*Acta Math. Hungar.*, **150** (1) (2016), 1–35
DOI: 10.1007/s10474-016-0653-7
First published online August 23, 2016

# WEAK AND STRONG STRUCTURES AND THE $T_{3.5}$ PROPERTY FOR GENERALIZED TOPOLOGICAL SPACES

E. MAKAI, JR.[1,*], E. PEYGHAN[2] and B. SAMADI[2]

[1]MTA Alfréd Rényi Institute of Mathematics, 1364 Budapest, Pf. 127, Hungary
e-mail: makai.endre@renyi.mta.hu

[2]Department of Mathematics, Faculty of Science, Arak University, Arak 38156-8-8349, Iran
e-mails: e-peyghan@araku.ac.ir, b-samadi@araku.ac.ir

*(Received July 8, 2013; revised April 20, 2016; accepted July 21, 2016)*

**Abstract.** We investigate weak and strong structures for generalized topological spaces, among others products, sums, subspaces, quotients, and the complete lattice of generalized topologies on a given set. Also we introduce $T_{3.5}$ generalized topological spaces and give a necessary and sufficient condition for a generalized topological space to be a $T_{3.5}$ space: they are exactly the subspaces of powers of a certain natural generalized topology on $[0, 1]$. For spaces with at least two points here we can have even dense subspaces. Also, $T_{3.5}$ generalized topological spaces are exactly the dense subspaces of compact $T_4$ generalized topological spaces. We show that normality is productive for generalized topological spaces. For compact generalized topological spaces we prove the analogue of the Tychonoff product theorem. We prove that also Lindelöfness (and $\kappa$-compactness) is productive for generalized topological spaces. On any ordered set we introduce a generalized topology and determine the continuous maps between two such generalized topological spaces: for $|X|, |Y| \geqq 2$ they are the monotonous maps continuous between the respective order topologies. We investigate the relation of sums and subspaces of generalized topological spaces to ways of defining generalized topological spaces.

## 1. Introduction

In this paper we do not require acquaintance with the terminology of category theory, although we use some of its concepts. These will be explained in the respective places.

In Section 2 we collect material needed later in our paper, and give the necessary definitions.

---

* Corresponding author.
*Key words and phrases:* generalized topology, weak and strong structure, product, sum, subspace, quotient, $T_{3.5}$, normal, compact, Lindelöf, $\kappa$-compact, ordered generalized topological space.
*Mathematics Subject Classification:* primary 54A05, secondary 54B30.

In Section 3 we investigate weak and strong structures for generalized topological spaces (GTS's), in particular, products (different from Császár's products), sums, subspaces, quotients, and the complete lattice of generalized topologies (GT's) on a set $X$. Our definition of the product is the categorical definition. It will turn out that GTS's form a topological category over sets (its definition cf. in Theorem 3.1). This is a slight variant of [26], Theorem 4.8.

In Section 4 we will investigate productivity of certain topological properties with respect to our definition of product. These include the natural analogues for GTS's of the separation axioms $T_0$, $T_1$, $T_2$ ,$T_3$, but also that of $T_4$. For compact GTS's there holds the analogue of the Tychonoff product theorem. However, also Lindelöf property and $\kappa$-compactness are productive for GT's. We will define $T_{3.5}$ GTS's that have an analogous relation to the GT on $[0,1]$ having a base $\{[0,x),(y,1] \mid x,y \in [0,1]\}$ as $T_{3.5}$ topological spaces (TS's) have to the usual topology on $[0,1]$: a GTS is $T_{3.5}$ if and only if it is a subspace of some power of the GTS $[0,1]$ if and only if it is a subspace of a normal $T_4$ GTS. For ordered spaces $(X,\leqq)$ there is a natural GT on $X$, and the continuous functions between two such spaces $X$, $Y$, for $|X|,|Y| \geqq 2$ are exactly the monotonous maps continuous in the respective order topologies.

In Section 5 we will investigate the relation of generating GT's by a monotonous map $\gamma\colon P(X) \to P(X)$, and by an enlargement $k\colon \mu \to P(X)$, to subspaces and sums of GTS's.

## 2. Preliminaries

**2.1.** The concept of generalized topology dates back to antiquity, then called "closure operator" (which could have still some additional properties, like, e.g., idempotence). A large number of such additional properties of closure operators and their interrelations are discussed in the monographs [19] and [5]. Early examples are the linear spans of a subset of a vector space, or more generally, subalgebras generated by subsets of some algebraic structure, like groups, semigroups, etc. For history and many properties of such closure operators cf. the papers [17] and [18] from 1987 and 1989, and particularly the monograph of D. Dikranjan and W. Tholen [19] from 1995, and the more recent monograph of G. Castellini [5] from 2003. Also cf. the monograph of E. Čech, Z. Frolík and M. Katětov [6], from 1966, but that deals only with one type of closure spaces, called Čech-closure spaces, or pretopologies (definition cf. later).

Let $X$ be a set and $P(X)$ its power set. [19], pp. (xiii) and 147, defined a *closure operator* $c\colon P(X) \to P(X)$ as follows. It should be *increasing* (called there *extensive*) i.e., $A \subset cA$ and *monotonous* i.e., $A \subset B \Longrightarrow cA \subset cB$. A *closure space*, also written as CS, is a pair $(X,c)$, where $X$ is a set and

$c\colon P(X) \to P(X)$ is a closure operator. [19], p. 147 also investigated *continuous maps between closure spaces* $f\colon (X,c) \to (Y,d)$, i.e., maps $X \to Y$, satisfying

$$A \subset X \Longrightarrow fcA \subset dfA \Longleftrightarrow cA \subset f^{-1}dfA\,, \tag{2.1}$$

or, equivalently,

$$B \subset Y \Longrightarrow cf^{-1}B \subset f^{-1}dB \Longleftrightarrow fcf^{-1}B \subset dB \tag{2.2}$$

(cf. [19], p. 25 and [5], p. 42, Proposition 4.2). All closure spaces and all continuous maps between them form a (so called) *category*, denoted by **CS**. (Actually the setting of [19] and [5] was more general: a category $\mathcal{X}$, with a distinguished class $\mathcal{M}$ of subobjects, and the closure operator mapped any distinguished subobject of any object $X$ of $\mathcal{X}$ to some distinguished subobject of the same object $X$. Additionally, all morphisms $f\colon X \to Y$ were required to be continuous from the closure operator on $X$ to the closure operator on $Y$. E.g., for topological groups, each morphism $f$ carries the closure of any subgroup $X_0$ of $X$ into the closure of the subgroup $f(X_0)$ of $Y$. Here $\mathcal{X}$ and $\mathcal{M}$ had to satisfy some natural hypotheses, which hold in our cases. However, their main topic is not a generalization of the investigation of generalized topological spaces. Namely, for $\mathcal{X} = \mathbf{Set}$ and $\mathcal{M}$ being all monomorphisms in **Set** their resulting category is just **Set**. If we let $\mathcal{X} = \mathbf{GenTop}$ and $\mathcal{M}$ all monomorphisms in **GenTop**, the category **GenTop** is already contained in the hypotheses, so this is no definition of **GenTop**. A reader not interested in category theory may just skip this point.) Initial, i.e., weak and final, i.e., strong structures for supratopological spaces — which are closely related to generalized topological spaces, cf. below — are proved to exist and are investigated in [26].

[6] required that a closure operator $c\colon P(X) \to P(X)$ should be increasing, and *preserve finite unions*, also called *finitely additive*, i.e., $c\emptyset = \emptyset$ (in [19] p. xiii *groundedness*) and $A, B \subset X \Longrightarrow c(A \cup B) = (cA) \cup (cB)$ (in [19] p. xiii and in [5], p. 65, Definition 6.1 *additivity*). Such an operator $c$ is called a *Čech-closure* and the pair $(X,c)$ a *Čech-closure space*, or more recently a *pretopology* and a *pretopological space*. The pretopological spaces with the corresponding continuous maps were investigated in great detail in [6]. They form the (so called) *category* **PrTop**. In particular, initial, i.e., weak and final, i.e., strong structures for pretopological spaces are proved to exist and are investigated in detail in [6], Section 32 and Section 33. In our paper pretopological spaces will not be investigated. We have to remark that also in [19] most of the concrete examples in topology were connected with pretopological spaces, while in [5] such examples are rare — just pretopological spaces are defined in p. 91, Example 7.12 — and GTS's and closure spaces were not systematically investigated from the topological point of view in [19] and [5].

**2.2.** In topology, generalized topologies $(X,\mu)$ formally seem (almost) to have been defined almost simultaneously by S. Lugojan [29], in 1982, under the very name "*generalized topological spaces*" (which remained almost unnoticed because of its language, although a bit of knowledge of French suffices to understand it) and by A. S. Mashhour, A. A. Allam, F. S. Mahmoud, F. H. Khedr [32], in 1983, under the name of "*supratopological spaces*", where however in both of these papers $X$ open was required (a strong generalized topology). The terminology supratopological spaces persists till now. Categorical topologists investigate them by this name, as one of the many types of structures in topology (the most well-known of these are beside topological spaces the uniform spaces), and investigate the relationships of these different types of structures in topology. However, unfortunately the terminologies collide: categorical topologists used to call supratopological spaces also as closure spaces (cf. e.g., [26]), which is in conflict with the usage of the monograph [19]. We will use the term *strong generalized topological space*. What is closure space in [19], yet satisfying the extra condition that the closure of the empty set is the empty set, is called by categorical topologists a *neighbourhood space*. It is given by a system of *neighbourhoods* $\mathcal{N}(x)$ for each point $x$ of a set $X$, such that $N\in\mathcal{N}(x)\Longrightarrow x\in N$, $N'\supset N\in\mathcal{N}(x)$ $\Longrightarrow N'\in\mathcal{N}(x)$, and $X\in\mathcal{N}(x)$.

Then Á. Császár [8] in 2002 introduced generalized topological spaces, which differ from supratopological spaces just by omitting the requirement of openness of $X$ from their definition. His motivation was the previous investigation of a number of generalizations of open sets in topological spaces, like semiopen sets ([28], 1963), $\alpha$-open sets ([35], 1965), preopen sets ([30], 1982), $\beta$-open sets ([1], 1983) defined by $A\subset\operatorname{cl}\operatorname{int}A$, $A\subset\operatorname{int}\operatorname{cl}\operatorname{int}A$, $A\subset\operatorname{int}\operatorname{cl}A$, $A\subset\operatorname{cl}\operatorname{int}\operatorname{cl}A$, for $A$ a subset of a topological space $X$, respectively. An extensive literature cf. in [8] from 2002. These definitions led Á. Császár [7] to introduce their common generalization, the so called $\gamma$-open sets, where $\gamma\colon P(X)\to P(X)$ is an arbitrary monotonous map, via the property $A\subset\gamma A$. The concept of $\gamma$-open sets already includes all generalized topologies (for suitable $\gamma$, namely for $\gamma$ the interior operator of the generalized topology). In [8] Á. Császár made a further step: he considered the system of $\gamma$-open sets, which is always a generalized topology, and disregarded from which $\gamma$ was it derived. Thus he [8] arrived to the concept of generalized topologies, and began their systematic topological investigation. The paper [8] was the basis for about 200 subsequent papers in this subject (by MathSciNet). This has been one of the important developments of general topology in the recent years.

We note that beginning with a topology, the first four above given generalizations of open sets form only generalized topologies, except for $\alpha$-openness, when we obtain a topology. Moreover, the first four above types of generalized open sets can be introduced also in generalized topological spaces, cf. [10].

We remark that the difference between supratopological spaces and generalized topological spaces is minor. Many proofs for supratopologies carry over to generalized topologies, sometimes with some notational complications. But of course, there are also differences between them.

Á. Császár [14] in 2008 introduced *generalized neighbourhood systems*, which is a generalization of the above mentioned neighbourhood spaces, by omitting the condition that $X \in \mathcal{N}(x)$. This concept is equivalent to that of the closure spaces. In fact, from a closure operator $c\colon P(X) \to P(X)$ one derives $\mathcal{N}(x)$ by $N \in \mathcal{N}(x) \iff x \notin c(X \setminus N)$, and the same formula derives $c\colon P(X) \to P(X)$ from $\langle \mathcal{N}(x) \mid x \in X \rangle$. Continuity can be rewritten as follows: $f\colon (X, \langle \mathcal{N}(x) \mid x \in X \rangle) \to (Y, \langle \mathcal{M}(y) \mid y \in Y \rangle)$ satisfies $x \in X \Longrightarrow f^{-1}\mathcal{M}(fx) \subset \mathcal{N}(x)$.

**2.3.** Let $X$ be a set and $\mu \subset P(X)$. (We observe that some authors require still $X \neq \emptyset$. However then e.g. intersections of subspaces are not subspaces, the empty sum does not exist, etc., so we must allow $X = \emptyset$.) Then $\mu$ is called a *generalized topology*, briefly GT on $X$ if $\emptyset \in \mu$ and any union of elements of $\mu$ belongs to $\mu$. A set $X$ with a GT $\mu$ is said to be a *generalized topological space* $(X, \mu)$, briefly *GTS*. The elements of $\mu$ are called *$\mu$-open sets*, and their complements are called *$\mu$-closed*. We say that $\mu$ is *strong* if $X \in \mu$. A *base of a GTS* $(X, \mu)$ is a subset $\beta$ of $\mu$ such that each $M \in \mu$ is a union of a subfamily (possibly empty) of $\beta$, cf. [11].

For $A \subset X$, we denote by $c_\mu(A)$ the intersection of all $\mu$-closed sets containing $A$ and by $i_\mu(A)$ the union of all $\mu$-open sets contained in $A$. Then the map $c_\mu\colon P(X) \to P(X)$ is increasing, monotonous and *idempotent* (i.e., $c^2 = c$). If some $c\colon P(X) \to P(X)$ has these properties, then it defines a GT via $\mu_c := \{X \setminus c(A) \mid A \subset X\}$. The description of GT's by open sets, or by the closure operator are equivalent: $\mu$ is sent to $c_\mu$, and $c$ to $\mu_c$, and these maps define bijections inverse to each other. For GT's we will use the notations $\mu, \nu, \varrho$ for the set of all open sets, and the notations $c, d, e$ for the associated closure operators. (The description by closed sets is clearly equivalent to the description by open sets, so we will not consider it in this paper.)

For maps, $f\colon (X, \mu) \to (Y, \nu)$ or $f\colon (X, c) \to (Y, d)$ is *continuous* if $f^{-1}(\nu) \subset \mu$, or in terms of closure operators if (2.1) or, equivalently, (2.2) holds. We will write also that $f$ is $(\mu, \nu)$-*continuous*, or $(c, d)$-*continuous*. Identifying the $\mu$'s and $c$'s on a set $X$ via the above bijections $\mu \mapsto c_\mu$ and $c \mapsto \mu_c$, these concepts become equivalent. The GTS's, with the continuous maps between any two of them form a (so called) *category*, denoted by **GenTop**. If $f\colon (X, \mu) \to (Y, \nu)$ is continuous, and has a continuous inverse $g\colon (Y, \nu) \to (X, \mu)$ (or we may use $(X, c), (Y, d)$), then it is called a *homeomorphism*.

For $\mathcal{A} \subset P(X)$ and $X_0 \subset X$ we write $\mathcal{A}|X_0 := \{A \cap X_0 \mid A \in \mathcal{A}\}$ (*trace of* $\mathcal{A}$ *on* $X_0$).

**2.4.** For concepts of category theory, we refer to [24], [2]. However, in this paper we do not want to suppose acquaintance with category theory. An exception is when we speak about limits or colimits, but then the reader may restrict himself to their special cases products or sums. However, because of this we have to recall the general concepts of products and sums in categories.

We begin with a notation. Recall that $\{(\cdot)_\alpha \mid \alpha \in J\}$ is the *family* (set, or class) of all $(\cdot)_\alpha$'s, for $\alpha \in J$. Here multiple occurrence of the same $(\cdot)_\alpha$ amounts to the same as if it occurred only once. If we write $\langle (\cdot)_\alpha \mid \alpha \in J \rangle$, this means the *indexed family* of all $(\cdot)_\alpha$'s, for $\alpha \in J$. That is, $\langle (\cdot)_\alpha \mid \alpha \in J \rangle$ is a function from $J$, whose values may coincide for different $\alpha$'s. When the indexed family is a set (i.e., $J$ is a set), we write *indexed set*.

In a category, like e.g. that of all sets (as objects) and all functions between them (as morphisms), or all generalized topological spaces (as objects) and all continuous maps between them (as morphisms), etc., one defines *products* and *sums* in the following way.

*For an indexed set of objects* $\langle X_\alpha \mid \alpha \in J \rangle$ *their product* $\prod_{\alpha \in J} X_\alpha$ is the up to isomorphism unique object, for which there exist so called *projections* $\pi_\alpha \colon \prod_{\alpha \in J} X_\alpha \to X_\alpha$, which have the following universality property. For any morphisms $\langle f_\alpha \colon Y \to X_\alpha \mid \alpha \in J \rangle$ there exists a unique morphism $g \colon Y \to \prod_{\alpha \in J} X_\alpha$ such that for each $\alpha \in J$ we have $f_\alpha = \pi_\alpha g$. The *underlying set* of $(X, \mu)$ (or of $(X, c)$) is $X$. (More details cf. in Section 3.) In our categories the product of an indexed set of objects exists and may be supposed to have as underlying set the product of the underlying sets. We will actually suppose this.

*For an indexed set of objects* $\langle X_\alpha \mid \alpha \in J \rangle$ *their sum* $\coprod_{\alpha \in J} X_\alpha$ is the up to isomorphism unique object, for which there exist so called *injections* $\iota_\alpha \colon X_\alpha \to \coprod_{\alpha \in J} X_\alpha$, which have the following universality property. For any morphisms $\langle f_\alpha \colon X_\alpha \to Y \mid \alpha \in J \rangle$ there exists a unique morphism $g \colon \coprod_{\alpha \in J} X_\alpha \to Y$ such that for each $\alpha \in J$ we have $f_\alpha = g\iota_\alpha$. In our categories the sum of an indexed set of objects exists and may be supposed to have as underlying set the sum (i.e., disjoint union) of the underlying sets. We will actually suppose this. Moreover, we may identify $X_\alpha$ and $\iota_\alpha X_\alpha$ via $\iota_\alpha$, and then we may consider the underlying set of $\coprod_{\alpha \in J} X_\alpha$ as the disjoint union of the underlying sets of the $X_\alpha$'s, that we will do also.

Analogously, if we have an injection $X_0 \to X$, then we may consider this as an inclusion of a subset, that we will do as well.

The *empty product* (or 0*'th power of a space*) is by this definition that up to isomorphism unique object $X_{\text{fin}}$ (*final object*) for which for any object $X$ there is exactly one morphism $X \to X_{\text{fin}}$. In **GenTop** this is $(X, \{\emptyset\})$, where $|X| = 1$. Similarly, the *empty sum* is that object $X_{\text{init}}$ (*initial object*) for which for any object $X$ there is exactly one morphism $X_{\text{init}} \to X$. In **GenTop** this is $(\emptyset, \{\emptyset\})$.

**2.5.** A way to produce GT's is given by the following ([8]). We call $\gamma\colon P(X) \to P(X)$ monotonous as in Section 2.1, (and write $\gamma A$ for $\gamma(A)$), and denote by $\Gamma(X)$ the family of all such mappings. A set $A \subset X$ is said to be *$\gamma$-open* if $A \subset \gamma A$. The $\gamma$-open sets constitute a GT on $X$ (cf. [7], 1.1), which we denote by $\mu(\gamma)$. Actually, all GT's on a given set $X$ can be obtained in this way (see Lemma 1.1 of [8] and also our Subsection 2.2).

Another way to produce GT's is given by the following (see [13]). A mapping $k\colon \mu \to P(X)$ is said to be an *enlargement on* $(X, \mu)$ if $M \subset kM$, whenever $M \in \mu$. A subset $A \subset X$ is *$\kappa(\mu, k)$-open* iff $x \in A$ implies the the existence of a $\mu$-open set $M$ such that $x \in M$ and $kM \subset A$. Császár in [13] proved that the collection $\kappa(\mu, k)$ of all $\kappa(\mu, k)$-open sets is a GT on $X$ that is *coarser than* $\mu$ (i.e. $\kappa(\mu, k) \subset \mu$) whenever $\mu$ is a GT on $X$. Some further aspects of enlargements are investigated in Y. K. Kim, W. K. Min [27].

[15] defined a sort of product of GTS's and obtained some of its basic properties. One can find more results related to this concept in [33], [37] and [42]. Its definition cf. in Section 3 of this paper, Definition 3.11. We will call this the *Császár product of GTS's*, but this will not be investigated in our paper.

**2.6.** There are some papers related to separation axioms on GTS's such as [29], [9], [12], [33] and [23]. In particular, $T_0$, $T_1$, $T_2$, *regularity*, $T_3$ (i.e., regular $T_1$, or equivalently regular $T_0$, like for topologies), *normality and* $T_4$ (i.e., normal $T_1$) are defined word for word as for topological spaces. (Observe that if $X$ is a $T_1$ — e.g., $T_2$ — GTS with $|X| \geqq 2$ or $X = \emptyset$ then $X$ is strong. For $|X| = 1$ there are two GT's on $X$: $(X, \{\emptyset\})$ and $(X, \{\emptyset, X\})$. Both are $T_2$ hence $T_1$, and the first one is not strong, the second one is strong. For normality the situation is converse: a not strong GTS is vacuously normal, since there are no two disjoint closed subsets (empty or non-empty). For regularity we have: a GTS of the form $(X, \{\emptyset\})$ is vacuously regular — but regularity of a GTS $(X, \mu)$ with $\mu \neq \{\emptyset\}$ implies strongness of $(X, \mu)$. Normal $T_0$ does not imply $T_1$, already for topologies, e.g. for $\mathbb{R}$ with open base $\{(-\infty, r) \mid r \in \mathbb{R}\}$.) Also, [12] studied normal GTS's and exhibited a suitable form of Urysohn lemma ([12], Theorem 3.3) by defining a suitable GT on $[0, 1]$: it has as a base $\{[0, x), (y, 1] \mid x, y \in [0, 1]\}$.

**2.7.** Let $X$ be a set. We say that $\mathcal{A} \subset P(X)$ is a *stack* (also called *ascending*) if $A \in \mathcal{A}$ and $A \subset B \subset X$ imply $B \in \mathcal{A}$ (cf. [26]). For $\mathcal{B} \subset P(X)$ we write $\operatorname{Stack} \mathcal{B} := \{C \subset X \mid \exists B \in \mathcal{B} \text{ such that } C \supset B\}$. This is called the *stack generated by* $\mathcal{B}$ (or the *ascending hull of* $\mathcal{B}$).

We recall that [14] investigated *generalized neighbourhood systems* (GNS's), i.e., functions $\psi\colon X \to P(P(X))$ with the property that $x \in X$ and $V \in \psi(x)$ imply $x \in V$. The filter property is not required, and also $\psi(x) = \emptyset$ is allowed. Moreover, not even the stack property is required. However, for what [14] uses these GNS's, remains invariant if we take the generated stack $\mathrm{Stack}\,\psi(x) := (\mathrm{Stack}\,\psi)(x) := \mathrm{Stack}\,(\psi x)$ rather than $\psi(x)$. That is, we may suppose that each $\psi(x)$ is a stack. These functions $\psi(\cdot)$ are equivalent to the closure operators defined above, cf. the end of Section 2.2.

[14] defined for such a $\psi$ a GT $\mu_\psi := \{M \subset X \mid x \in M \Longrightarrow \exists V \in \psi(x)\ V \subset M\}$, called the *GT generated by* $\psi$. By [14], Lemma 2.2 here for $\mathrm{Stack}\,\psi$ we have $\mu_\psi = \mu_{\mathrm{Stack}\,\psi}$, so here we may assume the stack property of $\psi$. [14], p. 396 established that each GT can be generated by at least one GNS, that can be supposed by [14], Lemma 2.2 to consist of stacks. [14] Example 2.1 and p. 397 showed that several different GNS's $\psi = \mathrm{Stack}\,\psi$ can generate the same $\mu_\psi$. Conversely, *a GT* $\mu$ *generates a GNS*, by the formula $\psi_\mu(x) := \mathrm{Stack}\,\{M \in \mu \mid x \in M\}$, [14], proof of Lemma 1.3. For two GNS's there can be defined the continuous maps: $f\colon (X,\psi) \to (Y,\psi')$ *is continuous* if and only if $x \in X \Rightarrow f^{-1}(\psi'(f(x))) \subset \psi(x)$. These continuous maps remain continuous if we replace the GNS's by the GT's generated by them, cf. [14], Proposition 2.1. However, the converse is not true: a map between GNS's which is continuous between the generated GT's is not necessarily continuous between the GNS's. Cf. [8], Example 2.2, where the generated GT's are even equal, so different GNS's may generate the same GT. These discrepancies between GT's and GNS's are the difference between the categories of GTS's and CS's.

W. K. Min [34] investigated the relationship of GNS's and GT's further.

## 3. Topologicity of GenTop over Set

Let **GenTop** be the *category* of all GTS's (called *objects*) and all continuous maps between them (called *morphisms*). Similarly, **Set** is the category of all sets (as objects) and all functions between sets (as morphisms). As well known, there are several ways to define the category **GenTop**. E.g., with generalized open sets, i.e., objects are pairs $(X,\mu)$, with $\{\emptyset\} \subset \mu \subset P(X)$, where $\mu$ is closed under arbitrary unions, and morphisms $f\colon (X,\mu) \to (Y,\nu)$ characterized by $f^{-1}\nu \subset \mu$. Or with closure operators $c\colon P(X) \to P(X)$, that are increasing, monotonous and idempotent. Then, as for topological spaces, $f\colon (X,c) \to (Y,d)$ is a morphism iff (2.1) holds, or, equivalently, iff (2.2) holds (cf. beside [19], p. 25 and [5], p. 42, Proposition 4.2 also [8], [39]).

A *source*, or *sink* in a category $\mathcal{C}$ is an indexed family (set or class) of morphisms with common domain $X$, or codomain $Y$, i.e., $\langle f_\alpha\colon X \to Y_\alpha \mid \alpha \in J\rangle$, or $\langle g_\alpha\colon X_\alpha \to Y \mid \alpha \in J\rangle$.

If we have an *indexed family of mappings* $\langle f_\alpha \colon P(X) \to P(Y) \mid \alpha \in J\rangle$, for some sets $X$ and $Y$, then their *union and intersection* are defined pointwise:

(3.1)
$$\Big(\bigcup_{\alpha\in J} f_\alpha\Big)(A) := \bigcup_{\alpha\in J}(f_\alpha A), \text{ and } \Big(\bigcap_{\alpha\in J} f_\alpha\Big)(A) := \bigcap_{\alpha\in J}(f_\alpha A), \text{ for } A \subset X\,.$$

The following theorem has to be preceded by some definitions.

The *underlying set functor* $U\colon \mathbf{GenTop} \to \mathbf{Set}$ maps the generalized topological space $(X,\mu)$ (or $(X,c)$) to the set $X$ and the continuous map $f\colon (X,\mu) \to (Y,\nu)$ (or $(X,c) \to (Y,d)$) to the function $f\colon X \to Y$. $U$ is called *faithful* if $f,g\colon (X,\mu) \to (Y,\nu)$ being different implies that also $Uf, Ug\colon X \to Y$ are different. This is evident for **GenTop**. $U$ is called *amnestic* if the following holds. If the identity map on $X$ is (underlies) a continuous map $(X,\mu) \to (X,\nu)$ and also is (underlies) a continuous map $(X,\nu) \to (X,\mu)$, then $\mu = \nu$. This is also evident for **GenTop**. $U$ is *fibre-small*, if any set $X$ is the underlying set of only set many $(X,\mu)$'s. This is also evident for **GenTop**, since the cardinality of GT's on a set $X$ is at most $\exp(\exp|X|))$.

The definition of *initial lifts*, also called *weak structures* is the following. If for a set $X$ there is an indexed class of morphisms $\langle \varphi_\alpha \colon X \to U(Y_\alpha,\nu_\alpha) = Y_\alpha \mid \alpha \in J\rangle$ in **Set** (i.e., a source in **Set**), then there is a (by the way, unique) so called *initial, or weak structure* $(X,\mu)$ on $X$, such that all $\varphi_\alpha$ underlie morphisms $f_\alpha\colon (X,\mu) \to (Y_\alpha,\nu_\alpha)$, and this source $\langle f_\alpha \mid \alpha \in J\rangle$ in **GenTop** has the following universality property. If for any $(Z,\lambda)$ there are morphisms $\langle g_\alpha \colon (Z,\varrho) \to (Y_\alpha,\nu_\alpha) \mid \alpha \in J\rangle$ (another source in **GenTop**) such that

$$Ug_\alpha = \varphi_\alpha h \quad \text{for all } \alpha \in J$$

for some $h\colon UZ \to X$, then there exists an $h'\colon (Z,\varrho) \to (X,\mu)$, such that

$$h = Uh' \text{ and for each } \alpha \in J \text{ we have } g_\alpha = f_\alpha h'.$$

If we reverse in this definition the direction of the maps (i.e., $\to$ is replaced by $\leftarrow$ and vice versa), we obtain the definition of *final lifts*, also called *strong structures*. In details, this is the following. If for a set $X$ there is an indexed class of morphisms $\langle \psi_\alpha \colon U(Y_\alpha,\nu_\alpha) = Y_\alpha \to X \mid \alpha \in J\rangle$ in **Set** (i.e., a sink in **Set**), then there is a (by the way, unique) so called *final structure* $(X,\mu)$ on $X$, such that all $\psi_\alpha$ underlie morphisms $f_\alpha\colon (Y_\alpha,\nu_\alpha) \to (X,\mu)$, and this sink $\langle f_\alpha \mid \alpha \in J\rangle$ in **GenTop** has the following universality property. If for any $(Z,\varrho)$ there are morphisms $\langle g_\alpha \colon (Y_\alpha,\nu_\alpha) \to (Z,\varrho) \mid \alpha \in J\rangle$ (another sink in **GenTop**) such that

$$Ug_\alpha = h\psi_\alpha \quad \text{for all } \alpha \in J$$

for some $h\colon X \to UZ$, then there exists a $h'\colon (X,\mu) \to (Z,\varrho)$, such that

$$h = Uh' \text{ and for each } \alpha \in J \text{ we have } g_\alpha = h' f_\alpha\,.$$

The existence of all initial (weak) structures is equivalent to the existence of all final (strong) structures (faithfulness and fibre-smallness supposed) [2], Theorem 21.9, [38], [21].

The initial lift of (i.e., weak structure for) the empty source for a set $X$ is the indiscrete GT, i.e., $(X, \{\emptyset\})$, i.e., $(X, c)$ with $\forall\, A \subset X\ cA = X$. The final lift of (i.e., strong structure for) the empty sink for a set $X$ is the discrete GT, i.e., $(X, P(X))$, i.e., $(X, c)$ with $\forall\, A \subset X\ cA = A$. (Therefore in our Propositions 3.2, 3.3, 3.4, 3.6 we might investigate only non-empty sources and sinks.)

For *topological categories over* **Set** we refer to J. Adámek, H. Herrlich, G. E. Strecker, Abstract and concrete categories: the joy of cats, [2], Ch. 21, G. Preuss, Theory of Topological Structures [38], or for a synopsis Encyclopedia of Math. Vol. 9 [21] pp. 201–202. cf. also the text of Theorem 3.1 for the definition.

THEOREM 3.1 (for supratopologies cf. [26], Theorem 4.8). *The category* **GenTop**, *with its underlying set functor* $U$: **GenTop** $\to$ **Set** *is a topological category over* **Set**. *That is*, $U$ *is faithful*, *amnestic*, *fibre-small*, *and there exist all initial lifts* (*i.e.*, *weak structures*) *or equivalently there exist all final lifts* (*i.e.*, *strong structures*). *Hence in* **GenTop** *there exist both limits and colimits of all diagrams*, *which can be obtained from the respective underlying diagrams in* **Set** *by initial/final lifts.*

Above we already observed faithfulness, amnesticity and fibre-smallness for $U$. Existence of all limits (e.g., products) and all colimits (e.g., sums) and the way of obtaining them hold in any topological category over **Set** [2], Proposition 21.15, [21]. So only the weak and strong structures need be given.

We give the simple proof of Theorem 3.1, even in several forms. We explicitly give all initial and all final lifts, i.e., weak and strong structures, both for the open sets and the closure operator definition.

PROPOSITION 3.2 (for supratopologies cf. [26], Proposition 4.1 and Theorem 4.4). *Let* $X$ *be a set. Let us have a source* $\langle \varphi_\alpha\colon X \to U(Y_\alpha, \nu_\alpha) = Y_\alpha \mid \alpha \in J\rangle$ *in* **Set**. *Then its initial lift* (*i.e.*, *weak structure*) *in* **GenTop** *is*

$$(X,\mu) := \bigg(X, \bigg\{ \bigcup_{\alpha \in J} \varphi_\alpha^{-1}(M_\alpha) \mid M_\alpha \in \nu_\alpha \bigg\}\bigg).$$

PROOF. Clearly $(X,\mu)$ is a GTS, and $\varphi_\alpha$ becomes (underlies) a continuous map $f_\alpha\colon (X,\mu) \to (Y_\alpha, \nu_\alpha)$ in **GenTop**, for each $\alpha \in J$.

We turn to show the universality property. Let us have for some $(Z,\lambda)$ morphisms $\langle g_\alpha\colon (Z,\varrho)\to(Y_\alpha,\nu_\alpha)\mid \alpha\in J\rangle$ (a source in **GenTop**) such that

$$Ug_\alpha=\varphi_\alpha h \quad \text{for all } \alpha\in J$$

for some $h\colon UZ\to X$. Then

$$\varrho\supset g_\alpha^{-1}(\nu_\alpha)=(Ug_\alpha)^{-1}(\nu_\alpha)=(\varphi_\alpha h)^{-1}(\nu_\alpha)=h^{-1}\varphi_\alpha^{-1}(\nu_\alpha)\,.$$

Hence

$$\varrho\supset\bigcup_{\alpha\in J}h^{-1}\varphi_\alpha^{-1}(\nu_\alpha)=h^{-1}\bigcup_{\alpha\in J}\varphi_\alpha^{-1}(\nu_\alpha)\,.$$

Thus $h^{-1}$ maps $\mu$ into $\varrho$, hence $h=Uh'$ for a continuous map $h'\colon(Z,\varrho)\to(X,\mu)$. □

PROPOSITION 3.3. *Let $X$ be a set. Let us have a sink $\langle\psi_\alpha\colon U(Y_\alpha,\nu_\alpha)=Y_\alpha\to X\mid\alpha\in J\rangle$ in* **Set**. *Then its final lift* (*strong structure*) *in* **GenTop** *is*

$$(X,\mu):=\bigl(X,\{M\subset X\mid\forall\,\alpha\in J\ \psi_\alpha^{-1}(M)\in\nu_\alpha\}\bigr)\,.$$

PROOF. Clearly $(X,\mu)$ is a GTS, and $\varphi_\alpha$ becomes (underlies) a continuous map $f_\alpha\colon(Y_\alpha,\nu_\alpha)\to(X,\mu)$ in **GenTop**, for each $\alpha\in J$.

We turn to show the universality property. Let us have for some $(Z,\lambda)$ morphisms $\langle g_\alpha\colon(Y_\alpha,\nu_\alpha)\to(Z,\varrho)\mid\alpha\in J\rangle$ (a sink in **GenTop**) such that

$$Ug_\alpha=h\psi_\alpha \quad \text{for all } \alpha\in J$$

for some $h\colon X\to UZ$. Then

$$\nu_\alpha\supset g_\alpha^{-1}(\varrho)=(Ug_\alpha)^{-1}(\varrho)=(h\psi_\alpha)^{-1}(\varrho)=\psi_\alpha^{-1}h^{-1}(\varrho)\,.$$

Hence

$$\forall\,\alpha\in J\ h^{-1}(\varrho)\subset\Bigl\{M\subset X\mid\psi_\alpha^{-1}(M)\in\nu_\alpha\Bigr\}$$

i.e.,

$$h^{-1}(\varrho)\subset\Bigl\{M\subset X\mid\forall\,\alpha\in J\ \psi_\alpha^{-1}(M)\in\nu_\alpha\Bigr\}\,.$$

Thus $h=Uh'$ for a continuous map $h'\colon(X,\mu)\to(Z,\varrho)$. □

We recall 3.1 for notations in the following proofs.

PROPOSITION 3.4. *Let $X$ be a set. Let us have a source*

$$\langle\varphi_\alpha\colon X\to U(Y_\alpha,d_\alpha)=Y_\alpha\mid\alpha\in J\rangle$$

*in* **Set**. *Then its initial lift* (*i.e., weak structure*) *in* **GenTop** *is* $(X,c)$, *where for* $A \subset X$ *we have*

$$cA := \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha(A).$$

PROOF. Obviously $c$ is increasing and monotonous. To show idempotence of $c$, it is sufficient to show $c^2(A) \subset c(A)$ for each $A \subset X$, i.e.,

$$\bigcap_{\beta \in J} \varphi_\beta^{-1} d_\beta \varphi_\beta \Big[ \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha(A) \Big] \subset \bigcap_{\beta \in J} \varphi_\beta^{-1} d_\beta \varphi_\beta(A).$$

It suffices to show the inclusion (3.2) obtained by deleting here $\bigcap_{\beta \in J}$ from both sides. We have

$$\varphi_\beta^{-1} d_\beta \varphi_\beta \Big[ \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha(A) \Big] \subset \varphi_\beta^{-1} d_\beta \varphi_\beta \big[ \varphi_\beta^{-1} d_\beta \varphi_\beta(A) \big]$$
$$\subset \varphi_\beta^{-1} d_\beta d_\beta \varphi_\beta(A) = \varphi_\beta^{-1} d_\beta \varphi_\beta(A)\,. \tag{3.2}$$

Here we used $\big[ \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha(A) \big] \subset \varphi_\beta^{-1} d_\beta \varphi_\beta(A)$ and $\varphi_\beta \varphi_\beta^{-1}(B_\beta) \subset B_\beta$ for $B_\beta \subset Y_\beta$ and $d_\beta^2 = d_\beta$. Thus we have obtained the claimed inclusion (3.2).

Let $A \subset X$. Then

$$cA = \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha(A) \subset \varphi_\alpha^{-1} d_\alpha \varphi_\alpha(A)$$

shows that $\varphi_\alpha \colon X \to Y_\alpha$ becomes (underlies) a continuous map $f_\alpha \colon (X,c) \to (Y, d_\alpha)$ for each $\alpha \in J$.

We turn to show the universality property. Let us have for some $(Z,e)$ morphisms $\langle g_\alpha \colon (Z,e) \to (Y_\alpha, d_\alpha) \mid \alpha \in J \rangle$ (a source in **GenTop**) such that

$$U g_\alpha = \varphi_\alpha h \quad \text{for all } \alpha \in J$$

for some $h \colon Z \to X$. We have to show that $h$ becomes (underlies) a continuous map $h' \colon (Z,e) \to (X,c)$. That is, we have to show for $C \subset Z$ that

$$eC \subset h^{-1} chC = h^{-1} \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha hC = h^{-1} \Big( \bigcap_{\alpha \in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha \Big) hC\,.$$

For $C \subset Z$ we have, for each $\alpha \in J$, by continuity of $g_\alpha$ that

$$eC \subset g_\alpha^{-1} d_\alpha g_\alpha C = (U g_\alpha)^{-1} d_\alpha (U g_\alpha) C$$
$$= (\varphi_\alpha h)^{-1} d_\alpha (\varphi_\alpha h) C = h^{-1} (\varphi_\alpha^{-1} d_\alpha \varphi_\alpha) hC\,.$$

Therefore we have

$$eC \subset \bigcap_{\alpha\in J} h^{-1}(\varphi_\alpha^{-1} d_\alpha \varphi_\alpha)hC = h^{-1}\Big(\bigcap_{\alpha\in J} \varphi_\alpha^{-1} d_\alpha \varphi_\alpha\Big)hC = h^{-1}chC\,,$$

which shows that $h = Uh'$ for a continuous map $h'\colon (Z,e) \to (X,c)$. This shows the universality property of $(X,c)$ and ends the proof of the proposition. $\square$

For the next proposition we will need the following concept.

Definition 3.5 (for the special case of **PrTop** [6], Ch. 16B, Topological modifications, in full generality [19], pp. xiv-xv, Ch. 4.6, Idempotent hull and weakly hereditary core, and [5], p. 74, Definition 6.9 and p. 87, Proposition 7.6). Let $(X,\gamma)$ be a CS (with $\gamma$ not necessarily idempotent). Let $\lambda$ be any ordinal (also $\kappa$ will denote here ordinals) and $A \subset X$. Then $\gamma^\lambda(A)$ is defined in the following way. We let $\gamma^0(A) := A$. For $\lambda = \kappa + 1$ we let $\gamma^\lambda(A) := \gamma(\gamma^\kappa(A))$. For $\lambda$ a limit ordinal we let $\gamma^\lambda(A) := \bigcup_{\kappa<\lambda}\gamma^\kappa(A)$. Then for any $A \subset X$ there is a smallest ordinal $\lambda_0$ such that $\gamma^{\lambda_0+1}(A) = \gamma^{\lambda_0}(A)$. (Clearly $\lambda_0$ has a cardinality at most $|X|$.) Then the operator $A \mapsto \gamma^\infty(A) := \gamma^{\lambda_0}(A)$ is called the *idempotent hull* (or *transfinite iteration*) *of* $\gamma$. Clearly $\gamma^\infty$ is increasing, monotonous and idempotent. Namely, it equals $\gamma^{\lambda_1}$, where $\lambda_1$ is the initial ordinal of the cardinal successor of $|X|$ for $X$ infinite, or $\lambda_1 = \omega$ for $X$ finite, for which these properties are obvious.

We will write id for the *identity operation* (its domain will be clear from the context).

Proposition 3.6. *Let $X$ be a set. Let us have a sink $\langle \psi_\alpha\colon U(Y_\alpha, d_\alpha) = Y_\alpha \to X \mid \alpha \in J\rangle$ in* **Set**. *Then its final lift* (*strong structure*) *in* **GenTop** *is* $(X,c)$, *where*

*$c$ is the idempotent hull $\gamma^\infty$ of the closure operator $\gamma$*

$$\text{on } P(X) \text{ given by } \gamma A := \Big[\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}(A)\Big] \cup A.$$

Proof. Clearly $\gamma$ is increasing and monotonous. Hence its idempotent hull is increasing, monotonous and idempotent. Let $A \subset X$. Then

$$cA = \Big[\Big[\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}\Big] \cup \mathrm{id}\Big]^\infty (A) \supset \Big[\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}(A)\Big] \cup A$$

$$\supset \bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}(A) \supset \psi_\alpha d_\alpha \psi_\alpha^{-1}(A)$$

shows that $\psi_\alpha$ becomes (underlies) a continuous map $f_\alpha\colon (Y_\alpha, d_\alpha) \to (X, c)$, for each $\alpha \in J$.

We turn to the universality property. Let us have for some $(Z, e)$ morphisms $\langle g_\alpha\colon (Y_\alpha, d_\alpha) \to (Z, e) \mid \alpha \in J\rangle$ (a sink in **GenTop**) such that

$$Ug_\alpha = h\psi_\alpha \quad \text{for all } \alpha \in J$$

for some $h\colon X \to Z$. We have to show that $h$ becomes (underlies) a continuous map $h'\colon (X, c) \to (Z, e)$. That is, we have to show for $C \subset Z$ that

$$eC \supset hch^{-1}C = h\left[\left[\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}(A)\right] \cup \mathrm{id}\right]^\infty h^{-1}C\,.$$

For $C \subset Z$ we have, for each $\alpha \in J$, by continuity of $g_\alpha$ that

$$\begin{aligned} eC \supset g_\alpha d_\alpha g_\alpha^{-1}C &= (Ug_\alpha) d_\alpha (Ug_\alpha)^{-1}C \\ &= (h\psi_\alpha) d_\alpha (h\psi_\alpha)^{-1}C = h(\psi_\alpha d_\alpha \psi_\alpha^{-1})h^{-1}C\,. \end{aligned}$$

Therefore we have

$$eC \supset \bigcup_{\alpha\in J} h(\psi_\alpha d_\alpha \psi_\alpha^{-1})h^{-1}C = h\left(\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}\right)h^{-1}C\,,$$

which implies together with $C \subset eC$ that for $\gamma = [\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}] \cup \mathrm{id}$ we have

$$h\gamma h^{-1}C = h\left(\left[\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}\right] \cup \mathrm{id}\right)h^{-1}C \tag{3.3}$$

$$= h\left(\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1} h^{-1}C\right) \cup hh^{-1}C \subset \left[h\left(\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1}\right)h^{-1}C\right] \cup C \subset eC\,.$$

Applying (3.3) to $eC$ rather than $C$, we obtain

$$h\gamma h^{-1}eC \subset e(eC) = eC\,. \tag{3.4}$$

Now we show by transfinite induction that for each ordinal $\lambda$ and any $C \subset Z$ we have

$$h\gamma^\lambda h^{-1}C \subset eC\,. \tag{3.5}$$

For $\lambda = 0$ (3.5) is evident (since $hh^{-1}C \subset C \subset eC$), and for $\lambda = 1$ this is (3.3)). Now let for $\lambda = \kappa + 1$ (3.5) hold for the ordinal $\kappa$ and any $C \subset Z$, and we prove it for $\lambda$ and any $C \subset Z$. We have

$$h\gamma^{\kappa+1}h^{-1}C = h\gamma\, \mathrm{id}\, \gamma^\kappa h^{-1}C \subset h\gamma h^{-1}(h\gamma^\kappa h^{-1}C) \subset h\gamma h^{-1}(eC) \subset eC\,.$$

In the first inclusion we used that $h^{-1}h$ was increasing, in the second inclusion we used the induction hypothesis and in the third inclusion we used (3.4). Now let $\lambda$ be a limit ordinal, and let us suppose that we know (3.5) for all ordinals $\kappa < \lambda$ and for all $C \subset Z$. Then

$$h\gamma^{\lambda}h^{-1}C = h \bigcup_{\kappa<\lambda} \gamma^{\kappa}h^{-1}C = \bigcup_{\kappa<\lambda} (h\gamma^{\kappa}h^{-1}C) \subset \bigcup_{\kappa<\lambda} eC = eC\,.$$

Therefore, by transfinite induction, for each ordinal $\lambda$ we have (3.5). In particular, for the ordinal $\lambda_0$ associated to the set $C$ (cf. Definition 3.5) and to the operation $\gamma$ we obtain

$$hch^{-1}C = h\gamma^{\infty}h^{-1}C = h\gamma^{\lambda_0}h^{-1}C \subset eC\,,$$

which shows that $h = Uh'$ for a continuous map $h'\colon (X, c) \to (Z, e)$. This shows the universality property of $(X, c)$, and ends the proof of the proposition. □

Proof. The proof of Theorem 3.1 follows from any of Propositions 3.2, 3.3, 3.4, 3.6. □

Remark 3.7. In Proposition 3.6 it is in fact necessary to use transfinite iteration over all ordinals. As an example, let us take the set $\{0, 1\}$, and for some set $J$ we consider $X := \{0, 1\}^J$. We consider the following sink to $X$. We let

$$\{Y_\alpha \mid \alpha \in J\} = \Big\{Y \subset X \mid |Y| = 2,\ Y = \{y_1, y_2\}, \text{ and } y_1 \text{ and } y_2 \text{ differ in exactly one coordinate}\Big\}.$$

On each $Y_\alpha$ we consider the indiscrete topology, i.e., $d_\alpha\emptyset = \emptyset$ and the closure of a non-empty subset is $Y_\alpha$. The maps $\psi_\alpha$ are the natural injections $Y_\alpha \to X$. We investigate the strong structure on $X$ associated to the sink $\langle \psi_\alpha\colon Y_\alpha \to X \mid \alpha \in J\rangle$. Let $x_0 \in X$ be the point with all coordinates 0. We write for $A \subset X$

$$\gamma A := \Big[\bigcup_{\alpha\in J} \psi_\alpha d_\alpha \psi_\alpha^{-1} A\Big] \cup A\,.$$

Then $\gamma\{x_0\} = \{x \in X \mid x \text{ has at most one non-zero coordinate}\}$. Similarly, $\gamma^2\{x_0\} = \{x \in X \mid x \text{ has at most two non-zero coordinates}\}$, and, in general, for any ordinal $\lambda$, $\gamma^{\lambda}\{x_0\} = \{x \in X \mid x \text{ has at most } |\lambda| \text{ non-zero coordinates}\}$. Therefore the smallest ordinal $\lambda_0$ such that $\gamma^{\lambda_0+1}\{x_0\} = \gamma^{\lambda_0}\{x_0\}$ is the initial ordinal belonging to the cardinal $|J|$, that can be arbitrarily large.

Now we give some corollaries to Theorem 3.1 and Propositions 3.2, 3.3, 3.4, 3.6, to some particular kinds of initial and final lifts (weak and strong structures), and of limits and colimits.

If the source consists of a single map $\varphi\colon X \to U(Y,\nu) = Y$, and $\varphi$ is injective, then the weak structure is called *subspace of* $(Y,\nu)$. Here we may suppose that $\varphi$ *is actually the embedding of a subset*, which we will suppose. If the sink consists of a single map $\psi\colon U(Y,\nu) = Y \to X$, and $\psi$ is surjective, then the strong structure is called *quotient of* $(Y,\nu)$.

For $(Y,\nu)$ a GTS and $X \subset Y$, we write $(Y,\nu)|X := (X,\nu|X)$, where for $\mathcal{A} \subset P(X)$ we have $\mathcal{A}|X = \{A \cap X \mid A \in \mathcal{A}\}$, Subsection 2.3 (cf. also [40]).

Corollary 3.8 (for supratopologies cf. [29], § 1, Example 2, and [26], Proposition 4.10, or [26], Theorem 4.4 and Proposition 4.9. Cf. also [40]). *Subspaces exist in the category* **GenTop**, *and they can be given as follows. For* $(Y,\nu)$ *and* $i\colon X \to Y$ *an embedding of a subset, the subspace structure on* $X$ *is* $(Y,\nu)|X = (X,\nu|X)$. *For* $(Y,d)$ *and* $i\colon X \to Y$ *an embedding of a subset, the subspace structure* $(X,c)$ *on* $X$ *is given by* $c(A) := d(A) \cap X$ *for* $A \subset X$.

Corollary 3.9. *Quotient spaces exist in the category* **GenTop**, *and they can be given as follows. For* $(Y,\nu)$ *and an onto map* $q\colon Y \to X$ *in the category* **Set** *the quotient space structure on* $X$ (*by the map* $q$) *is* $(X, \{M \subset X \mid q^{-1}(M) \in \nu\})$. *For* $(Y,d)$ *and an onto map* $q\colon Y \to X$ *in the category* **Set** *the quotient space structure on* $X$ (*by the map* $q$) *is* $(X,c)$, *where* $c$ *is the idempotent hull of the closure operator on* $P(X)$ *given by* $A \mapsto qdq^{-1}(A) \cup A$.

Corollary 3.10 (for supratopologies cf. [26], Proposition 4.10, or [26], Theorem 4.4 and Proposition 4.9). *Products exist in the category* **GenTop**, *and they can be given as follows. For* $\langle (Y_\alpha,\nu_\alpha) \mid \alpha \in J\rangle$ *the product is* $(\prod_{\alpha\in J} Y_\alpha, \mu)$ (*together with the natural projections* $\pi_\alpha$), *where* $\mu := \left\{ \bigcup_{\alpha\in J} \pi_\alpha^{-1}(N_\alpha) \mid N_\alpha \in \nu_\alpha \right\}$. *For* $\left\langle (Y_\alpha,d_\alpha) \mid \alpha \in J \right\rangle$ *the product is* $(\prod_{\alpha\in J} Y_\alpha, c)$, *where for* $M \subset \prod_{\alpha\in J} X_\alpha$ *we have* $c(M) := \prod_{\alpha\in J} d_\alpha \pi_\alpha M$.

Definition 3.11. The *Császár product of GTS's* is defined as follows. Let $J$ be an index set, let $Y_\alpha$ for $\alpha \in J$ be sets, and $X = \prod_{\alpha\in J} Y_\alpha$. Suppose that, for $\alpha \in J$, $\nu_\alpha$ is a GT on $Y_\alpha$. Let $\mathcal{B} := \left\{ \prod_{\alpha\in J} N_\alpha \mid N_\alpha \in \nu_\alpha \right.$ and, with the exception of finitely many indices $\alpha$, $\left. N_\alpha = M_{\nu_\alpha} \right\}$, where $M_{\nu_\alpha} := \bigcup_{B\in\nu_\alpha} B$. The GT on $X$ having $\mathcal{B}$ as a base is called the Császár product of the GTS's $\langle (Y_\alpha,\nu_\alpha) \mid \alpha \in J\rangle$.

Remark 3.12. Obviously, Császár's products of strong GTS's are finer than the product GTS's (they have the same underlying set $\prod_{\alpha\in J} Y_\alpha$), but if strongness is omitted, they are in general incomparable, even for $|J| = 2$.

Also the (categorical) product of GT's in general is not a topology, even if the factors are topological spaces, but the Császár product is the topological product in this last case.

Categorical products coincide with the Császár product only in some particular cases. The empty Császár product is $(X_0, \{\emptyset, X_0\})$ with $|X_0| = 1$, while the empty product is $(X_0, \{\emptyset\})$ with $|X_0| = 1$. For $|J| = 1$ both the Császár product and the product equal the unique factor. If some $Y_\alpha$ is empty, both the Császár product and the product are $(\emptyset, \{\emptyset\})$. For $|J| \geqq 2$ and $\forall\, \alpha \in J\ Y_\alpha \neq \emptyset$, the equality of the Császár product and the product is equivalent to that each $(Y_\alpha, \nu_\alpha)$ is strong, and there is at most one $\alpha \in J$ such that $\nu_\alpha \neq \{\emptyset, Y_\alpha\}$.

Corollary 3.13. *Sums exist in the category* **GenTop**, *and they can be given as follows. For* $\langle (Y_\alpha, \nu_\alpha) \mid \alpha \in J \rangle$ *the sum is* $(\coprod_{\alpha \in J} Y_\alpha, \mu)$ *(together with the natural injections* $\iota_\alpha$*), where* $\mu := \left\{ \bigcup_{\alpha \in J} \iota_\alpha N_\alpha \mid \forall\, \alpha \in J\ N_\alpha \in \nu_\alpha \right\}$. *For* $\langle (Y_\alpha, d_\alpha) \mid \alpha \in J \rangle$ *the sum is* $\left( \coprod_{\alpha \in J} Y_\alpha, c \right)$, *where for* $N_\alpha \subset Y_\alpha$ *we have* $c\left( \bigcup_{\alpha \in J} \iota_\alpha N_\alpha \right) := \bigcup_{\alpha \in J} \iota_\alpha d_\alpha N_\alpha$.

Corollary 3.14 (for supratopologies cf. [26], Proposition 4.1 and Theorem 4.4). *Let* $X$ *be a set. Then all generalized topologies on* $X$ *form a complete lattice, with* $(X, \mu) \leqq (X, \nu)$ *meaning that the identical map of* $X$ *is (underlies) a continuous map* $(X, \nu) \to (X, \mu)$. *The union, or intersection of a set of generalized topologies* $\{(X, \nu_\alpha) \mid \alpha \in J\}$ *is* $(X, \mu)$, *where* $\mu := \left\{ \bigcup_{\alpha \in J} N_\alpha \mid N_\alpha \in \nu_\alpha \right\}$, *or* $\mu := \bigcap_{\alpha \in J} \nu_\alpha$, *respectively. The union, or intersection of a set of generalized topologies* $\left\{ (X, d_\alpha) \mid \alpha \in J \right\}$ *is* $(X, c)$, *where* $c(A) := \bigcap_{\alpha \in J} d_\alpha(A)$ *for* $A \subset X$, *or* $c$ *is the idempotent hull of the closure operator* $A \mapsto \left[ \bigcup_{\alpha \in J} d_\alpha(A) \right] \cup A$ *for* $A \subset X$, *respectively.*

In the last formula "union with $A$" is necessary only for $J = \emptyset$.

## 4. $T_{3.5}$, normal, compact, Lindelöf GTS's and the analogue of Tychonoff's embedding theorem

Further on, products of GTS's will be meant in the categorical sense, cf. Subsection 2.4. Császár's products will not be used.

Császár [12] introduced a useful *GT on the set of real numbers* $\mathbb{R}$ as follows. It has as a base

$$\beta := \left\{ (-\infty, s) \mid s \in \mathbb{R} \right\} \cup \left\{ (t, \infty) \mid t \in \mathbb{R} \right\} .$$

This is a strong GT.

Henceforth, *we assign the notation* $\gamma$ *just for this GT.* We believe that this GTS is the appropriate choice for $\mathbb{R}$ as a GTS. Indeed, $(\mathbb{R}, \gamma)$ as a GTS has a similar role as the standard topology on $\mathbb{R}$ in general topology.

Similarly, *we use the notation* $([0,1],\gamma_0)$ *for the subspace* $[0,1]$ *of* $\mathbb{R}$ (i.e., $\gamma_0 = \gamma|[0,1]$, cf. Corollary 3.8).

Remark 4.1 ([3], Remark after Example 2.4). The GTS $(\mathbb{R},\gamma)$ is $T_4$. (Namely a simple discussion shows that its closed sets are just the convex subsets of $\mathbb{R}$ which are closed in the usual sense. Then two non-empty disjoint closed sets can be included into two disjoint halflines which are open in the usual sense. The $T_1$ property is evident.) By [12], Proposition 2.5 normality is closed-hereditary, as well as the $T_1$ property, hence $T_4$ property of $(\mathbb{R},\gamma)$ implies the $T_4$ property of $([0,1],\gamma_0)$.

Remark 4.2. Let $n \geqq 2$. We suppose that the appropriate GTS analogue of the $n$-dimensional real vector space is not the $n$'th power of $(\mathbb{R},\gamma)$ (this depends on some preassigned representation of the $n$-dimensional real vector space as a direct sum of 1-dimensional subspaces). We suppose the proper choice should be the GT with base all open (in the usual sense) halfspaces. Then the closed sets are exactly the closed (in the usual sense) convex subsets, and the associated closure operator is the closed (in the usual sense) convex hull of a subset (cf. e.g., [4], §1, 3, [20], V.2.7 Theorem 10, [41], Theorem 1.3.7). These have been extensively investigated in geometry and functional analysis, cf. e.g. the just cited three books. (For weak topologies on locally convex topological vector spaces there is an analogue of this construction: we consider inverse images of $(r,\infty)$, where $r \in \mathbb{R}$, by continuous linear functionals, as basic generalized open sets, and then the generalized closed sets are still the closed (in the usual sense) convex sets, and the associated closure operator is the closed convex hull of a subset. For this, including the necessary definitions, cf. [20].)

Observe that this space is not normal for $n \geqq 2$: the closed sets $F_1 :=$ $x_1 \dots x_{n-1}$-coordinate hyperplane and $F_2 := \{(x_1,\dots,x_n) \mid \forall\, i \in \{1,\dots,n\}$ $x_i > 0,\ x_1 \dots x_n \geqq 1\}$ are disjoint closed subsets which cannot be included into two disjoint open subsets. In fact, non-empty disjoint open sets $M_1, M_2$ are unions of open (in the usual sense) half-spaces $\{M_{1,\alpha} \mid \alpha \in A\}$ and $\{M_{2,\beta} \mid \beta \in B\}$. Then each $M_{1,\alpha}$ and each $M_{2,\beta}$, being disjoint, have parallel boundary hyperplanes. So $M_1, M_2$ also are open (in the usual sense) half-spaces, having parallel boundary hyperplanes. Therefore we may suppose that $|A| = |B| = 1$, $A = \{\alpha_0\}$, $B = \{\beta_0\}$, and the boundary hyperplanes of $M_{1,\alpha_0}$ and $M_{2,\beta_0}$ (in the usual sense) are parallel to $F_1$. If $F_1 \subset M_{1,\alpha_0}$, then $M_{1,\alpha_0}$ contains a parallel slab containing some usual open $\varepsilon$-neighbourhood $F_{1,\varepsilon}$ of $F_1$, and then $M_1 \cap M_2 \supset F_{1,\varepsilon} \cap F_2 \neq \emptyset$.

By the GTS $([0,1],\gamma_0)$ [12] exhibited a generalization of the Urysohn lemma for normal GTS's, as follows.

Theorem 4.3 ([12], Theorem 3.3). *Let* $(X,\mu)$ *be a normal GTS, and* $F, F' \subset X$ *be disjoint* $\mu$*-closed sets. Then there exists a continuous function* $f\colon (X,\mu) \to ([0,1],\gamma_0)$ *such that* $f(x) = 0$ *for* $x \in F$ *and* $f(x) = 1$ *for* $x \in F'$.

By the above discussions we are able to define and investigate $T_{3.5}$ spaces.

DEFINITION 4.4. A GTS $(X, \mu)$ is *completely regular* if for each $x \in X$ and each $\mu$-closed set $F$ of $X$ not containing $x$, there is a continuous function $f \colon (X, \mu) \to ([0,1], \gamma_0)$ such that $f(x) = 0$ and $f(F) \subset \{1\}$. We say that $(X, \mu)$ is $T_{3.5}$ (or *Tychonoff*) if it is a completely regular $T_1$ space.

We will write $T_{3.5}$ spaces rather than Tychonoff spaces. In Definition 4.4 we can write $f(F) = \{1\}$ if we require $F \neq \emptyset$, which we may suppose. (For $F = \emptyset$ $(X, \mu)$ is strong, and then we can take the identically 0 function to $([0,1], \gamma_0)$, which is now continuous.) In particular, $T_{3.5}$ is equivalent to completely regular $T_0$, since by Subsection 2.6, a regular $T_0$ space is $T_1$. Clearly, every $T_{3.5}$, or completely regular space is a $T_3$, or regular space (cf. Subsection 2.6), moreover, by Theorem 4.3, every $T_4$ space is a $T_{3.5}$ space. Also, as in the case of regularity (cf. Subsection 2.6), a GTS of the form $(X, \{\emptyset\})$ is vacuously completely regular, but complete regularity of a GTS $(X, \mu)$ with $\mu \neq \{\emptyset\}$ implies its regularity, and hence its strongness (cf. Subsection 2.6).
We are going to exhibit a generalized version of Tychonoff's embedding theorem to obtain a necessary and sufficient condition for a GTS to be a $T_{3.5}$ space.

EXAMPLE 4.5. Consider the GTS $([0,1], \gamma_0)$. Then

$$\gamma_0 = \big\{ \emptyset, [0,1] \big\} \cup \big\{ [0,p) \mid p \in (0,1] \big\}$$
$$\cup \big\{ (q,1] \mid q \in [0,1) \big\} \cup \big\{ [0,r) \cup (s,1] \mid r,s \in (0,1),\ r \leqq s \big\}$$

By Remark 4.1 $([0,1], \gamma_0)$ is $T_4$, hence it is a $T_{3.5}$ GTS as well.

DEFINITION 4.6. A source $\langle f_\alpha \colon X \to Y_\alpha \mid \alpha \in J \rangle$ in **GenTop**, **Set** is *point-separating* (also called *monosource*) if for $x_1, x_2 \in X$ distinct there is an $\alpha \in J$ such that $f_\alpha(x_1) \neq f_\alpha(x_2)$.

PROPOSITION 4.7 (for supratopologies, for $T_2$ for subspaces cf. [29], Section 3, Observation 1, and for regularity cf. Theorem 4.4). *Let $\langle f_\alpha \colon X \to U(Y_\alpha, \nu_\alpha) = Y_\alpha \mid \alpha \in J \rangle$ be a source in* **Set**.

(1) *If each $(Y_\alpha, \nu_\alpha)$ is regular or completely regular, then also the initial (weak) GTS structure on $X$ defined by this source is regular or completely regular.*

(2) *If each $(Y, \nu_\alpha)$ is $T_0$, $T_1$, $T_2$, $T_3$ or $T_{3.5}$, and still $\langle f_\alpha \mid \alpha \in J \rangle$ is point-separating, then also the initial (weak) GTS structure on $X$ defined by this source is $T_0$, $T_1$, $T_2$, $T_3$ or $T_{3.5}$. In particular, subspaces and products of $T_0$, $T_1$, $T_2$, regular, $T_3$, completely regular or $T_{3.5}$ GTS's are $T_0$, $T_1$, $T_2$, regular, $T_3$, completely regular or $T_{3.5}$.*

PROOF. In case (1) we give the proof for complete regularity, and in case (2) for $T_1$. All other proofs are analogous.

We prove (1) for complete regularity. Suppose we have a source $\langle \varphi_\alpha \colon X \to U(Y_\alpha, \nu_\alpha) = Y \mid \alpha \in J \rangle$ in **Set**, with all $(Y_\alpha, \nu_\alpha)$ completely regular. By Proposition 3.2 the weak structure w.r.t. this source is the GT on $X$ having as base $\bigcup_{\alpha \in J} \varphi_\alpha^{-1}(\nu_\alpha)$. Therefore it suffices to prove that for $x \in X$, $x \in N := \varphi_\alpha^{-1}(N_\alpha)$, $N_\alpha \in \nu_\alpha$, $\alpha \in J$ and $F := X \setminus N$ there is a continuous function $h \colon (X, \mu) \to ([0,1], \gamma_0)$ such that $h(x) = 0$ and $h(F) \subset \{1\}$, i.e., $h^{-1}[0,1) \subset N$.

Observe that $\varphi_\alpha(x) \in N_\alpha$. We have that $\varphi_\alpha$ becomes (underlies) a continuous map $f_\alpha \colon (X, \mu) \to (Y_\alpha, \nu_\alpha)$ in **GenTop**. By complete regularity of $(Y_\alpha, \nu_\alpha)$ there is a continuous function $g_\alpha \colon (Y_\alpha, \nu_\alpha) \to ([0,1], \gamma_0)$ such that $g_\alpha(f_\alpha(x)) = 0$ and $g_\alpha^{-1}[0,1) \subset N_\alpha$. Then we define $h := g_\alpha \circ f_\alpha$ which satisfies the claimed properties.

We prove (2) for $T_1$. Suppose that we have a source like above, which is additionally point-separating. We use the notation $f_\alpha$ like above. Let $x_1 \neq x_2$ be in $X$. Then there exists an index $\alpha \in J$ such that $f_\alpha(x_1) \neq f_\alpha(x_2)$. Then by the $T_1$ property of $Y_\alpha$ there is some open set $N_\alpha$ in $(Y_\alpha, \nu_\alpha)$ such that $f_\alpha(x_1) \in N_\alpha \not\ni f_\alpha(x_2)$. Then $x_1 \in \varphi_\alpha^{-1} N_\alpha \not\ni x_2$, and by continuity of $f_\alpha$ we have $f_\alpha^{-1} N_\alpha \in \mu$, proving the claimed $T_1$ property of $(X, \mu)$. $\square$

COROLLARY 4.8. *The GTS on the $n$-dimensional real vector space in Remark* 4.2 *is* $T_{3.5}$.

PROOF. This GTS is the weak structure w.r.t. all non-0 linear functionals, as functions to the GTS $(\mathbb{R}, \gamma)$. $\square$

Now, we are ready to prove the following variant of Tychonoff's embedding theorem that characterizes $T_{3.5}$ GTS's. As for topological spaces, *a map* $f \colon (X, \mu) \to (Y, \nu)$ *in* **GenTop** *is dense*, if $Z := f(X)$ *is dense in* $(Y, \nu)$, i.e., if the closure of $Z$ in $(Y, \nu)$ equals $Y$.

We begin with the analogue of the embedding lemma in topology. We call $f \colon (X, \mu) \to (Y, \nu)$ in **GenTop** *open*, if $f(\mu) \subset \nu$.

LEMMA 4.9. *Let $(X, \mu)$ be a GTS, and let us have a point-separating source (monosource) $\langle f_\alpha \colon (X, \mu) \to (Y_\alpha, \nu_\alpha) \mid \alpha \in J \rangle$ in* **GenTop**. *Suppose that for any $x \in M \in \mu$ there exists an $\alpha \in J$ and $f_\alpha(x) \in N_\alpha \in \nu_\alpha$ such that $(x \in)\ f_\alpha^{-1}(N_\alpha) \subset M$. Then the mapping $f \colon (X, \mu) \to \prod_{\alpha \in J} (Y_\alpha, \nu_\alpha)$ defined by $f(x) := \langle f_\alpha(x) \mid \alpha \in J \rangle$ satisfies that $f$ is a homeomorphism to its image $\big(\prod_{\alpha \in J} (Y_\alpha, \nu_\alpha)\big) | f(X)$.*

PROOF. The proof is analogous to the case of topological spaces. Recall that $\bigcup_{\alpha \in J} \pi_\alpha^{-1}(\nu_\alpha)$ is a base for $\prod_{\alpha \in J} (Y_\alpha, \nu_\alpha)$ (where the $\pi_\alpha$'s are the natural projections). Also, by hypothesis, $f$ is injective.

The hypothesis means that $\{ f_\alpha^{-1}(N_\alpha) \mid \alpha \in J,\ N_\alpha \in \nu_\alpha \}$ is a base of $\mu$. We factorize $f\colon (X,\mu) \to \prod_{\alpha\in J}(Y_\alpha,\nu_\alpha)$ across the image $f(X)$ as

$$(X,\mu) \xrightarrow{F} \Big( \prod_{\alpha\in J} (Y_\alpha,\nu_\alpha) \Big) \big| f(X) \hookrightarrow \prod_{\alpha\in J} (Y_\alpha,\nu_\alpha).$$

We are going to show that $F$ is open.

It suffices to show that for all $\alpha \in J$, and for all $N_\alpha \in \nu_\alpha$ we have that $f f_\alpha^{-1}(N_\alpha)$ is open in $\big( \prod_{\alpha\in J}(Y_\alpha,\nu_\alpha)\big) \,|f(X)$. We have evidently

$$f f_\alpha^{-1}(N_\alpha) = \langle f_\beta \mid \beta \in J\rangle f_\alpha^{-1}(N_\alpha) \subset f(X) \cap \pi_\alpha^{-1}(N_\alpha).$$

Next we show the converse inclusion. Let $f(x) \in f(X) \cap \pi_\alpha^{-1}(N_\alpha)$. Then $f_\alpha(x) = \pi_\alpha \langle f_\beta \mid \beta \in J\rangle(x) = \pi_\alpha f(x) \in N_\alpha$, hence $x \in f_\alpha^{-1}(N_\alpha)$, and $f(x) \in f f_\alpha^{-1}(N_\alpha)$. This shows the converse inclusion, hence $f f_\alpha^{-1}(N_\alpha) = f(X) \cap \pi_\alpha^{-1}(N_\alpha)$ is a basic open set in $\big( \prod_{\alpha\in J}(Y_\alpha,\nu_\alpha)\big) \,|f(X)$.

Thus $F\colon (X,\mu) \to \big( \prod_{\alpha\in J}(Y_\alpha,\nu_\alpha)\big) \,|f(X)$ is open. Thus $F$ is continuous, open and bijective, hence is a homeomorphism. □

Definition 4.10. For $(X,\mu)$ a GTS, we define its *$T_0$-reflection* as follows. We define $x, y \in X$ equivalent, written $x \equiv y$, by $\forall\, M \in \mu$ $(x \in M \iff y \in M)$. This is an equivalence relation, and we let $Y$ be the quotient of $X$ w.r.t. this equivalence relation. Hence we have an onto map $q\colon X \to Y$, where $q(x)$ is the equivalence class of $x$. The *quotient space structure on* $Y$ (cf. Corollary 3.9), *say,* $(Y,\nu)$, *is the $T_0$-reflection of* $(X,\mu)$. Then $(Y,\nu)$ is $T_0$.

Proposition 4.11. (1) *Definition* 4.10 *is correct. Also,* $(X,\mu)$ *is the initial* (*weak*) *structure associated to the one-element source* $Uq\colon X \to U(X,\mu)$ *in* **Set**. *Moreover, the map* $q\colon (X,\mu) \to (Y,\nu)$ *has the following universality property. If* $(Z,\varrho)$ *is a* $T_0$ *GTS, and* $f\colon (X,\mu) \to (Z,\varrho)$ *is continuous, then there exists a unique continuous* $h\colon (Y,\nu) \to (Z,\varrho)$, *such that* $hq = f$.

(2) *With the above notations, we have the following equivalences*: $(X,\mu)$ *is regular* (*completely regular*) $\iff (Y,\nu)$ *is regular* (*completely regular*) $\iff (Y,\nu)$ *is* $T_3$ $(T_{3.5})$.

Proof. (1) For $x_1, x_2, x_3 \in X$, with $x_1 \equiv x_2 \equiv x_3$ we have, for each $M \in \mu$, that $x_1 \in M \Longrightarrow x_2 \in M \Longrightarrow x_3 \in M$. The converse implication is proved in the same way, hence $x_1 \equiv x_3$.

The statement about the initial structure is evident.

Let $f\colon (X,\mu) \to (Z,\varrho)$, where $(Z,\varrho)$ is $T_0$. Then the equivalence relation on $(Z,\varrho)$ is the finest one (i.e., each equivalence class is a singleton). Then for $z_1 \neq z_2$ we have that $f^{-1}(z_1), f^{-1}(z_2) \subset X$ contain no $x_1$, $x_2$

such that $x_1 \equiv x_2$. Therefore the equivalence relation on $(X,\mu)$ (that corresponds to the partition $\{q^{-1}(y) \mid y \in Y\}$) is finer than the equivalence relation corresponding to the partition $\{f^{-1}(z) \mid z \in Z\}$. Let $C \in \varrho$. Then $f^{-1}(C) \in \mu$ is a union of some sets $f^{-1}(z)$ with $z \in Z$, hence is a union of some equivalence classes associated to $\mu$. Therefore $f\colon (X,\mu) \to (Z,\varrho)$ factors as $(X,\mu) \overset{q}{\to} (Y,\nu) \overset{h}{\to} (Z,\varrho)$. Unicity of such an $h$ follows since $q$ is onto.

(2) The first equivalences follow from the definition of $(Y,\nu)$. The second equivalences follow since $(Y,\nu)$ is $T_0$, which by regularity imply $T_1$, and hence $T_3$ ($T_{3.5}$). $\square$

THEOREM 4.12. *A GTS $(X,\mu)$ is $T_{3.5}$ (completely regular) if and only if it is homeomorphic to a subspace $Y$ of a power of the GTS $([0,1],\gamma_0)$ (has the weak structure w.r.t. a source $\varphi\colon X \to U([0,1],\gamma_0)^J = [0,1]^J$ in* **Set** *— consisting of a single map — for some $J$). For $|X| \geqq 2$ (for $\mu \not\subset \{\emptyset, X\}$) we may even suppose that $Y$ is dense ($\varphi$ is dense).*

PROOF. 1. We begin with the $T_{3.5}$ property. Necessity is a direct consequence of Proposition 4.7(2), because $([0,1],\gamma_0)$ is a $T_{3.5}$ space.

We turn to sufficiency. The space $(\emptyset, \{\emptyset\})$ is a subspace of any power of $([0,1],\gamma_0)$. For $|X| = 1$, $(X,\{\emptyset\}) \cong ([0,1],\gamma_0)^0$ (cf. Subsection 2.4), and $(X,\{\emptyset, X\})$ is a subspace of $([0,1],\gamma_0)$. So we may suppose $|X| \geqq 2$, that by $T_1$ implies strongness and $\mu \not\subset \{\emptyset, X\}$.

Put $J := \{(x,M) \mid x \in M \in \mu\}$. By $\mu \neq \{\emptyset\}$ we have $|J| \geqq 1$. Since $X$ is a completely regular space therefore for every $\alpha = (x,M) \in J$ there exists a continuous function $f_\alpha\colon (X,\mu) \to ([0,1],\gamma_0)$ such that $f_\alpha(x) = 0$ and $f_\alpha(X \setminus M) \subset \{1\}$. Then for $M \neq X$ we have $f_\alpha(X \setminus M) = \{1\}$, and for $M = X$ by strongness of $(X,\mu)$ we can choose for $f_\alpha$ the constant 0 function. Now we define $f\colon (X,\mu) \to ([0,1],\gamma_0)^J$ by $f(x) := \langle f_\alpha(x)) \mid \alpha \in J\rangle$. If $x, y \in X$ and $x \neq y$, then by the $T_1$ property of $(X,\mu)$, $\{y\}$ is a $\mu$-closed set not containing $x$ — therefore, there is $\alpha \in J$ such that $f_\alpha(x) = 0$ and $f_\alpha(y) = 1$. Hence, in presence of the $T_1$ property of $(X,\mu)$, $\langle f_\alpha \mid \alpha \in J\rangle$ is point-separating.

Applying Lemma 4.9 we obtain that the map $f$ is a homeomorphism to its image $(Y,\nu) := ([0,1],\gamma_0)^J|f(X)$.

Now we prove the addition about denseness. If $|X| \geqq 2$ and we have $T_1$, then $\mu \not\subset \{\emptyset, X\}$, and $|J| \geqq 1$, and $(Y,\nu)$ is a subspace of $([0,1],\gamma_0)^J$. Then $(Y,\nu)$ is a subspace of $\prod_{\alpha\in J} \pi_\alpha Y$. We distinguish the cases whether $M = X$ or $M \neq X$. For $M = X$ we have that $\pi_\alpha Y = (\{0\},\{\emptyset,\{0\}\})$. All such factors have product (up to isomorphism) this same space $(\{0\},\{\emptyset,\{0\}\})$, and we may omit all such factors. For $M \neq X$ we have that $\pi_\alpha Y \supset \{0,1\}$. Hence $c_{\gamma_0}\pi_\alpha Y = [0,1]$, for all non-omitted factors, which factors correspond to a set $J'$ ($\subset J$), of cardinality at least 1 (by $T_1$ and $|X| \geqq 2$). Then a homeomorphic copy $(Y',\nu')$ of $(Y,\nu)$ is contained in the product $([0,1],\gamma_0)^{J'}$ of the non-omitted factors — namely, $(Y', \nu') := ([0,1], \gamma_0)^{J'} | \langle f_\alpha \mid \alpha \in J' \rangle (X)$ — whose product with $(\{0\}, \{\emptyset, \{0\}\})$ is, up to isomorphism, $(Y, \nu)$. Since for all $\alpha \in J'$ we have $c_{\gamma_0} \pi_\alpha Y = [0,1]$, therefore by Corollary 3.10 the closure of $Y'$ in $([0,1], \gamma_0)^{J'}$ is $[0,1]^{J'}$, i.e., $Y'$ is dense in $([0,1], \gamma_0)^{J'}$.

2. We turn to the complete regularity. Spaces of the form $(X, \{\emptyset\})$ and (for $X \neq \emptyset$) $(X, \{\emptyset, X\})$ are completely regular, and have the weak structures w.r.t. the sources consisting of the single map to $([0,1], \gamma_0)^0 \cong (\{0\}, \{\emptyset\})$, and the identically 0 map to $([0,1], \gamma_0)$, respectively. Hence we may suppose $\mu \not\subset \{\emptyset, X\}$, when the $T_0$-reflection of $(X, \mu)$ has at least two points, and when the functions $f_\alpha \colon (X, \mu) \to ([0,1], \gamma_0)$ for $M \notin \{\emptyset, X\}$ satisfy $f_\alpha(x) = 0$ and $f_\alpha(X \setminus M) = \{1\}$. Hence $J \neq \emptyset$, and even $J' \neq \emptyset$ from part 1 of this proof.

Necessity follows, since for a source consisting of a single map $\varphi\colon X \to U(([0,1], \gamma_0)^J) = [0,1]^J$ we have that $([0,1], \gamma_0)^J | \varphi(X)$ is $T_{3.5}$ by Proposition 4.7(2). Then the initial lift (weak structure) for this source $\varphi\colon X \to U(([0,1], \gamma_0)^J)$ is the same as that for the source $X \xrightarrow{q} \varphi(X)$ $(\hookrightarrow U(([0,1], \gamma_0)^J)$, where $(\varphi(X), \gamma_0^J | \varphi(X))$ is the $T_0$-reflection of $(X, \mu)$, and $q$ from Definition 4.10 pointwise coincides with $\varphi$. By Proposition 4.11(2), the $T_{3.5}$ property of $\big(\varphi(X), \gamma_0^J | \varphi(X)\big)$ implies the complete regularity of $(X, \mu)$.

For sufficiency, let $(X, \mu)$ be completely regular, and let $(Y, \nu)$ be its $T_0$-reflection (with an onto map $q\colon X \to Y$ from Definition 4.10), which is $T_{3.5}$ by Proposition 4.11(2), hence is a subspace of some power $([0,1], \gamma_0))^J$ by part 1 of this proof. Let $i\colon (Y, \nu) \hookrightarrow ([0,1], \gamma_0)^J$ be the inclusion. Then, by Proposition 4.11(1), $(X, \mu)$ has the weak structure for the source consisting of the single map $iq\colon X \to U(([0,1], \gamma_0)^J)$. $\square$

It is interesting to observe that one can completely describe the continuous maps of $(\mathbb{R}, \gamma)$, or of $([0,1], \gamma_0)$, to itself. We do this in greater generality. Recall that for an ordered set $(X, \leqq)$ the *order topology* has as subbase $\big\{ \{x \in X \mid x < a\} \mid a \in X \big\} \cup \big\{ \{x \in X \mid x > b\} \mid b \in X \big\}$.

Definition 4.13. *Let $(X, \leqq)$ be an ordered set. Its order GT has as base* $\big\{ \{x \in X \mid x < a\} \mid a \in X \big\} \cup \big\{ \{x \in X \mid x > b\} \mid b \in X \big\}$. We will write these sets as $(-\infty, a)$ and $(b, \infty)$. Analogously we use the notations $(-\infty, a]$ and $[b, \infty)$ for $\{x \in X \mid x \leqq a\}$ and $\{x \in X \mid x \geqq b\}$, respectively. (Observe that $\pm\infty$ are just symbols and not elements of $X$, even if $X$ happens to have a minimal or a maximal element. Also, we will use the same symbols for any other ordered space $Y$ as well, but this will not cause misunderstanding.)

Remark 4.14. Letting $(\widetilde{X}, \leqq)$ be the Dedekind completion of $(X, \leqq)$, the open sets of the order GT of $X$ are exactly of the form $\emptyset$, or $X$ (this only for $|X| \neq 1$), $\{x \in X \mid x < \widetilde{x}_1\}$, or $\{x \in X \mid x > \widetilde{x}_2\}$, or $\{x \in X \mid x < \widetilde{x}_1$ or $x > \widetilde{x}_2\}$, where $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ with $\widetilde{x}_1 \leqq \widetilde{x}_2$, but excluding $\widetilde{x}_1 = \widetilde{x}_2 \in \widetilde{X} \setminus X$. Thus, the closed sets are of the form $X$, or $\emptyset$ (this only for $|X| \neq 1$), or $\{x \in X \mid x \leqq \widetilde{x}_1\}$ or $\{x \in X \mid x \geqq \widetilde{x}_2\}$, or $\{x \in X \mid \widetilde{x}_1 \leqq x \leqq \widetilde{x}_2\}$, where $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ with $\widetilde{x}_1 \leqq \widetilde{x}_2$, but excluding $\widetilde{x}_1 = \widetilde{x}_2 \in \widetilde{X} \setminus X$. I.e., for $|X| \neq 1$, the closed sets in $X$ are exactly the *convex* subsets (i.e., which contain with any two of their points the whole interval between them) which are closed in the (finer) order topology of $(X, \leqq)$.

For $X = \emptyset$ (with the void ordering) we have that the order GT is $(\emptyset, \{\emptyset\})$, that is strong. For $|X| = 1$ we have that the order GT is $(X, \{\emptyset\})$, that is not strong. For $|X| \geqq 2$ the order GT is strong (since it is $T_1$, cf. Subsection 2.6).

As a generalization of our Remark 4.1, [3] Remark after Example 2.4 claimed that the order GT associated to any ordered set $(X, \leqq)$ is $T_4$. (A detailed simple proof can be given for $|X| \leqq 1$ from the preceding paragraph, while for $|X| \geqq 2$ by using the case distinctions from the second preceding paragraph.)

Proposition 4.15. *Let $(X, \leqq)$ and $(Y, \leqq)$ be two ordered sets, with order GT's $\mu$ and $\nu$. Then the continuous functions $f\colon (X, \mu) \to (Y, \nu)$ are the following ones. For $|X| = 1 < |Y|$ there is no continuous map $f\colon (X, \mu) \to (Y, \nu)$. Else the continuous maps are the (non-strictly) monotonically increasing or monotonically decreasing maps, which are continuous between the respective order topologies.*

Proof. First we settle the case $\min\{|X|, |Y|\} = 0$. For $Y = \emptyset$ there exists an $f\colon (X, \mu) \to (Y, \nu)$ only if $X = \emptyset$. For $X = \emptyset$ we have $(X, \mu) = (\emptyset, \{\emptyset\})$, hence to any $(Y, \nu)$ there is exactly one set map $X \to Y$, that is continuous, and also is monotonous and continuous between the respective order topologies.

For $\min\{|X|, |Y|\} = 1$ we may have $|X| = |Y| = 1$ and then the unique set map $X \to Y$ is continuous, and also is monotonous and continuous between the respective order topologies. For $|X| = 1 < |Y|$ we have that $(Y, \nu)$ is strong, while $X$ is not strong, hence there is no continuous map $f\colon (X, \mu) \to (Y, \nu)$. For $|X| > 1 = |Y|$ there is a unique set map $X \to Y$, that is continuous, and also is monotonous and continuous between the respective order topologies.

*From now on let $|X|, |Y| \geqq 2$.* Suppose that $f$ is not monotonous. Then there are $a, b, c, d \in X$ such that

$$a < b \text{ and } f(a) < f(b), \text{ and } c < d \text{ and } f(c) > f(d). \tag{4.1}$$

Let $\{a, b, c, d\} = \{x_1, \ldots, x_k\}$, where $x_i < x_{i+1}$. By (4.1) $3 \leqq k \leqq 4$. If for some $i$ we have $f(x_i) = f(x_{i+1})$ then we delete from $\{x_1, \ldots, x_k\}$ the point $x_{i+1}$. Thus we obtain $x_1', \ldots, x_l'$, where by (4.1) $x_i' < x_{i+1}'$ and $3 \leqq l \leqq 4$. Now for each $i$ we have $f(x_i) < f(x_{i+1})$ or $f(x_i) > f(x_{i+1})$. By (4.1) there is some $i$ such that $f(x_i) > f(x_{i+1}) < f(x_{i+2})$ or $f(x_i) < f(x_{i+1}) > f(x_{i+2})$. We may suppose that we have the first case (else we consider the converse ordering on $Y$).

By Remark 4.14 the closed sets in $X$ and $Y$ are exactly the convex subsets which are closed in the (finer) order topology. Then the set $f^{-1}\big([\min\{f(x_i), f(x_{i+2})\}, \infty)\big)$ contains $x_i$ and $x_{i+2}$, but does not contain $x_{i+1}$. Thus the inverse image of a closed set is not convex, hence is not closed, a contradiction.

So $f$ is monotonous. We may suppose that it is monotonically increasing (else we take the converse ordering on $Y$). Then it suffices to prove that *in the sense of topology, $f$ is continuous from the left at any point $x \in X$.* (Namely then a similar reasoning shows that in the sense of topology, $f$ is also continuous from the right, hence is, in the sense of topology, continuous.) Observe that in the sense of topology $f$ is continuous from the left at $x \in X$ if $x$ has an immediate predecessor in $X$. Therefore we suppose that $(-\infty, x)$ has no largest element (in particular, $x$ is not a minimal element of $X$).

Suppose the contrary, namely that for some $x \in X$ and some $X \ni x' < x$, we have $f((x', x)) \subset (-\infty, y]$, for some $Y \ni y < f(x)$, hence also $f((-\infty, x)) \subset (-\infty, y]$, for some $Y \ni y < f(x)$. Then $(y, \infty)$ is a neighbourhood of $f(x)$ in the sense of GT's, such that even no topological neighbourhood $N$ of $x$ satisfies $f(N \cap (-\infty, x)) \subset (y, \infty)$, hence no such $N$ satisfies $f(N) \subset (y, \infty)$, although $(y, \infty)$ is also a topological neighbourhood of $f(x)$.

Conversely, continuity of a (non-strictly) monotonically increasing or decreasing function $f$ between the order topologies of $(X, \leqq)$ and $(Y, \leqq)$ implies its continuity between the order GT's of $(X, \leqq)$ and $(Y, \leqq)$ (recall $|X|, |Y| \geqq 2$). It is sufficient to prove this for $f$ monotonically increasing. It is sufficient to show that the inverse image by $f$ of a basic open set in the order GT of $(Y, \leqq)$ is open in the order GT of $(X, \leqq)$. It is sufficient to prove this for a basic open set of the form $(-\infty, y) \subset Y$. By continuity of $f$ in the topological sense we have that $f^{-1}(-\infty, y)$ is open in the topology of $(X, \leqq)$. Therefore it is the union of non-empty basic open sets in the sense of topology, say $f^{-1}(-\infty, y) = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha)$. Also, by the monotonically increasing property of $f$, we have that $f^{-1}(-\infty, y)$ is downward closed (i.e., $x' < x \in f^{-1}(-\infty, y)$ implies $x' \in f^{-1}(-\infty, y)$). Therefore we have also $f^{-1}(-\infty, y) = \bigcup_{\alpha \in J} (-\infty, b_\alpha)$. Thus $f^{-1}(-\infty, y)$ is a union of open sets in the order GT of $(X, \leqq)$, hence it is open in the order GT of $(X, \leqq)$ as well. □

For topological spaces $X$, $Y$ where $Y$ is $T_2$, and continuous maps $f, g \colon X \to Y$, if $f$, $g$ coincide on a dense subset of $X$, then they are equal. For GTS's this is false.

Example 4.16. Let $X = Y = ([0,1], \gamma_0)$. Let $f, g \colon [0,1] \to [0,1]$ be any two different strictly monotonically increasing maps in **GenTop**, continuous in the topological sense, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then $f$, $g$ are even homeomorphisms of $([0,1], \gamma_0)$ in **GenTop**, coinciding on the dense subset $\{0,1\}$ (cf. Remark 4.14), but $f \neq g$. Even, we can choose $f$ and $g$ so, that $\{x \in [0,1] \mid f(x) = g(x)\} = \{0,1\}$.

As is well known ([22], Example 2.3.12 and Theorem 5.2.8, Hist. and Bibl. Notes to Section 5.2) $T_4$ is not even finitely productive for topological spaces. Also, all powers of some topological space $X$ are $T_4$ if and only if $X$ is compact $T_2$ ([36]). However, for GTS's we have

PROPOSITION 4.17. *Let* $\langle (X_\alpha, \mu_\alpha) \mid \alpha \in J\}$ *be an indexed set of normal* (*or* $T_4$) *GTS's. Then their product is also normal* (*or* $T_4$).

PROOF. Since $T_1$ is productive (cf. Proposition 4.7), we investigate the case of normal spaces only.

First we settle the case of the empty product. By the Subsection 2.4, this is the GTS $(X_0, \{\emptyset\})$, where $|X_0| = 1$, which is normal.

Now we suppose $J \neq \emptyset$. Let $F_1$, $F_2$ be disjoint closed sets in $(X,\mu) := \prod_{\alpha\in J}(X_\alpha, \mu_\alpha)$. By Corollary 3.10 we have $F_i = \prod_{\alpha\in J} F_{i,\alpha}$, where $F_{i,\alpha} \subset X_\alpha$ is $\mu_\alpha$-closed. If for each $\alpha \in J$ we have $F_{1,\alpha} \cap F_{2,\alpha} \neq \emptyset$, then $F_1 \cap F_2 \neq \emptyset$. Hence for some $\alpha_0 \in J$ we have $F_{1,\alpha_0} \cap F_{2,\alpha_0} = \emptyset$. Then by normality of $(X_{\alpha_0}, \mu_{\alpha_0})$ there are disjoint $\mu_{\alpha_0}$-open sets $M_{1,\alpha_0}$, $M_{2,\alpha_0}$, such that $F_{i,\alpha_0} \subset M_{i,\alpha_0}$ for $i = 1,2$. Then $F_i \subset M_{i,\alpha_0} \times \prod_{\alpha\in J\setminus\{\alpha_0\}} X_\alpha$, for $i = 1,2$. These last sets are disjoint $\mu$-open sets in $X$. $\square$

We say that a GTS $(X,\mu)$ is *compact* (cf. [29], Section 3, Definition 2), or *Lindelöf* if any open cover of $X$ has a finite, or at most countably infinite subcover of $X$, respectively. This is the exact analogue of compactness and the Lindelöf property for topological spaces. In particular, if $X \notin \mu$, then $X$ is compact. More generally, for $\kappa$ an infinite cardinal, we say that a GTS $(X,\mu)$ is $\kappa$-*compact* if any open cover of $(X,\mu)$ has an (open) subcover of $X$, of cardinality less than $\kappa$. This is the analogue of $\kappa$-compactness for topological spaces, cf. [25], p. 6. For $\kappa = \aleph_0$ this is compactness, for $\kappa = \aleph_1$ this is the Lindelöf property.

By Tychonoff's theorem compactness is *productive* for topological spaces (i.e., products of compact spaces are compact), but Lindelöfness is not even finitely productive, cf. [22], 3.8.15. For GTS's the situation is very different, as shown by the next Proposition.

PROPOSITION 4.18. *For any infinite cardinal* $\kappa$, $\kappa$*-compactness is closed-hereditary and productive for GTS's*, *and is also inherited by surjective images for GTS's.*

PROOF. Closed-hereditariness and inheriting by surjective images (for compactness cf. [29], Section 3, Propositions 5 and 6) are proved exactly as for topological spaces.

We turn to products. Let $\langle (X_\alpha, \mu_\alpha) \mid \alpha \in J\rangle$ be an indexed set of $\kappa$-compact GTS's, with product $(X,\mu)$. We may suppose that each $X_\alpha$

is non-empty. Let $\mathcal{G} = \{G_\beta \mid \beta \in B\}$ be an open cover of $(X, \mu)$. By Corollary 3.10 we have for each $\beta \in B$ that

$$G_\beta = \bigcup_{\alpha \in J} \pi_\alpha^{-1}(M_{\alpha,\beta}),$$

where $M_{\alpha,\beta} \in \nu_\alpha$. Also by Corollary 3.10 we have that an open base of $(X, \mu)$ is $\bigcup\{\pi_\alpha^{-1}(\nu_\alpha) \mid \alpha \in J\}$. We define the set system $\mathcal{H}$ on $X$ as follows:

$$\mathcal{H} := \left\{ \pi_\alpha^{-1}(M_{\alpha,\beta}) \mid \alpha \in J,\ \beta \in B \right\}.$$

Then $\mathcal{H}$ is a cover of $X$ since $\cup\mathcal{H} = \cup\mathcal{G}$ and $\mathcal{G}$ is a cover of $X$, and consists of open sets in $(X, \mu)$ since each element of $\mathcal{H}$ belongs to the above mentioned base of $(X, \mu)$.

We distinguish two cases.

(i) Either for each $\alpha \in J$ we have $\bigcup\{M_{\alpha,\beta} \mid \beta \in B\} \neq X_\alpha$, or

(ii) for some $\alpha_0 \in J$ we have $\bigcup\{M_{\alpha_0,\beta} \mid \beta \in B\} = X_{\alpha_0}$.

In case (i) we choose for each $\alpha \in J$ a point $x_\alpha \in X_\alpha$ not contained by the union in (i). Then the point $\langle x_\alpha \mid \alpha \in J\rangle$ is not covered by $\mathcal{H}$, a contradiction. In case (ii) the union in (ii) is an open cover of $X_{\alpha_0}$, hence by $\kappa$-compactness of $X_{\alpha_0}$ it has an open subcover $\{M_{\alpha_0,\beta} \mid \beta \in B'\}$ of $X_{\alpha_0}$ with $|B'| < \kappa$. Then $\{\pi_{\alpha_0}^{-1}(M_{\alpha_0,\beta}) \mid \beta \in B'\}$ is an open cover of $(X, \mu)$, of cardinality less than $\kappa$. Now recall that for each $\beta \in B' \subset B$ we have $\pi_{\alpha_0}^{-1}(M_{\alpha_0,\beta}) \subset G_\beta$. Therefore $\{G_\beta \mid \beta \in B'\}$ is a subset of $\mathcal{G}$, which is also an open cover of $X$, and has cardinality less than $\kappa$. $\square$

A topological space $X$ is $T_{3.5}$ if and only if it is a subspace or a dense subspace of some (compact) $T_4$ space. Namely, one can consider the Stone–Čech compactification of $X$ that is compact $T_2$ hence $T_4$. It is interesting that an analogue of this statement holds also for GTS's, although with a completely different proof.

PROPOSITION 4.19. *A GTS* $(X, \mu)$ *is* $T_{3.5}$ (*completely regular*) *if and only if it is homeomorphic to a subspace, or equivalently to a dense subspace of a* $T_4$ *GTS, which can be supposed to be also compact* (*has the weak structure w.r.t. a map, or equivalently w.r.t. a dense map from* $UX$ *to a* $T_4$ *GTS, which can be supposed to be also compact*).

PROOF. 1. We begin with the "if" part. $T_4$ implies the hereditary property $T_{3.5}$ (cf. Proposition 4.7), which implies complete regularity, and complete regularity is inherited by initial (weak) structures, in particular for sources consisting of one map, cf. Proposition 4.7. Hence all subspaces of $T_4$ spaces are $T_{3.5}$, and initial (weak) structures for all sources consisting of a single map $\varphi \colon X \to U(Y, \nu) = Y$ with $Y$ $T_4$ are completely regular.

2. Conversely, for the "only if" part, we begin with the trivial cases $|X| \leqq 1$, i.e., with $(\emptyset, \{\emptyset\})$, and with $(X, \{\emptyset\})$ and $(X, \{\emptyset, X\})$ where $|X| = 1$.

Each of these three spaces are finite hence compact, are $T_1$ and normal (the second space has no disjoint closed sets, and the other ones are discrete, i.e., of the form $(X, P(X))$, which are normal). Then they are dense subspaces of themselves, which proves for them the "only if" part of the theorem.

Now let $(X, \mu)$ be $T_{3.5}$ with $|X| \geqq 2$. By Theorem 4.12, $(X, \mu)$ is homeomorphic to a dense subspace of some power of $([0,1], \gamma_0)$. By Remark 4.1, $([0,1], \gamma_0)$ is $T_4$, and then by Proposition 4.17 each power of $([0,1], \gamma_0)$ is $T_4$. Since $([0,1], \gamma_0)$ is coarser than the topological $[0,1]$ space, it is compact as well, hence all its powers are compact as well by Proposition 4.18. This ends the proof of the "only if" part for $T_{3.5}$ GTS's.

3. We turn to complete regularity, to the "only if" part. Let $(X, \mu)$ be completely regular, let $(Y, \nu)$ be its $T_0$-reflection, with canonical quotient map $q\colon (X, \mu) \to (Y, \nu)$ from Definition 4.10. Then by Proposition 4.11(2), $(Y, \nu)$ is $T_{3.5}$, hence by part 2 of this proof it admits a dense embedding $i\colon (Y, \nu) \to (Z, \varrho)$ to some compact $T_4$ GTS $(Z, \varrho)$. Then the source consisting of the single map $U(iq)\colon U(X, \mu) = X \to U(Z, \varrho)$ has as initial lift (weak structure) $(X, \mu)$, and $iq\colon (X, \mu) \to (Z, \varrho)$ is a dense map. This ends the proof of the "only if" part for completely regular GTS's. □

As known, a $T_2$ topological space of density $\kappa$ has cardinality at most $\exp(\exp(\kappa))$, cf. [25], p. 13. For GTS's the situation is completely different.

Example 4.20. For GTS's we may have a two-point dense subset in a compact $T_4$ GTS of arbitrarily large cardinality. Namely, in $([0,1], \gamma_0)^\kappa$, which is compact by Proposition 4.18 and is $T_4$ by Proposition 4.17, the two points having all coordinates 0, or all coordinates 1, form a dense subspace in $([0,1], \gamma_0)^\kappa$ (cf. Corollary 3.10).

## 5. Subspaces and sums of GTS's

We recall Corollary 3.13 about the construction of the sum of an indexed set $\langle (X_\alpha, \mu_\alpha) \mid \alpha \in J \rangle$ or $\langle (X_\alpha, c_\alpha) \mid \alpha \in J \rangle$ of GTS's.

As mentioned in Subsection 2.5, a common way to produce GT's is given by the GT's $\mu(\gamma)$ (see [7] and also our Subsection 2.5), where $X$ is a set and $\gamma \in \Gamma(X)$.

Let $\gamma_\alpha\colon X_\alpha \to X_\alpha$ be monotonous. We define

$$\gamma\colon P\Big(\coprod_{\alpha \in J} X_\alpha\Big) \to P\Big(\coprod_{\alpha \in J} X_\alpha\Big)$$

by

$$\gamma(A) := \coprod_{\alpha \in J} \gamma_\alpha(A \cap X_\alpha)\,.$$

Then $\gamma \in \Gamma(\coprod_{\alpha \in J} X_\alpha)$.

Now we can consider two GT's on the same set $X := \coprod_{\alpha\in J} X_\alpha$. The first one is $\mu(\gamma)$ and the second one is $\coprod_{\alpha\in J}\mu(\gamma_\alpha)$ (notations cf. in Subsection 2.5).

Similarly, for $\gamma\colon P(X)\to P(X)$ monotonous, and $X_0\subset X$, for $\gamma_0\colon P(X_0) \to P(X_0)$ defined by

$$\gamma_0 A_0 := (\gamma A_0)\cap X_0\,,$$

we have $\gamma_0\in\Gamma(X_0)$. Then we have two GT's on $X_0$, the first one is $\mu(\gamma_0)$ and the second one is $\mu(\gamma)|X_0$.

We are going to compare these two pairs of GT's by the following proposition. We recall that $\gamma\colon P(X)\to P(X)$ is *completely additive* if for any $\{A_\alpha \mid \alpha\in J\}\subset P(X)$ we have $\gamma\big(\bigcup_{\alpha\in J} A_\alpha\big) = \bigcup_{\alpha\in J}\gamma(A_\alpha)$.

Proposition 5.1. (1) *Let $\langle X_\alpha \mid \alpha\in J\rangle$ be an indexed set of sets and $\gamma_\alpha\colon P(X_\alpha)\to P(X_\alpha)$, for $\alpha\in J$ be monotonous. Then, with the notations introduced before this Proposition,*

$$\Big(\coprod_{\alpha\in J} X_\alpha, \mu(\gamma)\Big) = \coprod_{\alpha\in J}(X_\alpha,\mu(\gamma_\alpha))\,.$$

(2) *Let $X$ be a set and $\gamma\colon P(X)\to P(X)$ be monotonous, and let $X_0\subset X$. Then, with the notations introduced before this Proposition,*

$$\mu(\gamma_0)\subset\mu(\gamma)|X_0\,.$$

*The converse inclusion is false even if $\gamma$ is completely additive and $\gamma(P(X)) = \{\emptyset, X\}$.*

Proof. 1. As in Subsection 2.4, we suppose that the sets $X_\alpha$, for $\alpha\in J$, form a partition of $\coprod_{\alpha\in J} X_\alpha$. Let $A\subset\coprod_{\alpha\in J} X_\alpha$. We write $A_\alpha := A\cap X_\alpha$; hence $A = \coprod_{\alpha\in J} A_\alpha$. We have $A\subset\gamma A \iff \forall\,\alpha\in J\ A_\alpha\subset\gamma_\alpha A_\alpha$. Hence $\mu(\gamma) = \coprod_{\alpha\in J}\mu(\gamma_\alpha)$.

2. First we show $\mu(\gamma_0)\subset\mu(\gamma)|X_0$. We have for $A_0\subset X_0$ that $A_0\in\mu(\gamma_0) \iff A_0\subset\gamma_0 A_0 \iff A_0\subset\gamma(A_0)\cap X_0 \iff A_0\subset\gamma(A_0) \iff A_0\in\mu(\gamma)$. Then $A_0\in\mu(\gamma_0) \iff A_0\in\mu(\gamma) \implies A_0 = A_0\cap X_0\in\mu(\gamma)|X_0$, therefore $\mu(\gamma_0)\subset\mu(\gamma)|X_0$.

About the converse inclusion we give the following counterexample. Let $|X|\geqq 2$, $X_0\subset X$ and $X_0\not\in\{\emptyset, X\}$. Let $\gamma\in\Gamma(X)$ be defined as follows. For $A\subset X_0$ we have $\gamma(A)=\emptyset$, for $A\subset X$ and $A\not\subset X_0$ we have $\gamma(A) = X$. Thus $\gamma$ is completely additive and $\gamma\left(P(X)\right) = \{\emptyset, X\}$. Then by $X_0\neq X$ we have $\mu(\gamma) = \{A\subset X \mid A\subset\gamma A\} = \{\emptyset\}\cup\{A\subset X \mid A\not\subset X_0\}$, hence

$$\mu(\gamma)|X_0 = P(X_0)\,.$$

On the other hand

$$\mu(\gamma_0) = \left\{ A_0 \subset X_0 \mid A_0 \subset \gamma_0 A_0 \right\} = \left\{ A_0 \subset X_0 \mid A_0 \subset \gamma(A_0) \cap X_0 \right\}$$
$$= \left\{ A_0 \subset X_0 \mid A_0 \subset \gamma(A_0) \right\} = \left\{ \emptyset \right\}.$$

Therefore, by $X_0 \neq \emptyset$ we have

$$\mu(\gamma_0) = \{\emptyset\} \not\supset P(X_0) = \mu(\gamma)|X_0 . \quad \square$$

As mentioned in Subsection 2.5, another common way to produce GT's is given by the GT's $\kappa(\mu, k)$ (see [13] and also our Subsection 2.5), where $(X, \mu)$ is a GTS and $k\colon \mu \to P(X)$ is an enlargement on $(X, \mu)$.

Now, let $\langle (X_\alpha, \mu_\alpha) \mid \alpha \in J \rangle$ be an indexed set of GTS's and $k_\alpha\colon \mu_\alpha \to P(X_\alpha)$ be an enlargement on $X_\alpha$, for $\alpha \in J$. Let

$$(5.1) \qquad \begin{cases} (X, \mu) := \coprod_{\alpha \in J} (X_\alpha, \mu_\alpha) \quad \text{and} \\ \Big( \coprod_{\alpha \in J} X_\alpha, \coprod_{\alpha \in J} \kappa(\mu_\alpha, k_\alpha) \Big) := \coprod_{\alpha \in J} \big( X_\alpha, \kappa(\mu_\alpha, k_\alpha) \big) . \end{cases}$$

We can ask if there is an enlargement $k$ on the sum set $X := \coprod_{\alpha \in J} X_\alpha$, such that $(X, \kappa(\mu, k)) = \coprod_{\alpha \in J} (X_\alpha, \kappa(\mu_\alpha, k_\alpha))$. Actually, here we will have only a one-sided inclusion. The converse inclusion is in general false, but the necessary and sufficient condition for equality will be given.

Similarly, for subspaces $(X_0, \mu_0)$ of $(X, \mu)$, one could ask if on $(X_0, \mu_0)$ there is an enlargement $k_0\colon \mu_0 \to P(X_0)$, such that

$$(X_0, \kappa(\mu_0, k_0)) = (X, \kappa(\mu, k))|X_0.$$

Actually, here we will have only a one-sided inclusion, and only under some additional hypotheses. The converse inclusion is false, even under more restrictive additional hypotheses.

PROPOSITION 5.2. *Let $\langle (X_\alpha, \mu_\alpha) \mid \alpha \in J \rangle$ be an indexed set of GTS's and $k_\alpha\colon \mu_\alpha \to P(X_\alpha)$ be an enlargement on $(X_\alpha, \mu_\alpha)$, for $\alpha \in J$. Then the enlargement $k$ on $(X, \mu) := \coprod_{\alpha \in J} (X_\alpha, \mu_\alpha)$ defined for $M \in \mu$ by $kM := \coprod_{\alpha \in J} k_\alpha(M \cap X_\alpha)$ satisfies $\kappa(\mu, k) \subset \coprod_{\alpha \in J} \kappa(\mu_\alpha, k_\alpha)$. Here equality holds if and only if either $|\{\alpha \in J \mid X_\alpha \neq \emptyset\}| \leqq 1$ or $|\{\alpha \in J \mid X_\alpha \neq \emptyset\}| \geqq 2$ and for each $\alpha \in J$ we have $k_\alpha \emptyset = \emptyset$.*

*Let $(X, \mu)$ be a GTS and $k : \mu \to P(X)$ be an enlargement on $(X, \mu)$. Let $X_0 \subset X$. If $k$ is also monotonous and $\mu$ is closed under the intersections of pairs of elements and $X_0$ is $\mu$-open, then the enlargement $k_0$ on $(X_0, \mu_0)$ defined for $M_0 \in \mu_0$ by $k_0 M_0 \colon = (kM_0) \cap X_0$ satisfies $\kappa(\mu, k)|X_0 \subset \kappa(\mu_0, k_0)$. The converse inclusion is false even for $\mu$ a topology, $X_0$ $\mu$-open, and $k$ a topological closure.*

PROOF. 1. We begin with the case of sums. As in Subsection 2.4, we suppose that the sets $X_\alpha$ for $\alpha \in J$ form a partition of $\coprod_{\alpha\in J} X_\alpha$.

We have $kM = \coprod_{\alpha\in J} k_\alpha(M_\alpha)$, where $M \in \mu$ and $M_\alpha := M \cap X_\alpha$. Then $M = \coprod_{\alpha\in J} M_\alpha \subset \coprod_{\alpha\in J} k_\alpha M_\alpha = kM \subset X$, thus $k$ is an enlargement on $(X, \mu)$.

Since for $X_\alpha = \emptyset$ we have $\mu_\alpha = \kappa(\mu_\alpha, k_\alpha) = \{\emptyset\}$, these contribute nothing to either $\mu$, or to $k$, or to $\kappa(\mu, k)$, or to $\coprod_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha)$, therefore we may omit all empty $X_\alpha$'s simultaneously. Therefore we will suppose that each $X_\alpha$ is non-empty. If we have at most one non-empty $X_\alpha$, then evidently $\coprod_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha) = \kappa(\mu, k)$. Therefore we suppose that there are at least two non-empty $X_\alpha$'s.

First we prove

$$\kappa(\mu, k) \subset \coprod_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha)\,. \tag{5.2}$$

Let $A \in \kappa(\mu, k) \subset \mu$. We write $A_\alpha := A \cap X_\alpha \in \mu_\alpha$. If $A_\alpha = \emptyset$, then $A_\alpha \in \kappa(\mu_\alpha, k_\alpha)$, so we need to deal only with such $\alpha$'s, for which $A_\alpha \neq \emptyset$. Let $x_\alpha \in A_\alpha \subset A$. Then there exists an $M \in \mu$ such that $x_\alpha \in M$ and $kM \subset A$. Writing $M_\alpha := M \cap A_\alpha$ and $M_\beta := M \cap X_\beta$ for $\beta \in J \setminus \{\alpha\}$, this means that $x_\alpha \in M_\alpha \in \mu_\alpha$ and $k_\alpha M_\alpha \subset A_\alpha$ and for $\beta \in J \setminus \{\alpha\}$ that $M_\beta \in \mu_\beta$ and $k_\beta M_\beta \subset A_\beta$. (For $\beta \neq \alpha$ the condition for $A_\beta$ is weaker than the condition for $A_\alpha$, so we disregard the condition about all $\beta \in J \setminus \{\alpha\}$. Observe that any $\beta \in J \setminus \{\alpha\}$ — with $X_\beta \neq \emptyset$ — can occur in the role of $\alpha$.) Then we obtain $A_\alpha \in \kappa(\mu_\alpha, k_\alpha)$, that is, (5.2) is proved.

Second we deal with the validity of the inclusion

$$\coprod_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha) \subset \kappa(\mu, k)\,.$$

Since $\coprod_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha)$ has as base $\bigcup_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha)$, this inclusion is equivalent to

$$\bigcup_{\alpha\in J} \kappa(\mu_\alpha, k_\alpha) \subset \kappa(\mu, k), \quad \text{i.e., to } \ \forall\, \alpha \in J \ \kappa(\mu_\alpha, k_\alpha) \subset \kappa(\mu, k)\,.$$

Let $A_\alpha \in \kappa(\mu_\alpha, k_\alpha)$, i.e.,

$$\begin{cases} \text{for each } x_\alpha \in A_\alpha \text{ there exists an } M_\alpha \in \mu_\alpha \\ \text{such that } x_\alpha \in M_\alpha \subset k_\alpha M_\alpha \subset A_\alpha \subset X_\alpha. \end{cases} \tag{5.3}$$

Then $A_\alpha \in \kappa(\mu, k)$ if and only if

$$\begin{cases} \text{for each } x_\alpha \in A_\alpha \text{ there exists an } M \in \mu \\ \text{such that } x_\alpha \in M \subset kM \subset A_\alpha \subset X_\alpha. \end{cases} \tag{5.4}$$

Here $M \in \mu$ and $M \subset X_\alpha$ means $M =: M^\alpha \in \mu_\alpha$, hence (5.4) is equivalent to

$$\text{(5.5)} \quad \begin{cases} \text{for each } x_\alpha \in A_\alpha \text{ there exists an } M \in \mu \text{ such that} \\ \qquad X_\alpha \supset A_\alpha \supset kM = kM^\alpha = \bigcup_{\beta \in J} k_\beta M^\alpha \\ \quad = (k_\alpha M^\alpha) \coprod \Big( \coprod_{\beta \in J \setminus \{\alpha\}} k_\beta \emptyset \Big) \supset M^\alpha = M \ni x_\alpha. \end{cases}$$

Then $k_\alpha M_\alpha \subset A_\alpha$ is satisfied for $M^\alpha := M_\alpha$, thus we need to satisfy yet that for all $\beta \in J \setminus \{\alpha\}$ we have for $k_\beta \emptyset \subset X_\beta$ that also $k_\beta \emptyset \subset X_\alpha$ (cf. (5.5). But then $k_\beta \emptyset \subset X_\alpha \cap X_\beta = \emptyset$, i.e., $k_\beta \emptyset = \emptyset$. However, this is just the necessary and sufficient hypothesis for the case $|\{\alpha \in J \mid X_\alpha \neq \emptyset\}| \geqq 2$, given in this Theorem. (Observe that here we have $k_\beta \emptyset = \emptyset$ only for $\beta \in J \setminus \{\alpha\}$. However, since now $|J| \geqq 2$, we change $\alpha \in J$ to another element $\alpha' \in J$, and then we obtain $k_\alpha \emptyset = \emptyset$ as well.)

Conversely, $|\{\alpha \in J \mid X_\alpha \neq \emptyset\}| \geqq 2$ and $\forall\, \alpha \in J\ k_\alpha \emptyset = \emptyset$ implies

$$\kappa(\mu_\alpha, k_\alpha) \subset \kappa(\mu, k),$$

i.e., (5.3) $\Longrightarrow$ (5.5) $\Longleftrightarrow$ (5.4).

2. We turn to the case of subspaces. We have $k_0 M_0 = k(M_0) \cap X_0$ for $M_0 \in \mu_0$ (thus $M_0 \subset X_0$). Here $k(M_0)$ is defined since $M_0$ is the intersection of some open set of $(X, \mu)$ and $X_0$, hence is $\mu$-open by hypothesis. Then $k_0 M_0 \supset M_0 \cap X_0 = M_0$, thus $k_0$ is an enlargement on $(X_0, \mu_0)$.

Let $A \subset X$ be $\kappa(\mu, k)$-open, i.e., $x \in A$ implies $\exists M \in \mu$ such that $x \in M$ and $kM \subset A$. Let $A_0 := A \cap X_0$. Then for $x_0 \in A_0$ ($\subset X_0$) there exists $x_0 \in M \subset A$ such that $kM \subset A$. Then also

$$x_0 \in M \cap X_0 \subset A \cap X_0 \ \text{ and } \ k(M \cap X_0) \subset kM \subset A$$

by monotony of $k$. Then also $k_0(M \cap X_0) = k(M \cap X_0) \cap X_0 \subset A \cap X_0$ while $M \cap X_0 \in \mu_0$ by hypothesis. Hence $A_0 = A \cap X_0$ is $\kappa(\mu_0, k_0)$-open.

About the converse inclusion we give the following counterexample.

Let $X := \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ and $X_0 := \{1/n \mid n \in \mathbb{N}\}$ its open subspace, with the usual topologies. We define $k \colon P(X) \to P(X)$ as follows: $k\emptyset = \emptyset$, and for $\emptyset \neq M \subset X$ we define $kM := M \cup \{0\}$. This is a topological closure.

Then for $A \subset X$ we have $A \in \kappa(\mu, k)$ if and only if

$$\text{(5.6)} \qquad x \in A \implies \exists M \in \mu \text{ such that } x \in M \text{ and } kM = M \cup \{0\} \subset A.$$

Now we will determine $\kappa(\mu, k)$.

(1) If $A = \emptyset$ then $A \in \kappa(\mu, k)$.

(2) If $A \neq \emptyset$ and $0 \notin A$ then supposing (5.6) we have $\exists x \in A$, and then $\{0\} \subset A$, which is a contradiction. Hence such an $A$ does not belong to $\kappa(\mu, k)$.

(3) If $A \neq \emptyset$ and $0 \in A$, then $x \in A$ can be either $1/n$ for some $n \in \mathbb{N}$, or it can be 0. For $x = 1/n$ (5.6) is satisfied for $M := \{1/n\}$, since $0 \in A$. For $x = 0$ (5.6) means $\exists\, m \in \mathbb{N}$ such that 0 has a neighbourhood $0 \in \{1/m, 1/(m+1), \ldots\} \cup \{0\}$ in $X$, such that $\{1/m, 1/(m+1), \ldots\} \cup \{0\} \subset A$. This means that $0 \in A$ and $A \setminus \{0\}$ is cofinite in $X \setminus \{0\} = X_0$. That is,

(5.7)
$$\kappa(\mu, k) = \{\emptyset\} \cup \big\{\, A \subset X \mid 0 \in A \text{ and } A \setminus \{0\} \text{ is cofinite in } X \setminus \{0\} = X_0 \big\}\,.$$

Hence

(5.8)
$$\kappa(\mu, k)|X_0 \ \text{ is the cofinite topology on } \ X_0\,.$$

Turning to $X_0$, we have that $\mu_0$ is the discrete topology on $X_0$, and, for $M_0 \subset X_0$, we have for $M_0 = \emptyset$ that $k_0\emptyset := k(\emptyset) \cap X_0 = \emptyset$, and for $M_0 \neq \emptyset$ we have $k_0 M_0 := k(M_0) \cap X_0 = (M_0 \cup \{0\}) \cap X_0 = M_0$. Therefore $k_0$ is the closure associated to the discrete topology on $X_0$. Hence each $A_0 \subset X_0$ is $\kappa(\mu_0, k_0)$-open, since for $x_0 \in A_0$ we can choose $M_0 := \{x_0\}$ and then $x_0 \in \{x_0\}$ and $k_0\{x_0\} = \{x_0\} \subset A_0$. That is, we have

(5.9)
$$\kappa(\mu_0, k_0) = P(X_0)\,.$$

By (5.8) and (5.9) we have

$$\kappa(\mu_0, k_0) \not\subset \kappa(\mu, k)|X_0\,. \qquad \square$$

Remark 5.3. The construction of the counterexample is a special case of the $\theta$-modification of a bitopological space (which is itself a special case of the $\theta$-modification of a bi-GTS, cf. [31] and [16]).

## References


[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, $\beta$-open sets and $\beta$-continuous mappings, *Bull. Fac. Sci. Assiut Univ.* A, **12** (1983), 77–90.

[2] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories: the Joy of Cats*, Reprint of the 1990 original, Wiley (New York), Repr. Theory Appl. Categ. **17** (2006).

[3] M. Arar, On countably $\mu$-paracompact spaces, *Acta Math. Hungar.*, **146** (2016), 50–57.

[4] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Berichtigter Reprint, Springer (Berlin etc., 1974); English transl.: Theory of Convex Bodies, Transl. from German and ed. by L. Boron, C. Christenson, B. Smith, with collab. of W. Fenchel, BCS Associates (Moscow, ID, 1987).

[5] G. Castellini, *Categorical Closure Operators*, Math.: Theory and Appl., Birkhäuser (Boston, MA, 2003).

[6] E. Čech, *Topological Spaces*, revised edition by Z. Frolík and M. Katětov, Scientific ed. V. Pták, Ed. of the English transl. C. O. Junge, Publishing House of Czechoslovak Acad. Sci. (Praha), Interscience, Wiley (London etc., 1966).

[7] Á. Császár, Generalized open sets, *Acta Math. Hungar.*, **75** (1997), 65–87.

[8] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351–357.

[9] Á. Császár, Separation axioms for generalized topologies, *Acta Math. Hungar.*, **104** (2004), 63–69.

[10] Á. Császár, Generalized open sets in generalized topologies, *Acta Math. Hungar.*, **106** (2005), 53–66.

[11] Á. Császár, $\lambda$-open sets in generalized topological spaces, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **49** (2006), 59–63.

[12] Á. Császár, Normal generalized topologies, *Acta Math. Hungar.*, **115** (2007), 309–313.

[13] Á. Császár, Enlargements and generalized topologies, *Acta Math. Hungar.*, **120** (2008), 351–354.

[14] Á. Császár, On generalized neighbourhood systems, *Acta Math. Hungar.*, **121** (2008), 395–400.

[15] Á. Császár, *Product of generalized topologies*, *Acta Math. Hungar.*, **123** (2009), 127–132.

[16] Á. Császár and E. Makai, Jr., Further remarks on $\delta$- and $\theta$-modifications, *Acta Math. Hungar.*, **123** (2009), 223–228.

[17] D. Dikranjan, E. Giuli, Closure operators. I, in: *Proc. 8th Internat. Conf. on Categ. Top.* (L'Aquila, 1986), *Topology Appl.*, **27** (1987), pp. 129–143.

[18] D. Dikranjan, E. Giuli and W. Tholen, Closure operators. II, in: *Categorical Topology and its Relations to Analysis, Algebra and Combinatorics* (Prague, 1988), World Sci. Publ. (Teaneck, NJ, 1989), pp. 287–335.

[19] D. Dikranjan and W. Tholen, *Categorical Structure of Closure Operators, With Applications to Topology, Algebra and Discrete Mahematics*, Math. and its Appl. **346**, Kluwer (Dordrecht, 1995).

[20] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, with the assistance of W. G. Bade and R. G. Bartle, Pure and Appl. Math. **7**, Interscience Inc., (New York–London, 1958).

[21] *Encyclopaedia of Mathematics*, vol. 9, Translated from Russian, updated, Ed. M. Hazewinkel, Kluwer (Dordrecht, etc., 1993), Topological structures, pp. 197–204.

[22] R. Engelking, *General Topology*, Volume **6** of Sigma Series in Pure Mathematics, Heldermann Verlag (Berlin, 1989).

[23] X. Ge and Y. Ge, $\mu$-separations in generalized topological spaces, *Appl. Math. J. Chinese Univ. Ser. B*, **25** (2010), 243–252.

[24] H. Herrlich and G. E. Strecker, *Category Theory. An introduction*, third ed., Sigma Series in Pure Math. **1**, Heldermann (Lemgo, 2007).

[25] I. Juhász, *Cardinal Functions in Topology — Ten Years Later*, 2nd ed., Math. Centre Tracts **123**, Math. Centrum (Amsterdam, 1980).

[26] D. C. Kent and W. K. Min, Neighborhood spaces, *Int. J. Math. Math. Sci.*, **32** (2002), 387–399.

[27] Y. K. Kim and W. K. Min, Remarks on enlargements of generalized topologies, *Acta Math. Hungar.*, **130** (2011), 390–395.

[28] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36–41.

[29] S. Lugojan, Generalized topology (Romanian. English summary), *Stud. Cerc. Mat.*, **34** (1982), 348–360.
[30] A. S. Mashhour, Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, **53** (1982), 47–53.
[31] A. S. Mashhour, A. A. Allam and I. A. Hasanein, Bisupratopological spaces, *Proc. Math. Phys. Soc. Egypt*, **62** (1986), 39–47.
[32] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supratopological spaces, *Indian J. Pure Appl. Math.*, **14** (1983), 502–510.
[33] W. K. Min, Remarks on separation axioms on generalized topological spaces, *J. Chungcheong Math. Soc.*, **23** (2010), 293–298.
[34] W. K. Min, On ascending generalized neighbourhood spaces, *Acta Math. Hungar.*, **127** (2010), 391–396.
[35] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.*, **15** (1965), 961–970.
[36] N. Noble, Products with closed projections. II, *Trans. Amer. Math. Soc.*, **160** (1971), 169–183.
[37] V. Pavlović and A. S. Cvetković, On generalized topologies arising from mappings, *Bull. Iranian Math. Soc.*, **38** (2012), 553–565.
[38] G. Preuss, *Theory of Topological Structures*, Math. and its Appl. **39**, Reidel (Dordrecht, 1988).
[39] M. S. Sarsak, Weakly $\mu$-compact spaces, *Demonstratio Math.*, **45** (2012), 929–938.
[40] M. S. Sarsak, On $\mu$-compact sets in $\mu$-spaces, *Questions Answers Gen. Topology*, **31** (2013), 49–57.
[41] R. Schneider, *Convex bodies: the Brunn–Minkowski Theory*, Second expanded edition, Encyclopedia of Math. and its Appl. **44**, **151**, Cambridge Univ. Press (Cambridge, 1993, 2014).
[42] R. Shen, Remarks on products of generalized topologies, *Acta Math. Hungar.*, **124** (2009), 363–369.