda_{k-2} y_{k-2}| \big) \big]^{p_k} < \infty.$$

This proves that $[\Lambda^2, F, u, p]_\infty$ is a linear space. Similarly, we can prove that $[\Lambda^2, F, u, p]$ and $[\Lambda^2, F, u, p]_0$ are linear spaces. $\square$

THEOREM 2.2. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ a sequence of strictly positive real numbers. Then $[\Lambda^2, F, u, p]_0$ is a paranormed space with paranorm*

$$g(x) = \sup_n \left\{ \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}| \big) \big]^{p_k} \right\}^{1/M},$$

where $H = \sup_k p_k < \infty$ and $M = \max\{1, H\}$.

PROOF. Clearly, $g(x) = g(-x)$, $x = \theta$ implies $(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|) = \theta$ and $f_k(0) = 0$, where θ is the zero sequence. Therefore, $g(\theta) = 0$. Since $p_k/M \le 1$ and $M \ge 1$, using the Minkowski's inequality and definition of modulus function, we have

$$\left\{ \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}| + |\lambda_k y_k - \lambda_{k-2} y_{k-2}| \big) \big]^{p_k} \right\}^{1/M}$$

$$\le \left\{ \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k x_k - \lambda_{k-2} x_{k-2}| \big) + u_k f_k\big(|\lambda_k y_k - \lambda_{k-2} y_{k-2}| \big) \big]^{p_k} \right\}^{1/M}$$

$$\le \left\{ \frac{1}{\lambda_n} \sum_{k=0}^{n} \left[u_k f_k\left(|\lambda_k x_k - \lambda_{k-2} x_{k-2}| \right) \right]^{p_k} \right\}^{1/M}$$

$$+ \left\{ \frac{1}{\lambda_n} \sum_{k=0}^{n} \left[u_k f_k\left(|\lambda_k y_k - \lambda_{k-2} y_{k-2}| \right) \right]^{p_k} \right\}^{1/M}.$$

Now, it follows that g is subadditive. Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition of $F = (f_k)$, we have

$$g(\mu x) = \sup_n \left\{ \frac{1}{\lambda_n} \sum_{k=0}^{n} \left[u_k f_k\left(|\mu(\lambda_k x_k - \lambda_{k-2} x_{k-2})| \right) \right]^{p_k} \right\}^{1/M} \le K_\mu^{H/M} g(x),$$

where K_μ is an integer such that $|\mu| < K_\mu$. Let $\mu \to 0$ for any fixed x with $g(x) = 0$. By definition for $|\mu| < 1$, we have

$$\frac{1}{\lambda_n} \sum_{k=0}^{n} \left[u_k f_k\left(|(\lambda_k x_k - \lambda_{k-2} x_{k-2})| \right) \right]^{p_k} < \varepsilon \quad \text{for } n > N(\varepsilon). \tag{2.1}$$

Also for $1 \le n \le N$, taking μ small enough, since $F = (f_k)$ is continuous, we have

$$\frac{1}{\lambda_n} \sum_{k=0}^{n} \left[u_k f_k(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|) \right]^{p_k} < \varepsilon. \tag{2.2}$$

Equations (2.1) and (2.2) together imply that $g(\mu x) \to 0$ as $\mu \to 0$. This completes the proof of the theorem. $\square$

THEOREM 2.3. *Suppose $F = (f_k)$, $F' = (f'_k)$, $F'' = (f''_k)$ are sequences of modulus functions, $p = (p_k)$ is a bounded sequence of positive real numbers, $u = (u_k)$ is a sequence of strictly positive real numbers and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. Then*

(i) $[\Lambda^2, F', u, p]_0 \subseteqq [\Lambda^2, F \circ F', u, p]_0$,

(ii) $[\Lambda^2, F', u, p]_0 \cap [\Lambda^2, F'', u, p]_0 \subseteqq [\Lambda^2, F' + F'', u, p]_0$.

PROOF. (i) Let $x \in [\Lambda^2, F', u, p]_0$. Then, we have

$$\frac{1}{\lambda_n} \sum_{k=0}^{n} \left[u_k f_k(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|) \right]^{p_k} = 0.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $f_k(t) < \varepsilon$ for $0 \le t \le \delta$. We write $y_k = [u_k f'_k(|\lambda_k x_k - \lambda_{k-2} x_{k-2}|)]$ and let us consider

$$\sum_{k=0}^{n} [f_k(y_k)]^{p_k} = \sum_{1} [f_k(y_k)]^{p_k} + \sum_{2} [f_k(y_k)]^{p_k},$$

where the first summation is over $y_k \le \delta$ and the second over $y_k > \delta$. Since $F = (f_k)$ is continuous, we have

$$\frac{1}{\lambda_n} \sum_{1} [f_k(y_k)]^{p_k} < \varepsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

By definition of modulus function, we have for $y_k > \delta,\ f_k(y_k) < 2 f_k(1) \frac{y_k}{\delta}$. Hence,

$$\frac{1}{\lambda_n} \sum_{2} [f_k(y_k)]^{p_k} \le \max\left(1, (2 f_k(1) \delta^{-1})^H\right) \frac{1}{\lambda_n} \sum_{k=0}^{n} [y_k]^{p_k}. \tag{2.4}$$

So by equations (2.3) and (2.4), we have $[\Lambda^2, F', u, p]_0 \subseteqq [\Lambda^2, F \circ F', u, p]_0$.

(ii) Let $x \in [\Lambda^2, F', u, p]_0 \cap [\Lambda^2, F'', u, p]_0$. Then using inequality (1.1) it can be shown that $x \in [\Lambda^2, F' + F'', u, p]_0$. Hence,

$$[\Lambda^2, F', u, p]_0 \cap [\Lambda^2, F'', u, p]_0 \subseteqq [\Lambda^2, F' + F'', u, p]_0. \quad \square$$

Corollary 2.4. *Suppose $F = (f_k)$, $F' = (f'_k)$, $F'' = (f''_k)$ are sequences of modulus functions, $p = (p_k)$ is a bounded sequence of positive real numbers and $u = (u_k)$ is a sequence of strictly positive real numbers. Then*

(i) $[\Lambda^2, F', u, p] \subseteqq [\Lambda^2, F \circ F', u, p]$,

(ii) $[\Lambda^2, F', u, p] \cap [\Lambda^2, F'', u, p] \subseteqq [\Lambda^2, F' + F'', u, p]$,

(iii) $[\Lambda^2, F', u, p]_\infty \subseteqq [\Lambda^2, F \circ F', u, p]_\infty$,

(iv) $[\Lambda^2, F', u, p]_\infty \cap [\Lambda^2, F'', u, p]_\infty \subseteqq [\Lambda^2, F' + F'', u, p]_\infty$.

Proof. It is easy to prove by using Theorem 2.3, so we omit the details. $\square$

Theorem 2.5. *Let $F = (f_k)$ be a sequence of modulus functions. Then for any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers, we have*

(i) $[\Lambda^2, F, u, p]_0 \cap [\Lambda^2, F, u, t]_0 \ne \phi$,

(ii) $[\Lambda^2, F, u, p] \cap [\Lambda^2, F, u, t] \neq \phi$,

(iii) $[\Lambda^2, F, u, p]_\infty \cap [\Lambda^2, F, u, t]_\infty \neq \phi$.

PROOF. (i) Since the zero element belongs to $[\Lambda^2,F,u,p]_0$ and $[\Lambda^2,F,u,t]_0$, thus the intersection is non-empty. Similarly, we can prove (ii) and (iii). □

PROPOSITION 2.6. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then, we have*

(i) $[\Lambda^2, u, p]_0 \subseteqq [\Lambda^2, F, u, p]_0$,

(ii) $[\Lambda^2, u, p] \subseteqq [\Lambda^2, F, u, p]$,

(iii) $[\Lambda^2, u, p]_\infty \subseteqq [\Lambda^2, F, u, p]_\infty$.

PROOF. It is obvious, so we omit the details. □

THEOREM 2.7. *Let $0 < p_k \leq r_k$ and $(\frac{r_k}{p_k})$ be bounded, then $[\Lambda^2, F, u, r] \subseteqq [\Lambda^2, F, u, p]$.*

PROOF. Let $x \in [\Lambda^2, F, u, r]$, $t_k = [u_k f_k(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x))]^{r_k}$ and $\mu_k = (\frac{p_k}{r_k})$ for all $k \in \mathbb{N}$ so that $0 < \mu \leq \mu_k \leq 1$. Define the sequence (v_k) and (w_k) as follows: For $t_k \geq 1$, let $v_k = t_k$ and $w_k = 0$ and for $t_k < 1$, let $v_k = 0$ and $w_k = t_k$. Then, clearly for all $k \in \mathbb{N}$, we have $t_k = v_k + w_k$, $t_k^{\mu_k} = v_k^{\mu_k} + w_k^{\mu_k}$, $v_k^{\mu_k} \leq v_k \leq t_k$ and $w_k^{\mu_k} \leq w_k^{\mu}$. Therefore,

$$\frac{1}{\lambda_n}\sum_{k=0}^{n} t_k^{\mu_k} \leq \frac{1}{\lambda_n}\sum_{k=0}^{n} t_k + \left[\frac{1}{\lambda_n}\sum_{k=0}^{n} w_k\right]^{\mu}.$$

Hence, $x \in [\Lambda^2, F, u, p]$. Thus, $[\Lambda^2, F, u, r] \subseteqq [\Lambda^2, F, u, p]$. This completes the proof of the theorem. □

We now turn to the characterizing of strongly convergent series.

Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Let $\Lambda = (\lambda_k)$ be a non-decreasing sequence of positive numbers tending to infinity. A sequence (x_k) of complex numbers is said to converge strongly to a complex number x if

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k=0}^{n} \big[\, u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k} = 0,$$

with the agreement $\lambda_{-1} = \lambda_{-2} = x_{-1} = x_{-2} = 0$.

LEMMA 2.8. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence* *of strictly positive real numbers. Then a sequence* (x_k) *of complex numbers converges strongly to a number* x *if and only if*

(i) $F(x_k)$ *converges to* $F(x)$ *in the ordinary sense; and*

(ii) $\lim\limits_{n\to\infty} \frac{1}{\lambda_n} \sum\limits_{k=2}^{n} [u_k f_k(\lambda_{k-2}|x_k - x_{k-2}|)]^{p_k} = 0.$

PROOF. The representation

$$\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x) = (\lambda_k - \lambda_{k-2})(x_k - x) + \lambda_{k-2}(x_k - x_{k-2})$$

implies both

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)| \big) \big]^{p_k} \\ &\quad\le \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big((\lambda_k - \lambda_{k-2})|x_k - x| \big) \big]^{p_k} \\ &\qquad + \frac{1}{\lambda_n} \sum_{k=2}^{n} \big[u_k f_k\big(\lambda_{k-2}|x_k - x_{k-2}| \big) \big]^{p_k} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k=2}^{n} \big[u_k f_k\big(\lambda_{k-2}|x_k - x_{k-2}| \big) \big]^{p_k} \\ &\le \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)| \big) \big]^{p_k} \\ &\quad + \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big((\lambda_k - \lambda_{k-2})|x_k - x| \big) \big]^{p_k}. \end{aligned}$$

By using these inequalities together with the fact that $F(x_k)$ converging to $F(x)$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big((\lambda_k - \lambda_{k-2})|x_k - x| \big) \big]^{p_k} = 0.$$

Hence, we get the necessity and sufficiency of both (i) and (ii). □

LEMMA 2.9. *Let* $F = (f_k)$ *be a sequence of modulus functions,* $p = (p_k)$ *be a bounded sequence of positive real numbers and* $u = (u_k)$ *be a sequence* *of strictly positive real numbers. Then a sequence* (x_k) *of complex numbers converges strongly to a number* x *if and only if*

(i) $F(\sigma_n) = \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2|n-k}} [u_k f_k((\lambda_k - \lambda_{k-2}) x_k)]^{p_k}$ *converges to* $F(x)$ *in the ordinary sense and*

(ii) $\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=2}^{n} [u_k f_k(\lambda_{k-2}|x_k - x_{k-2}|)]^{p_k} = 0.$

Proof. Clearly,

$$F(x_n) - F(\sigma_n) = \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2|n-k}} \left[u_k f_k\big((\lambda_k - \lambda_{k-2})(x_n - x_k)\big)\right]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2|n-k}} \left[(\lambda_k - \lambda_{k-2})\right]^{p_k} \sum_{\substack{k+2 \leqq j \leqq n \\ 2|n-j}} \left[u_k f_k(x_j - x_{j-2})\right]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_{\substack{2 \leqq j \leqq n \\ 2|n-j}} [u_k f_k(x_j - x_{j-2})]^{p_k} \sum_{\substack{0 \leqq k \leqq j-2 \\ 2|n-k}} \left[(\lambda_k - \lambda_{k-2})\right]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_{\substack{2 \leqq j \leqq n \\ 2|n-j}} \left[u_k f_k\big(\lambda_{j-2}(x_j - x_{j-2})\big)\right]^{p_k}.$$

Hence,

$$\limsup_{n\to\infty} \big|F(x_n) - F(\sigma_n)\big| \leqq \limsup_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=2}^{n} \left[u_k f_k\big(\lambda_{k-2}|x_k - x_{k-2}|\big)\right]^{p_k}.$$

According to Lemma 2.8, for the necessity part, it is easy to see that $\lim_n F(\sigma_n) = F(x)$ which comes from the above inequality, condition (ii) of this lemma and $\lim_n F(x_n) = F(x)$. For the sufficient part, we only need $\lim_n F(x_n) = F(x)$ which also comes from the above inequality, condition (ii) of this lemma and $\lim_n F(\sigma_n) = F(x)$. $\square$

3. Statistical convergence

The concept of statistical convergence was introduced by Fast [3] and Schoenberg [18] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was investigated from the sequence space point of view and linked with summability theory by Fridy [4], Connor [2], Salat [17], Mursaleen [8], Fridy and Orhan [5] and many others. The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists, where χ_E is the characteristic function of E.

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \big| \{ k \le n : |x_k - L| \ge \varepsilon \} \big| = 0,$$

In this case, we write $S\text{-}\lim_k x_k = L$ or $x \to L\,(S)$. The set of all statistical convergent sequences is denoted by S.

DEFINITION 3.1. A sequence $x = (x_k)$ is said to be $[\Lambda^2, F, u, p]$-statistically convergent to x if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \Big| \Big\{ k \le n : \big| \big[u_k f_k(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)) \big]^{p_k} \big| \ge \varepsilon \Big\} \Big| = 0,$$

where the vertical bars indicate the number of elements in the closed set. In this case, we write $S\text{-}\lim_k \Lambda_k^2(x) = x$ and the set of all statistically convergent sequences is denoted by $S(\Lambda^2)$.

THEOREM 3.2. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers and* $\sup_k p_k = H < \infty$. *Then* $[\Lambda^2, F, u, p] \subset (S(\Lambda^2))$.

PROOF. Let $x \in [\Lambda^2, F, u, p]$. Take $\varepsilon > 0$, $\sum_1$ denote the sum over $k \le n$ with

$$\big(\big| \big[u_k f_k(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)) \big]^{p_k} \big| \big) \ge \varepsilon$$

and $\sum_2$ denote the sum over $k \le n$ with

$$\big(\big| \big[u_k f_k(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)) \big]^{p_k} \big| \big) < \varepsilon.$$

Then

$$\frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)| \big) \big]^{p_k}$$

$$= \frac{1}{\lambda_n}\Big(\sum\nolimits_1 \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k}$$

$$+ \sum\nolimits_2 \big[u_k f_k\big(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)\big)\big]^{p_k}\Big)$$

$$\geq \frac{1}{\lambda_n}\sum\nolimits_1 \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k}$$

$$\geq \frac{1}{\lambda_n}\sum\nolimits_1 \big[u_k f_k(\varepsilon)\big]^{p_k} \geq \frac{1}{\lambda_n}\sum\nolimits_1 \min\big([u_k f_k(\varepsilon)]^h, [u_k f_k(\varepsilon)]^H\big)$$

$$= \frac{1}{\lambda_n}\Big|\Big\{k \leq n : \big|\big[u_k f_k(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x))\big]^{p_k}\big| \geq \varepsilon\Big\}\Big|$$

$$\times \min\big([u_k f_k(\varepsilon)]^h, [u_k f_k(\varepsilon)]^H\big).$$

Hence, $x \in (S(\Lambda^2))$. $\square$

THEOREM 3.3. *Let $F = (f_k)$ be a bounded sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers and $0 < \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $(S(\Lambda^2)) \subset [\Lambda^2, F, u, p]$.*

PROOF. Suppose that $F = (f_k)$ be bounded. For given $\varepsilon > 0$, denote $\sum_1$ and $\sum_2$ the same as in the proof of Theorem 3.2. Since $F = (f_k)$ be bounded there exists an integer M such that $f_k(x) < M$ for all $x \geq 0$. Then

$$\frac{1}{\lambda_n}\sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k}$$

$$\leq \frac{1}{\lambda_n}\Big(\sum\nolimits_1 \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k}$$

$$+ \sum\nolimits_2 \big[u_k f_k\big(|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)|\big)\big]^{p_k}\Big)$$

$$\leq \frac{1}{\lambda_n}\sum\nolimits_1 \max(M^h, M^H) + \frac{1}{\lambda_n}\sum\nolimits_2 \big[u_k f_k(\varepsilon)\big]^{p_k} \leq \max(M^h, M^H)\frac{1}{\lambda_n}$$

$$\times \Big|\Big\{k \leq n : \big|\big[u_k f_k\big(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)\big)\big]^{p_k}\big| \geq \varepsilon\Big\}\Big|$$

$$+ \max\big([u_k f_k(\varepsilon)]^h, [u_k f_k(\varepsilon)]^H\big).$$

Hence, $x \in [\Lambda^2, F, u, p]$. $\square$

4. Results for Fourier series

In this section by using the concept of strongly convergence we shall show some results related to Fourier series. The space of all 2π periodic complex-valued continuous functions is a Banach space endowed with the norm $\|f\|_C = \max_t |f(t)|$ and is denoted by C. Let

$$\frac{1}{2}a_0(f) + \sum_{k=1}^{\infty} \big(a_k(f)\cos kt + b_k(f)\sin kt\big) \tag{4.1}$$

be the Fourier series of $f \in C$ and denote by $s_k(f)$ the k-th partial sum of the series (4.1). We shall denote by $U_{F,u,p}$, $A_{F,u,p}$ and $S(\Lambda^2_{F,u,p})$ respectively the classes of functions $f \in C$ whose Fourier series converges uniformly, absolutely and strongly on $[0, 2\pi)$. In other words, a function $f \in S(\Lambda^2_{F,u,p})$ if

$$\lim_n \Big\| \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(s_k(f) - f) - \lambda_{k-2}(s_{k-2}(f) - f)|\big)\big]^{p_k} \Big\|_C = 0. \tag{4.2}$$

and $\lambda_{-1} = \lambda_{-2} = x_{-1} = x_{-2} = 0$. The space $U_{F,u,p}$ is a Banach space with the norm

$$\|f\|_{U_{F,u,p}} = \sup_k \big\| \big[u_k f_k(s_k(f))\big]^{p_k} \big\|_C$$

and the space $A_{F,u,p}$ is also a Banach space with the norm

$$\|f\|_{A_{F,u,p}} = \frac{1}{2}\big[u_k f_k(|a_0(f)|)\big]^{p_k} + \sum_{k=1}^{\infty} \big[u_k f_k\big(|a_k(f)| + |b_k(f)|\big)\big]^{p_k}$$

One can easily prove that $U_{F,u,p}$ and $A_{F,u,p}$ are Banach spaces. We shall give the proof only for $S(\Lambda^2_{F,u,p})$ in the next theorem. Now, we define the norm

$$\|f\|_{S(\Lambda^2_{F,u,p})} = \sup_n \Big\| \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(s_k(f) - f) - \lambda_{k-2}(s_{k-2}(f) - f)|\big)\big]^{p_k} \Big\|_C,$$

which is finite for every $f \in S(\Lambda^2_{F,u,p})$. By using triangle inequality, we have

$$\|f\|_{S(\Lambda^2_{F,u,p})}$$
$$\le \|f\|_C + \sup_n \Big\| \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(|\lambda_k(s_k(f) - f) - \lambda_{k-2}(s_{k-2}(f) - f)|\big)\big]^{p_k} \Big\|_C$$

and this sup is due to equation (4.2). The norm inequalities corresponding to in [11, equation (3.6)] are

$$\|f\|_{U_{F,u,p}} \leq \|f\|_{S(\Lambda^2_{F,u,p})} \leq 2\|f\|_{A_{F,u,p}}, \tag{4.3}$$

which implies that $A_{F,u,p} \subset S(\Lambda^2_{F,u,p}) \subset U_{F,u,p}$.

LEMMA 4.1. *Let $F=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $f \in S(\Lambda^2_{F,u,p})$ if and only if*

(i) $\lim_k \|s_k(f)-f\|_C = 0$ *and*

(ii) $\lim_k \Big\| \frac{1}{\lambda_n} \sum_{k=2}^{n} \big[u_k f_k\big(\lambda_{k-2}|a_k(f)\cos kt + b_k(f)\sin kt|\big)\big]^{p_k} \Big\|_C = 0.$

PROOF. It follows from Lemma 2.8, so we omit it. $\square$

Let

$$\sigma_n(f) = \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big((\lambda_k - \lambda_{k-2}) s_k(f)\big)\big]^{p_k} \quad (n=0,1,\dots). \tag{4.4}$$

LEMMA 4.2. *Let $F=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $f \in S(\Lambda^2_{F,u,p})$ if and only if*

(i) $\lim_k \|\sigma_k(f)-f\|_C = 0$ *and*

(ii) $\lim_k \Big\| \frac{1}{\lambda_n} \sum_{k=2}^{n} \big[u_k f_k\big(\lambda_{k-2}|a_k(f)\cos kt + b_k(f)\sin kt|\big)\big]^{p_k} \Big\|_C = 0.$

THEOREM 4.3. *The set $S(\Lambda^2_{F,u,p})$ endowed with norm*

$$\|f\|_{S(\Lambda^2_{F,u,p})} = \sup_n \Big\| \frac{1}{\lambda_n} \sum_{k=0}^{n} \big[u_k f_k\big(\big| \lambda_k (s_k(f)-f) - \lambda_{k-2}(s_{k-2}(f)-f) \big| \big)\big]^{p_k} \Big\|_C$$

is a Banach space.

PROOF. The only thing we have to prove is completeness. For this, let $\{s_j\}_{j\geq 1}$ be a Cauchy sequence in the norm $\|\cdot\|_{S(\Lambda^2_{F,u,p})}$. Then by equation (4.3), $\{s_j\}$ is a Cauchy sequence in the norm $\|\cdot\|_{U_{F,u,p}}$ as well so there exists a sequence $s \in S(\Lambda^2_{F,u,p})$ such that $\lim_{j\to\infty} \|s_j - s\|_{U_{F,u,p}} = 0$.

Now, we show that $s \in S(\Lambda^2_{F,u,p})$. Suppose $\varepsilon > 0$, then by assumption there exists $v = v(\varepsilon)$ such that

$$\|s_j - s_i\|_{S(\Lambda^2_{F,u,p})} \leq \varepsilon \quad \text{for all } i,j \geq v. \tag{4.5}$$

Let $s_j = \{s_{jk} : k = 0, 1, \dots\}$ and $s = \{s_k : k = 0, 1, \dots\}$. We shall fix i, n. Similarly by equation (4.3), we have

$$\tag{4.6} \begin{aligned} &\frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(|\lambda_k(s_{jk}-s_k)-\lambda_{k-2}(s_{j(k-2)}-s_{k-2})|\big)\big]^{p_k} \\ &\le \|s_j - s\|_{U_{F,u,p}}\frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k(\lambda_k+\lambda_{k-2})\big]^{p_k} \le \varepsilon, \end{aligned}$$

provided j is large enough, due to $\lim_{j\to\infty}\|s_j - s\|_{U_{F,u,p}} = 0$. Here j depends on n and ε and assume that $j \ge v$. Applying triangle inequality by taking equations (4.5) and (4.6) into account we obtain that

$$\begin{aligned} &\frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(\big|\lambda_k(s_{jk}-s_k)-\lambda_{k-2}(s_{j(k-2)}-s_{k-2})\big|\big)\big]^{p_k} \\ &\le \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(\big|\lambda_k(s_{jk}-s_{ik})-\lambda_{k-2}(s_{j(k-2)}-s_{i(k-2)})\big|\big)\big]^{p_k} \\ &\quad + \frac{1}{\lambda_n}\sum_{k=0}^{n}\big[u_k f_k\big(\big|\lambda_k(s_{ik}-s_k)-\lambda_{k-2}(s_{i(k-2)}-s_{k-2})\big|\big)\big]^{p_k} \\ &\le \|s_j - s_i\|_{S(\Lambda^2_{F,u,p})} + \varepsilon = 2\varepsilon, \end{aligned}$$

for $j \ge v$. Since this hold for any $n \ge 0$, by definition $\|s_j - s_i\|_{S(\Lambda^2_{F,u,p})} \le 2\varepsilon$ for $j \ge v$. This proves $\lim_{j\to\infty}\|s_j - s\|_{S(\Lambda^2_{F,u,p})} = 0$ and $s \in S(\Lambda^2_{F,u,p})$ which completes the proof. $\square$

The next result indicates that strong convergence exhibits some of the characteristics of absolute values. Such fact is provided in [11] and [19].

LEMMA 4.4. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. If a trigonometric series*

$$\sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

converges strongly for t belonging to a set of positive measure or of second category, then

$$\lim_n \frac{1}{\lambda_n}\sum_{k=2}^{n}\big[u_k f_k\big(\lambda_{k-2}(|a_k|+|b_k|)\big)\big]^{p_k} = 0.$$

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