

HARMONIC ANALYSIS OPERATORS RELATED TO SYMMETRIZED JACOBI EXPANSIONS FOR ALL ADMISSIBLE PARAMETERS

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Abstract. This is an ultimate completion of our earlier paper [3] where mapping properties of several fundamental harmonic analysis operators in the setting of symmetrized Jacobi trigonometric expansions were investigated under certain restrictions on the underlying parameters of type. In the present article we take advantage of very recent results due to Nowak, Sjögren and Szarek to fully release those restrictions, and also to provide shorter and more transparent proofs of the previous restricted results. Moreover, we also study mapping properties of analogous operators in the parallel context of symmetrized Jacobi function expansions. Furthermore, as a consequence of our main results we conclude some new results related to the classical non-symmetrized Jacobi polynomial and function expansions.

1. Introduction

In [11] Nowak and Stempak postulated a unified conjugacy scheme in the context of general orthogonal expansions related to a second order differential operator, a ‘Laplacian’. Later the same authors in [12] proposed a symmetrization procedure pertaining to the theory of [11] that allowed them to overcome the lack of symmetry in a decomposition of the related Laplacian and, consequently, eliminate substantial deviations from the classical theory. It was shown in [12] that the symmetrization is reasonable as far as L^2 theory is concerned. However, the question of validity from the L^p theory perspective was left open, being practically impossible to be answered

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on the assumed level of generality. Thus a natural problem arose, namely to test L^p flavor of the symmetrization in selected concrete, possibly classical contexts.

To a large extent, this motivated our paper [3] where the symmetrization was applied in the framework of Jacobi trigonometric polynomial expansions. More precisely, we proved that fundamental harmonic analysis operators in the Jacobi symmetrized setting, including Riesz transforms, Littlewood–Paley–Stein type square functions, Jacobi–Poisson semigroup maximal operator and certain spectral multipliers, are bounded on weighted L^p spaces and are of weighted weak type $(1, 1)$. Analogous results in the original non-symmetrized Jacobi context were obtained earlier by Nowak and Sjögren [8], by means of the Calderón–Zygmund theory. The results of [3] heavily depend on the techniques developed in [8] and, consequently, inherit the restriction $\alpha, \beta \geq -1/2$ on the Jacobi parameters of type. The reason of this restriction in [8] was the lack of suitable integral representation of the Jacobi–Poisson kernel for other values of α and β .

The latter obstacle was recently overcome by Nowak, Sjögren and Szarek [10]. The authors established an integral formula for the Jacobi–Poisson kernel valid for all admissible $\alpha, \beta > -1$. This new general formula is essentially more complicated comparing to the one from [8] (actually, it contains the one from [8] as a special case), nevertheless it turned out to be suitable for developing methods parallel to those from [8] and then proving weighted L^p boundedness of several fundamental harmonic analysis operators. In fact, [10] offers some new ideas and improvements which, in particular, simplify and clarify the earlier analysis in the restricted case $\alpha, \beta \geq -1/2$.

The main purpose of the present paper is to take advantage of [10] in order to remove the restriction on α and β imposed in [3], see Theorem 3.1, and also to provide shorter and more transparent proofs of the main results from [3]. All this completes the research undertaken in [3].

Another aim of this work is to examine an alternative Jacobi symmetrized setting, this time originating from the discrete system of Jacobi trigonometric functions rather than polynomials. Sobolev and potential spaces in this symmetrized context were investigated recently by the author in [6]. Here we focus on weighted L^p mapping properties of the fundamental harmonic analysis operators mentioned above. It occurs that weighted L^p boundedness of these operators, see Theorem 3.7, can be concluded in a rather straightforward manner from the analogous results in the symmetrized Jacobi polynomial setting.

We point out that analysis in Jacobi settings received a considerable attention in recent years, see for instance [1–10, 13] and numerous other references given in these articles. In particular, mapping properties of harmonic analysis operators in the classical (non-symmetrized) discrete Jacobi polynomial and function contexts were extensively studied. Our present results in

the symmetrized Jacobi settings also contribute to the latter line of research since they imply new mapping properties in the non-symmetrized Jacobi contexts, see Propositions 3.9 and 3.10 and the accompanying comments in Section 3.3.

The paper is organized as follows. The remaining part of this section describes the notation used throughout the paper. In Section 2 we introduce the Jacobi settings to be investigated and the associated basic notions. Section 3 contains statements of the main results. These are either immediately justified or their proofs are reduced to certain kernel estimates. The latter are technically involved and their proofs require some preparation that is done in Section 4. After that, in Section 5, we finally prove the kernel estimates.

Notation. We always assume that $\alpha, \beta > -1$, unless explicitly stated otherwise. Let

$$\Psi^{\alpha, \beta}(\theta) := \left| \sin \frac{\theta}{2} \right|^{\alpha+1/2} \left(\cos \frac{\theta}{2} \right)^{\beta+1/2}.$$

Throughout the paper we use fairly standard notation, with all symbols referring either to the metric measure spaces $((-\pi, \pi), \mu_{\alpha, \beta}, |\cdot|)$ and $((-\pi, \pi), d\theta, |\cdot|)$ or to their subspaces $((0, \pi), \mu_{\alpha, \beta}^+, |\cdot|)$ and $((0, \pi), d\theta, |\cdot|)$, depending on the context. Here the measure $\mu_{\alpha, \beta}$ on the interval $(-\pi, \pi)$ is given by

$$d\mu_{\alpha, \beta}(\theta) = [\Psi^{\alpha, \beta}(\theta)]^2 d\theta,$$

and $\mu_{\alpha, \beta}^+$ is the restriction of $\mu_{\alpha, \beta}$ to the interval $(0, \pi)$. All the measures appearing above are doubling, hence the four metric measure spaces are actually spaces of homogeneous type.

For any function f on $(-\pi, \pi)$, by f^+ we mean its restriction to $(0, \pi)$. Further, \check{f} denotes the reflection of f , i.e. $\check{f}(\theta) = f(-\theta)$, and f_{even} and f_{odd} the even and odd parts of f , respectively,

$$f_{\text{even}} = \frac{1}{2}(f + \check{f}), \quad f_{\text{odd}} = \frac{1}{2}(f - \check{f}).$$

By $\langle f, g \rangle_{d\mu_{\alpha, \beta}}$ we mean $\int_{-\pi}^{\pi} f(\theta) \overline{g(\theta)} d\mu_{\alpha, \beta}(\theta)$ whenever the integral makes sense, and similarly for $\langle f, g \rangle_{d\mu_{\alpha, \beta}^+}$. Analogously, $\langle f, g \rangle$ stands for the L^2 scalar product with respect to the Lebesgue measure in $(-\pi, \pi)$ or $(0, \pi)$, depending on the context. Weighted L^p spaces with respect to $\mu_{\alpha, \beta}$ or $\mu_{\alpha, \beta}^+$ will be written as $L^p(w d\mu_{\alpha, \beta})$ and $L^p(w d\mu_{\alpha, \beta}^+)$, respectively, w being a non-negative weight. We simply write $L^p(w)$ when the underlying measure is the Lebesgue measure in $(-\pi, \pi)$ or $(0, \pi)$.

The Muckenhoupt classes of A_p weights related to the measures $\mu_{\alpha,\beta}$ and $\mu_{\alpha,\beta}^+$ will be denoted by $A_p^{\alpha,\beta}$ and $(A_p^{\alpha,\beta})^+$, respectively (see e.g. [3, Section 1] for the definitions). Note that a double power even weight $w_{r,s}$ defined by

$$w_{r,s} := \Psi^{r-1/2, s-1/2}$$

belongs to $A_p^{\alpha,\beta}$ for a fixed $1 \leq p < \infty$ if and only if

$$-(2\alpha + 2) < r < (2\alpha + 2)(p - 1) \text{ and } -(2\beta + 2) < s < (2\beta + 2)(p - 1),$$

with the upper inequalities weakened in case $p = 1$. For our purposes, we also define classes of double power even weights $B_p^{\alpha,\beta}$, $1 \leq p < \infty$, by the requirement that $w_{r,s} \in B_p^{\alpha,\beta}$ if and only if $w_{r+(\alpha+1/2)(p-2), s+(\beta+1/2)(p-2)} \in A_p^{\alpha,\beta}$. Thus

$$B_p^{\alpha,\beta} := \{w_{r,s} : -1 - (\alpha + 1/2)p < r < p - 1 + (\alpha + 1/2)p \\ \text{and } -1 - (\beta + 1/2)p < s < p - 1 + (\beta + 1/2)p\}$$

with the upper inequalities weakened in case $p = 1$. Observe that $B_p^{\alpha,\beta} \neq \emptyset$ and the trivial weight $w_{0,0} \equiv 1$ belongs to $B_p^{\alpha,\beta}$ if and only if $\alpha, \beta \geq -1/2$ or $\min(\alpha, \beta) < -1/2$ and p is restricted by the condition $-\min(\alpha, \beta) - 1/2 < 1/p < \min(\alpha, \beta) + 3/2$.

While writing estimates, we will frequently use the notation $X \lesssim Y$ to indicate that $X \leq CY$ with a positive constant C independent of significant quantities. We shall write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$.

2. Preliminaries

In this section we briefly describe the Jacobi settings we investigate. For the facts we present and also for further details concerning the non-symmetrized Jacobi contexts the reader is referred, for instance, to [8] and [4,13]. In case of the symmetrized settings we refer to the author's papers [3,6].

2.1. Non-symmetrized Jacobi settings. Let $P_n^{\alpha,\beta}$ denote the classical Jacobi polynomials. It is natural and convenient to apply the trigonometric parametrization $x = \cos \theta$ and consider the normalized trigonometric polynomials

$$(1) \quad \mathcal{P}_n^{\alpha,\beta}(\theta) = c_n^{\alpha,\beta} P_n^{\alpha,\beta}(\cos \theta),$$

where $c_n^{\alpha,\beta} > 0$ are suitable normalizing constants. It is well known that the system $\{\mathcal{P}_n^{\alpha,\beta} : n \geq 0\}$ is an orthonormal basis in $L^2(d\mu_{\alpha,\beta}^+)$. Moreover, $\mathcal{P}_n^{\alpha,\beta}$ are eigenfunctions of the Jacobi differential operator

$$(2) \quad \mathcal{J}_{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{\alpha - \beta + (\alpha + \beta + 1) \cos \theta}{\sin \theta} \frac{d}{d\theta} + \lambda_0^{\alpha,\beta},$$

being

$$\mathcal{J}_{\alpha,\beta} \mathcal{P}_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta} \mathcal{P}_n^{\alpha,\beta}, \quad n \geq 0,$$

where the eigenvalues are given by

$$\lambda_n^{\alpha,\beta} = \left(n + \frac{\alpha + \beta + 1}{2} \right)^2, \quad n \geq 0.$$

The decomposition

$$\mathcal{J}_{\alpha,\beta} = \delta_{\alpha,\beta}^* \delta + \lambda_0^{\alpha,\beta},$$

determines a natural derivative δ associated with $\mathcal{J}_{\alpha,\beta}$. Here

$$\delta = \frac{d}{d\theta}, \quad \delta_{\alpha,\beta}^* = -\frac{d}{d\theta} - \left(\alpha + \frac{1}{2} \right) \cot \frac{\theta}{2} + \left(\beta + \frac{1}{2} \right) \tan \frac{\theta}{2}$$

and $\delta_{\alpha,\beta}^*$ is the formal adjoint of δ in $L^2(d\mu_{\alpha,\beta}^+)$. Notice that $\delta \neq -\delta_{\alpha,\beta}^*$ in general.

The system of Jacobi functions arises by adjusting the system of Jacobi trigonometric polynomials so that the orthogonality measure is the Lebesgue measure $d\theta$ in $(0, \pi)$. Thus the Jacobi functions are given by

$$(3) \quad \phi_n^{\alpha,\beta} = \Psi^{\alpha,\beta} \mathcal{P}_n^{\alpha,\beta}, \quad n \geq 0,$$

and the resulting system $\{\phi_n^{\alpha,\beta} : n \geq 0\}$ constitutes an orthonormal basis in $L^2(d\theta)$. Moreover, each $\phi_n^{\alpha,\beta}$ is an eigenfunction of another Jacobi differential operator

$$L_{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{1 - 4\alpha^2}{16 \sin^2 \frac{\theta}{2}} - \frac{1 - 4\beta^2}{16 \cos^2 \frac{\theta}{2}},$$

the corresponding eigenvalue being again $\lambda_n^{\alpha,\beta}$. We note that $L_{\alpha,\beta}$ admits the decomposition

$$L_{\alpha,\beta} = D_{\alpha,\beta}^* D_{\alpha,\beta} + \lambda_0^{\alpha,\beta},$$

where

$$(4) \quad D_{\alpha,\beta} = \frac{d}{d\theta} - \frac{2\alpha + 1}{4} \cot \frac{\theta}{2} + \frac{2\beta + 1}{4} \tan \frac{\theta}{2}, \quad D_{\alpha,\beta}^* = D_{\alpha,\beta} - 2 \frac{d}{d\theta},$$

are the first order derivative naturally associated with $L_{\alpha,\beta}$ and its formal adjoint in $L^2(d\theta)$, respectively.

Observe that the Laplacians in the two Jacobi frameworks are conjugated via the identity $L_{\alpha,\beta}(\Psi^{\alpha,\beta}f) = \Psi^{\alpha,\beta}\mathcal{J}_{\alpha,\beta}f$, for suitable f . Similar relations hold for the derivatives and many other operators emerging from $\mathcal{J}_{\alpha,\beta}$ and $L_{\alpha,\beta}$.

2.2. Symmetrized Jacobi polynomial setting. This framework is related to the larger interval¹ $(-\pi, \pi)$ and emerges from applying the symmetrization procedure proposed in [12] to the system of Jacobi trigonometric polynomials. The symmetrized Jacobi operator $\mathbb{J}_{\alpha,\beta}$ is given by

$$\mathbb{J}_{\alpha,\beta}f = \mathcal{J}_{\alpha,\beta}f + \frac{(\alpha + \beta + 1) + (\alpha - \beta) \cos \theta}{\sin^2 \theta} f_{\text{odd}},$$

where $\mathcal{J}_{\alpha,\beta}$ is naturally extended to $(-\pi, \pi)$ by (2). This operator can be decomposed as

$$\mathbb{J}_{\alpha,\beta} = -\mathbb{D}_{\alpha,\beta}^2 + \lambda_0^{\alpha,\beta},$$

with the derivative $\mathbb{D}_{\alpha,\beta}$ given by

$$\mathbb{D}_{\alpha,\beta}f = \frac{df}{d\theta} + \frac{\alpha - \beta + (\alpha + \beta + 1) \cos \theta}{\sin \theta} f_{\text{odd}}.$$

It is remarkable that $\mathbb{D}_{\alpha,\beta}$ is formally skew-adjoint in $L^2(d\mu_{\alpha,\beta})$.

The orthonormal and complete in $L^2(d\mu_{\alpha,\beta})$ system $\{\Phi_n^{\alpha,\beta} : n \geq 0\}$ associated with $\mathbb{J}_{\alpha,\beta}$ is defined as

$$(5) \quad \Phi_n^{\alpha,\beta}(\theta) = \frac{1}{\sqrt{2}} \begin{cases} \mathcal{P}_{n/2}^{\alpha,\beta}(\theta), & n \text{ even,} \\ \frac{1}{2} \sin \theta \mathcal{P}_{(n-1)/2}^{\alpha+1,\beta+1}(\theta), & n \text{ odd,} \end{cases}$$

where $\mathcal{P}_k^{\alpha,\beta}$ are extended to even functions on $(-\pi, \pi)$ by (1). Each $\Phi_n^{\alpha,\beta}$ is an eigenfunction of $\mathbb{J}_{\alpha,\beta}$, we have

$$\mathbb{J}_{\alpha,\beta}\Phi_n^{\alpha,\beta} = \lambda_{\langle n \rangle}^{\alpha,\beta}\Phi_n^{\alpha,\beta}, \quad n \geq 0,$$

with the notation

$$\langle n \rangle = \left\lfloor \frac{n+1}{2} \right\rfloor = \max \left\{ k \in \mathbb{Z} : k \leq \frac{n+1}{2} \right\}.$$

¹Formally, the space is the union $(-\pi, 0) \cup (0, \pi)$. However, usually we will identify it with the interval $(-\pi, \pi)$, since for the aspects of the theory we are interested in, such as L^p inequalities, the single point $\theta = 0$ is negligible. This remark concerns also the symmetrized function setting discussed in the next subsection.

Thus $\mathbb{J}_{\alpha,\beta}$, considered initially on $C_c^2((-\pi, \pi) \setminus \{0\})$, has a natural self-adjoint extension in $L^2(d\mu_{\alpha,\beta})$, still denoted by $\mathbb{J}_{\alpha,\beta}$ and given by

$$(6) \quad \mathbb{J}_{\alpha,\beta} f = \sum_{n=0}^{\infty} \lambda_{\langle n \rangle}^{\alpha,\beta} \langle f, \Phi_n^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}} \Phi_n^{\alpha,\beta}$$

on the domain $\text{Dom } \mathbb{J}_{\alpha,\beta}$ consisting of all functions $f \in L^2(d\mu_{\alpha,\beta})$ for which the defining series converges in $L^2(d\mu_{\alpha,\beta})$. Clearly, the spectral decomposition of $\mathbb{J}_{\alpha,\beta}$ is given by (6).

The semigroup of operators generated by the square root of $\mathbb{J}_{\alpha,\beta}$ will be denoted by $\{\mathbb{H}_t^{\alpha,\beta}\}$. We have, for $f \in L^2(d\mu_{\alpha,\beta})$ and $t \geq 0$,

$$(7) \quad \mathbb{H}_t^{\alpha,\beta} f = \exp(-t\sqrt{\mathbb{J}_{\alpha,\beta}}) f = \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_{\langle n \rangle}^{\alpha,\beta}}\right) \langle f, \Phi_n^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}} \Phi_n^{\alpha,\beta},$$

the convergence being in $L^2(d\mu_{\alpha,\beta})$. In fact, for $t > 0$ the last series converges pointwise for any $f \in L^p(wd\mu_{\alpha,\beta})$, $w \in A_p^{\alpha,\beta}$, $1 \leq p < \infty$, and defines a smooth function of $(t, \theta) \in (0, \infty) \times (-\pi, \pi)$; see [3, Section 2]. Thus (7) can be regarded as the definition of $\{\mathbb{H}_t^{\alpha,\beta}\}_{t>0}$ on the weighted spaces $L^p(wd\mu_{\alpha,\beta})$, $w \in A_p^{\alpha,\beta}$, $1 \leq p < \infty$. The integral representation of $\{\mathbb{H}_t^{\alpha,\beta}\}_{t>0}$, valid on the weighted L^p spaces mentioned above, is

$$\mathbb{H}_t^{\alpha,\beta} f(\theta) = \int_{-\pi}^{\pi} \mathbb{H}_t^{\alpha,\beta}(\theta, \varphi) f(\varphi) d\mu_{\alpha,\beta}(\varphi), \quad \theta \in (-\pi, \pi), t > 0,$$

with the symmetrized Jacobi–Poisson kernel

$$\mathbb{H}_t^{\alpha,\beta}(\theta, \varphi) = \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_{\langle n \rangle}^{\alpha,\beta}}\right) \Phi_n^{\alpha,\beta}(\theta) \Phi_n^{\alpha,\beta}(\varphi).$$

The last series converges absolutely and defines a smooth function of $(t, \theta, \varphi) \in (0, \infty) \times (-\pi, \pi)^2$.

The central objects of our study are the following linear or sublinear operators associated with $\mathbb{J}_{\alpha,\beta}$.

- (i) Symmetrized Riesz–Jacobi transforms of orders $N = 1, 2, \dots$

$$\mathbb{R}_N^{\alpha,\beta} f = \sum_{n=0}^{\infty} (\lambda_{\langle n \rangle}^{\alpha,\beta})^{-N/2} \langle f, \Phi_n^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}} \mathbb{D}_{\alpha,\beta}^N \Phi_n^{\alpha,\beta}.$$

- (ii) Multipliers of Laplace and Laplace–Stieltjes transform type

$$\mathbb{M}_m^{\alpha,\beta} f(\theta) = \sum_{n=0}^{\infty} m\left(\sqrt{\lambda_{\langle n \rangle}^{\alpha,\beta}}\right) \langle f, \Phi_n^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}} \Phi_n^{\alpha,\beta},$$

where either $m(z) = m_\phi(z) = \int_0^\infty z e^{-tz} \phi(t) dt$ with $\phi \in L^\infty(\mathbb{R}_+, dt)$ or $m(z) = m_\nu(z) = \int_{\mathbb{R}_+} e^{-tz} d\nu(t)$ with ν being a signed or complex Borel measure on $\mathbb{R}_+ = (0, \infty)$ whose total variation satisfies

$$(8) \quad \int_{\mathbb{R}_+} \exp\left(-t\sqrt{\lambda_0^{\alpha,\beta}}\right) d|\nu|(t) < \infty.$$

(iii) The symmetrized Jacobi–Poisson semigroup maximal operator

$$\mathbb{H}_*^{\alpha,\beta} f(\theta) = \|\mathbb{H}_t^{\alpha,\beta} f(\theta)\|_{L^\infty(\mathbb{R}_+, dt)}, \quad \theta \in (-\pi, \pi).$$

(iv) Symmetrized mixed square functions of arbitrary orders M, N

$$\mathbb{G}_{M,N}^{\alpha,\beta}(f)(\theta) = \|\partial_t^M \mathbb{D}_{\alpha,\beta}^N \mathbb{H}_t^{\alpha,\beta} f(\theta)\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}, \quad \theta \in (-\pi, \pi),$$

where $M, N = 0, 1, 2, \dots$ and $M + N > 0$.

The operators $\mathbb{R}_N^{\alpha,\beta}$ and $\mathbb{M}_m^{\alpha,\beta}$ are well defined and bounded on $L^2(d\mu_{\alpha,\beta})$. In case of $\mathbb{M}_m^{\alpha,\beta}$ this follows from Plancherel’s theorem and the boundedness of m . The case of the Riesz transforms is covered by [12, Proposition 4.4]. We note that if $\alpha + \beta = -1$ then $0 = \lambda_0^{\alpha,\beta}$ is the eigenvalue of $\mathbb{J}_{\alpha,\beta}$ and we actually need to interpret the series defining $\mathbb{R}_N^{\alpha,\beta}$ as the sum over $n \geq 1$, in view of the identity $\mathbb{D}_{\alpha,\beta} \Phi_0^{\alpha,\beta} \equiv 0$. As for the remaining operators $\mathbb{H}_*^{\alpha,\beta}$ and $\mathbb{G}_{M,N}^{\alpha,\beta}$, their definitions are understood pointwise and make sense for general $f \in L^p(w d\mu_{\alpha,\beta})$, $W \in A_p^{\alpha,\beta}$, $1 \leq p < \infty$, since, for such f , $\mathbb{H}_t^{\alpha,\beta} f(\theta)$ is a smooth function of $(t, \theta) \in (0, \infty) \times (-\pi, \pi)$.

In the definitions of $\mathbb{R}_N^{\alpha,\beta}$ and $\mathbb{G}_{M,N}^{\alpha,\beta}$ we use the non-local operator $\mathbb{D}_{\alpha,\beta}$ instead of the seemingly more natural ordinary derivative δ . This is because $\mathbb{D}_{\alpha,\beta}$, in contrast with δ , is formally skew-adjoint in $L^2(d\mu_{\alpha,\beta})$ and it provides a (symmetric) decomposition of $\mathbb{J}_{\alpha,\beta}$. Moreover, it gives rise to a conjugacy scheme that is very close to the classical one. For a thorough discussion clarifying the relevance of the choice of $\mathbb{D}_{\alpha,\beta}$ we refer the reader to [12].

2.3. Symmetrized Jacobi function setting. This context emerges from applying the symmetrization procedure from [12] to the system of Jacobi functions. The extended measure space is $(-\pi, \pi)$ equipped with Lebesgue measure $d\theta$. We arrive at the symmetrized Laplacian

$$\mathbb{L}_{\alpha,\beta} = -\overline{\mathbb{D}}_{\alpha,\beta}^2 + \lambda_0^{\alpha,\beta},$$

where the associated derivative is given by

$$\overline{\mathbb{D}}_{\alpha,\beta} f = \frac{df}{d\theta} - \left(\frac{2\alpha + 1}{4} \cot \frac{\theta}{2} - \frac{2\beta + 1}{4} \tan \frac{\theta}{2} \right) \check{f} = D_{\alpha,\beta} f_{\text{even}} - D_{\alpha,\beta}^* f_{\text{odd}},$$

with $D_{\alpha,\beta}$ and $D_{\alpha,\beta}^*$ given on $(-\pi, \pi)$ by (4).

The orthonormal basis in $L^2(d\theta)$ of eigenfunctions of $\mathbb{L}_{\alpha,\beta}$ is $\{\Theta_n^{\alpha,\beta} : n \geq 0\}$,

$$\Theta_n^{\alpha,\beta}(\theta) = \frac{1}{\sqrt{2}} \begin{cases} \phi_{n/2}^{\alpha,\beta}(\theta), & n \text{ even,} \\ \text{sign}(\theta)\phi_{(n-1)/2}^{\alpha+1,\beta+1}(\theta), & n \text{ odd,} \end{cases}$$

where $\phi_n^{\alpha,\beta}$ are even functions on $(-\pi, \pi)$ given by (3) and implicitly by (1). The corresponding eigenvalues are $\lambda_{\langle n \rangle}^{\alpha,\beta}$,

$$\mathbb{L}_{\alpha,\beta}\Theta_n^{\alpha,\beta} = \lambda_{\langle n \rangle}^{\alpha,\beta}\Theta_n^{\alpha,\beta}, \quad n \geq 0.$$

Consequently, $\mathbb{L}_{\alpha,\beta}$ has a natural self-adjoint extension from $C_c^2((-\pi, \pi) \setminus \{0\})$ to $L^2(-\pi, \pi)$, still denoted by $\mathbb{L}_{\alpha,\beta}$, whose spectral decomposition is given by the $\Theta_n^{\alpha,\beta}$, see (6).

The semigroup of operators generated by the square root of $\mathbb{L}_{\alpha,\beta}$ is denoted by $\{\overline{\mathbb{H}}_t^{\alpha,\beta}\}$. For $f \in L^2(-\pi, \pi)$ and $t \geq 0$ one has

$$(9) \quad \overline{\mathbb{H}}_t^{\alpha,\beta} f = \exp(-t\sqrt{\mathbb{L}_{\alpha,\beta}})f = \sum_{n=0}^{\infty} \exp(-t\sqrt{\lambda_{\langle n \rangle}^{\alpha,\beta}}) \langle f, \Theta_n^{\alpha,\beta} \rangle \Theta_n^{\alpha,\beta},$$

with the convergence in $L^2(-\pi, \pi)$. Moreover, for $t > 0$ the last series converges pointwise on $(-\pi, \pi) \setminus \{0\}$ and defines a smooth function of $(t, \theta) \in (0, \infty) \times [(-\pi, \pi) \setminus \{0\}]$ provided that $f \in L^p(w)$, $1 \leq p < \infty$, and $w = w_{r,s}$ is an even double power weight satisfying $r < p - 1 + (\alpha + 1/2)p$ and $s < p - 1 + (\beta + 1/2)p$, with the last two inequalities weakened in case $p = 1$; see [6, Section 4] and [4, Section 2], and also [13, Section 2], for the relevant arguments. This means, in particular, that (9) defines $\overline{\mathbb{H}}_t^{\alpha,\beta} f$, $t > 0$, on $L^p(w)$, $w \in B_p^{\alpha,\beta}$, $1 \leq p < \infty$.

As in the previous symmetrized setting, we consider the following operators related to $\mathbb{L}_{\alpha,\beta}$.

(i) Symmetrized Riesz–Jacobi transforms of orders $N = 1, 2, \dots$

$$\overline{\mathbb{R}}_N^{\alpha,\beta} f = \sum_{n=0}^{\infty} (\lambda_{\langle n \rangle}^{\alpha,\beta})^{-N/2} \langle f, \Theta_n^{\alpha,\beta} \rangle \overline{\mathbb{D}}_{\alpha,\beta}^N \Theta_n^{\alpha,\beta}.$$

(ii) Multipliers of Laplace and Laplace–Stieltjes transform type

$$\overline{\mathbb{M}}_m^{\alpha,\beta} f(\theta) = \sum_{n=0}^{\infty} m(\sqrt{\lambda_{\langle n \rangle}^{\alpha,\beta}}) \langle f, \Theta_n^{\alpha,\beta} \rangle \Theta_n^{\alpha,\beta},$$

where $m = m_\phi$ or $m = m_\nu$, with m_ϕ and m_ν as in Section 2.2.

(iii) The symmetrized Jacobi–Poisson semigroup maximal operator

$$\overline{\mathbb{H}}_*^{\alpha,\beta} f(\theta) = \|\overline{\mathbb{H}}_t^{\alpha,\beta} f(\theta)\|_{L^\infty(\mathbb{R}_+, dt)}, \quad \theta \in (-\pi, \pi) \setminus \{0\}.$$

(iv) Symmetrized mixed square functions of arbitrary orders M, N

$$\overline{\mathbb{G}}_{M,N}^{\alpha,\beta}(f)(\theta) = \|\partial_t^M \overline{\mathbb{D}}_{\alpha,\beta}^N \overline{\mathbb{H}}_t^{\alpha,\beta} f(\theta)\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}, \quad \theta \in (-\pi, \pi) \setminus \{0\},$$

where $M, N = 0, 1, 2, \dots$ and $M + N > 0$.

The operators $\overline{\mathbb{R}}_N^{\alpha,\beta}$ and $\overline{\mathbb{M}}_m^{\alpha,\beta}$ are well defined on $L^2(-\pi, \pi)$, by [12, Proposition 4.4] and Plancherel’s theorem, respectively (in case $\alpha + \beta = -1$ the bottom eigenvalue is 0 and a proper interpretation of $\overline{\mathbb{R}}_N^{\alpha,\beta}$ is needed, see the case of $\mathbb{R}_N^{\alpha,\beta}$). The definitions of $\overline{\mathbb{H}}_*^{\alpha,\beta}$ and $\overline{\mathbb{G}}_{M,N}^{\alpha,\beta}$ are understood pointwise for any $f \in L^p(w)$, $w \in B_p^{\alpha,\beta}$, $1 \leq p < \infty$. This indeed makes sense since, for such f , $\overline{\mathbb{H}}_t^{\alpha,\beta} f(\theta)$ is a smooth function of $(t, \theta) \in (0, \infty) \times [(-\pi, \pi) \setminus \{0\}]$.

The following final observations are in order. The two symmetrized Jacobi settings are conjugated by means of $\Psi^{\alpha,\beta}$. We have

$$\mathbb{L}_{\alpha,\beta}(\Psi^{\alpha,\beta} f) = \Psi^{\alpha,\beta} \mathbb{J}_{\alpha,\beta} f$$

for suitable f and analogous relations hold for the operators (i)–(iv). This allows us to transmit certain mapping properties of the relevant operators between the two frameworks, see Section 3.2 below. Moreover, the non-symmetrized settings, viz. those related to $(0, \pi)$, are naturally embedded in the corresponding symmetrized ones. Consequently, essentially any results in the spirit of this paper in the symmetrized situations can be projected suitably (by restricting them to even functions) onto the non-symmetrized frameworks. Some new results following from this transference are presented in Section 3.3.

3. Main results

In this section we state the main results of the paper. For clarity and the reader’s convenience, we arrange them into three subsections corresponding to the two symmetrized Jacobi contexts, and the non-symmetrized situations.

3.1. Results in the symmetrized Jacobi polynomial setting.

The theorem below is the principal result of the paper, since most of our results in the other settings can be viewed as its consequences.

THEOREM 3.1. *Let $\alpha, \beta > -1$ and w be an even weight on $(-\pi, \pi)$. Then the maximal operator $\mathbb{H}_*^{\alpha, \beta}$ and the square functions $\mathbb{G}_{M, N}^{\alpha, \beta}$, $M, N = 0, 1, 2, \dots$, $M + N > 0$, are bounded on $L^p(w d\mu_{\alpha, \beta})$, $w \in A_p^{\alpha, \beta}$, $1 < p < \infty$ and from $L^1(w d\mu_{\alpha, \beta})$ to weak $L^1(w d\mu_{\alpha, \beta})$, $w \in A_1^{\alpha, \beta}$. Furthermore, the Riesz transforms $\mathbb{R}_N^{\alpha, \beta}$, $N = 1, 2, \dots$ and the multipliers $\mathbb{M}_m^{\alpha, \beta}$ extend uniquely to bounded linear operators on $L^p(w d\mu_{\alpha, \beta})$, $w \in A_p^{\alpha, \beta}$, $1 < p < \infty$ and from $L^1(w d\mu_{\alpha, \beta})$ to weak $L^1(w d\mu_{\alpha, \beta})$, $w \in A_1^{\alpha, \beta}$.*

Following [3], we now outline a reduction of the proof of Theorem 3.1 that allows us to approach the problem by means of the powerful Calderón–Zygmund theory. Then the main difficulty will be showing suitable kernel estimates, a tricky technical task to which we devote Section 5. In the first step, proving Theorem 3.1 is reduced to showing analogous mapping properties for suitably defined ‘restricted’ operators related to the smaller space $((0, \pi), \mu_{\alpha, \beta}^+, |\cdot|)$. This proceeds as follows.

Using (5) we decompose

$$\begin{aligned} \mathbb{H}_t^{\alpha, \beta}(\theta, \varphi) &= \frac{1}{2} \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_n^{\alpha, \beta}}\right) \mathcal{P}_n^{\alpha, \beta}(\theta) \mathcal{P}_n^{\alpha, \beta}(\varphi) \\ &+ \frac{1}{8} \sin \theta \sin \varphi \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_{n+1}^{\alpha, \beta}}\right) \mathcal{P}_n^{\alpha+1, \beta+1}(\theta) \mathcal{P}_n^{\alpha+1, \beta+1}(\varphi) \\ &\equiv H_t^{\alpha, \beta}(\theta, \varphi) + \tilde{H}_t^{\alpha, \beta}(\theta, \varphi). \end{aligned}$$

The restriction of $H_t^{\alpha, \beta}(\theta, \varphi)$ to $\theta, \varphi \in (0, \pi)$ coincides, up to the factor $1/2$, with the standard (non-symmetrized) Jacobi–Poisson kernel related to $\mathcal{J}_{\alpha, \beta}$ and studied recently in [8–10]. Furthermore, since each $\mathcal{P}_n^{\alpha, \beta}$ is an even function on $(-\pi, \pi)$, we see that $H_t^{\alpha, \beta}(\theta, \varphi)$ and $\tilde{H}_t^{\alpha, \beta}(\theta, \varphi)$ are even and odd, respectively, functions both of θ and φ . Notice also that, since $\lambda_{n+1}^{\alpha, \beta} = \lambda_n^{\alpha+1, \beta+1}$, we have

$$\tilde{H}_t^{\alpha, \beta}(\theta, \varphi) = \frac{1}{4} \sin \theta \sin \varphi H_t^{\alpha+1, \beta+1}(\theta, \varphi).$$

Using the sharp description of the Jacobi–Poisson kernel obtained in [9, Theorem A.1] and [10, Theorem 6.1] it is straightforward to see that $\tilde{H}_t^{\alpha, \beta}(\theta, \varphi)$ is controlled pointwise by $H_t^{\alpha, \beta}(\theta, \varphi)$,

$$(10) \quad \left| \tilde{H}_t^{\alpha, \beta}(\theta, \varphi) \right| \lesssim H_t^{\alpha, \beta}(\theta, \varphi), \quad \theta, \varphi \in (-\pi, \pi), \quad t > 0.$$

Next, we consider the operators acting on $L^2(d\mu_{\alpha,\beta}^+)$ and defined by

$$(H_t^{\alpha,\beta})^+ f = \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_n^{\alpha,\beta}}\right) \langle f, \Phi_{2n}^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}^+} \Phi_{2n}^{\alpha,\beta},$$

$$(\tilde{H}_t^{\alpha,\beta})^+ f = \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_{n+1}^{\alpha,\beta}}\right) \langle f, \Phi_{2n+1}^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}^+} \Phi_{2n+1}^{\alpha,\beta},$$

for $t > 0$. Similarly as in the case of $\mathbb{H}_t^{\alpha,\beta}$, the series defining $(H_t^{\alpha,\beta})^+$ and $(\tilde{H}_t^{\alpha,\beta})^+$ converge pointwise for any $f \in L^p(w d\mu_{\alpha,\beta}^+)$, $w \in (A_p^{\alpha,\beta})^+$, $1 \leq p < \infty$, and give rise to smooth functions of $(t, \theta) \in (0, \infty) \times (0, \pi)$. The integral representations of $(H_t^{\alpha,\beta})^+$ and $(\tilde{H}_t^{\alpha,\beta})^+$ are

$$(H_t^{\alpha,\beta})^+ f(\theta) = \int_0^\pi H_t^{\alpha,\beta}(\theta, \varphi) f(\varphi) d\mu_{\alpha,\beta}^+(\varphi),$$

$$(\tilde{H}_t^{\alpha,\beta})^+ f(\theta) = \int_0^\pi \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) f(\varphi) d\mu_{\alpha,\beta}^+(\varphi).$$

Denote

$$\delta_N^{\text{even}} = \underbrace{\dots \delta \delta_{\alpha,\beta}^* \delta \delta_{\alpha,\beta}^* \delta}_{N \text{ components}}, \quad \delta_N^{\text{odd}} = \underbrace{\dots \delta_{\alpha,\beta}^* \delta \delta_{\alpha,\beta}^* \delta \delta_{\alpha,\beta}^*}_{N \text{ components}},$$

with the natural convention for the case $N = 0$. These derivatives correspond to the action of $\mathbb{D}_{\alpha,\beta}^N$ on even and odd functions, respectively. In particular,

$$\mathbb{D}_{\alpha,\beta}^N f = (-1)^{\lfloor N/2 \rfloor} \delta_N^{\text{even}} f_{\text{even}} + (-1)^{\lfloor (N+1)/2 \rfloor} \delta_N^{\text{odd}} f_{\text{odd}}.$$

We take this opportunity to clarify certain issue that occurred in our previous article [3]. Namely, in [3, p. 260, line-5] the factors $(-1)^{\lfloor N/2 \rfloor}$ and $(-1)^{\lfloor (N+1)/2 \rfloor}$ coming from the above decomposition of $\mathbb{D}_{\alpha,\beta}^N$ are missing. This does not affect the line of reasoning in any manner, however some identities are valid only up to a sign. This remark concerns [3, Corollary 3.2] and following it [3, Proof of Proposition 2.4].

Now we are ready to define the ‘restricted’ operators we need:

(i)

$$(R_N^{\alpha,\beta})^+ f = \sum_{n=1}^{\infty} (\lambda_n^{\alpha,\beta})^{-N/2} \langle f, \Phi_{2n}^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}^+} \delta_N^{\text{even}} \Phi_{2n}^{\alpha,\beta}, \quad f \in L^2(d\mu_{\alpha,\beta}^+),$$

$$(\tilde{R}_N^{\alpha,\beta})^+ f = \sum_{n=0}^{\infty} (\lambda_{n+1}^{\alpha,\beta})^{-N/2} \langle f, \Phi_{2n+1}^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}^+} \delta_N^{\text{odd}} \Phi_{2n+1}^{\alpha,\beta}, \quad f \in L^2(d\mu_{\alpha,\beta}^+),$$

(ii)

$$(M_m^{\alpha,\beta})^+ f = \sum_{n=0}^{\infty} m \left(\sqrt{\lambda_n^{\alpha,\beta}} \right) \langle f, \Phi_{2n}^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}^+} \Phi_{2n}^{\alpha,\beta}, \quad f \in L^2(d\mu_{\alpha,\beta}^+),$$

$$(\widetilde{M}_m^{\alpha,\beta})^+ f = \sum_{n=0}^{\infty} m \left(\sqrt{\lambda_{n+1}^{\alpha,\beta}} \right) \langle f, \Phi_{2n+1}^{\alpha,\beta} \rangle_{d\mu_{\alpha,\beta}^+} \Phi_{2n+1}^{\alpha,\beta}, \quad f \in L^2(d\mu_{\alpha,\beta}^+),$$

(iii)

$$(H_*^{\alpha,\beta})^+ f(\theta) = \| (H_t^{\alpha,\beta})^+ f(\theta) \|_{L^\infty(\mathbb{R}_+, dt)},$$

$$(\widetilde{H}_*^{\alpha,\beta})^+ f(\theta) = \| (\widetilde{H}_t^{\alpha,\beta})^+ f(\theta) \|_{L^\infty(\mathbb{R}_+, dt)},$$

(iv)

$$(G_{M,N}^{\alpha,\beta})^+(f)(\theta) = \| \partial_t^M \delta_N^{\text{even}} (H_t^{\alpha,\beta})^+ f(\theta) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)},$$

$$(\widetilde{G}_{M,N}^{\alpha,\beta})^+(f)(\theta) = \| \partial_t^M \delta_N^{\text{odd}} (\widetilde{H}_t^{\alpha,\beta})^+ f(\theta) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}.$$

The proof of Theorem 3.1 reduces to showing the following statement, see [3, Section 2] for the details.

THEOREM 3.2. *Let $\alpha, \beta > -1$. Then the operators*

$$(H_*^{\alpha,\beta})^+, (\widetilde{H}_*^{\alpha,\beta})^+, (G_{M,N}^{\alpha,\beta})^+, (\widetilde{G}_{M,N}^{\alpha,\beta})^+, \quad M, N = 0, 1, 2, \dots, \quad M + N > 0,$$

are bounded on $L^p(w d\mu_{\alpha,\beta}^+)$, $w \in (A_p^{\alpha,\beta})^+$, $1 < p < \infty$, and from $L^1(w d\mu_{\alpha,\beta}^+)$ to weak $L^1(w d\mu_{\alpha,\beta}^+)$, $w \in (A_1^{\alpha,\beta})^+$. Furthermore, the operators

$$(R_N^{\alpha,\beta})^+, (\widetilde{R}_N^{\alpha,\beta})^+, \quad N = 1, 2, \dots, \quad (M_m^{\alpha,\beta})^+, (\widetilde{M}_m^{\alpha,\beta})^+$$

extend uniquely to bounded linear operators on $L^p(w d\mu_{\alpha,\beta}^+)$, $w \in (A_p^{\alpha,\beta})^+$, $1 < p < \infty$, and from $L^1(w d\mu_{\alpha,\beta}^+)$ to weak $L^1(w d\mu_{\alpha,\beta}^+)$, $w \in (A_1^{\alpha,\beta})^+$.

A part of Theorem 3.2 is covered by the existing literature. More precisely, $(H_*^{\alpha,\beta})^+$ and $(M_m^{\alpha,\beta})^+$ were proved to possess the desired boundedness properties in [10, Corollary 5.2], and $(R_N^{\alpha,\beta})^+$ was treated in [2, Proposition 3.7] (here we implicitly identify these ‘restricted’ operators with the corresponding operators in the non-symmetrized Jacobi polynomial setting). Moreover, in view of (10), the mapping properties of $(H_*^{\alpha,\beta})^+$ in question imply the same mapping properties for $(\widetilde{H}_*^{\alpha,\beta})^+$. Therefore, to prove Theorem 3.2 it remains to deal with $(\widetilde{R}_N^{\alpha,\beta})^+$, $(\widetilde{M}_m^{\alpha,\beta})^+$, $(G_{M,N}^{\alpha,\beta})^+$ and $(\widetilde{G}_{M,N}^{\alpha,\beta})^+$. Finally, we remark that Theorem 3.2 was stated and justified in [3] under

the restriction $\alpha, \beta \geq -1/2$. Here, apart from extending that result, we take the opportunity to simplify and shorten the reasoning given in [3] by means of the techniques elaborated recently in [10].

The part of Theorem 3.2 that needs to be proved is a consequence of a more general result stated below. In case of $(\tilde{R}_N^{\alpha,\beta})^+$ and $(\tilde{M}_m^{\alpha,\beta})^+$ this claim follows directly from the general Calderón–Zygmund theory for spaces of homogeneous type. The implication in case of $(G_{M,N}^{\alpha,\beta})^+$ and $(\tilde{G}_{M,N}^{\alpha,\beta})^+$ is easily justified by arguments analogous to those given in the proof of [8, Corollary 2.5]. Notice that $(G_{M,N}^{\alpha,\beta})^+$ and $(\tilde{G}_{M,N}^{\alpha,\beta})^+$ are not linear, but can be naturally interpreted as vector-valued linear operators taking values in the Banach space $L^2(\mathbb{R}_+, t^{2M+2N-1} dt)$.

THEOREM 3.3. *Assume that $\alpha, \beta > -1$. The operator $s(\tilde{R}_N^{\alpha,\beta})^+$, $N = 1, 2, \dots$ and $(\tilde{M}_m^{\alpha,\beta})^+$ are Calderón–Zygmund operators in the sense of the space of homogeneous type $((0, \pi), d\mu_{\alpha,\beta}^+, |\cdot|)$. Further, the operators $(G_{M,N}^{\alpha,\beta})^+$ and $(\tilde{G}_{M,N}^{\alpha,\beta})^+$, $M, N = 0, 1, 2, \dots$, $M + N > 0$, viewed as vector-valued linear operators, are Calderón–Zygmund operators, in the sense of the same space of homogeneous type, associated with the Banach space $L^2(\mathbb{R}_+, t^{2M+2N-1} dt)$.*

The proof of Theorem 3.3 splits naturally into showing the following three results. The first two of them are essentially contained in [3].

PROPOSITION 3.4. *Let $\alpha, \beta > -1$. The operators $(\tilde{R}_N^{\alpha,\beta})^+$, $N = 1, 2, \dots$, $(\tilde{M}_m^{\alpha,\beta})^+$, $(G_{M,N}^{\alpha,\beta})^+$ and $(\tilde{G}_{M,N}^{\alpha,\beta})^+$, $M, N = 0, 1, 2, \dots$, $M + N > 0$, are bounded on $L^2(d\mu_{\alpha,\beta}^+)$. In particular, the operators*

$$(G_{M,N}^{\alpha,\beta})^+, (\tilde{G}_{M,N}^{\alpha,\beta})^+, \quad M, N = 0, 1, 2, \dots, \quad M + N > 0,$$

viewed as vector-valued linear operators, are bounded from $L^2(d\mu_{\alpha,\beta}^+)$ to the Bochner–Lebesgue space $L^2_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}(d\mu_{\alpha,\beta}^+)$.

PROOF. It suffices to observe that all the arguments given in the proof of [3, Proposition 2.4] remain valid for the full range $\alpha, \beta > -1$. \square

For $\theta, \varphi \in (0, \pi)$ define the kernels

$$\begin{aligned} \tilde{M}_\phi^{\alpha,\beta}(\theta, \varphi) &= - \int_0^\infty \partial_t \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \phi(t) dt, \\ \tilde{M}_\nu^{\alpha,\beta}(\theta, \varphi) &= \int_{(0,\infty)} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) d\nu(t), \\ \tilde{R}_N^{\alpha,\beta}(\theta, \varphi) &= \frac{1}{\Gamma(N)} \int_0^\infty \delta_N^{\text{odd}} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) t^{N-1} dt, \quad N \geq 1. \end{aligned}$$

Here and elsewhere the derivatives δ_N^{even} and δ_N^{odd} act always on the θ variable. The next result establishes the weak association (see [3, p. 262]) of the operators in question with the corresponding integral kernels.

PROPOSITION 3.5. *Let $\alpha, \beta > -1$. The operators $(\tilde{R}_N^{\alpha, \beta})^+$, $N = 1, 2, \dots$ and $(\tilde{M}_m^{\alpha, \beta})^+$ are associated in the Calderón–Zygmund theory sense with the following scalar-valued kernels:*

$$(\tilde{R}_N^{\alpha, \beta})^+ \sim \tilde{R}_N^{\alpha, \beta}(\theta, \varphi), \quad (\tilde{M}_m^{\alpha, \beta})^+ \sim \tilde{M}_m^{\alpha, \beta}(\theta, \varphi),$$

where $\tilde{M}_m^{\alpha, \beta}(\theta, \varphi)$ is equal either to $\tilde{M}_\phi^{\alpha, \beta}(\theta, \varphi)$ or $\tilde{M}_\nu^{\alpha, \beta}(\theta, \varphi)$, depending on whether $m = m_\phi$ or $m = m_\nu$, respectively. Further, the operators $(G_{M, N}^{\alpha, \beta})^+$, $(\tilde{G}_{M, N}^{\alpha, \beta})^+$, $M, N = 0, 1, 2, \dots$, $M + N > 0$, viewed as vector-valued linear operators, are associated with the following kernels taking values in $L^2(\mathbb{R}_+, t^{2M+2N-1} dt)$:

$$(G_{M, N}^{\alpha, \beta})^+ \sim \{\partial_t^M \delta_N^{\text{even}} H_t^{\alpha, \beta}(\theta, \varphi)\}_{t>0}, \quad (\tilde{G}_{M, N}^{\alpha, \beta})^+ \sim \{\partial_t^M \delta_N^{\text{odd}} \tilde{H}_t^{\alpha, \beta}(\theta, \varphi)\}_{t>0}.$$

PROOF. When $\alpha, \beta \geq -1/2$, this is contained in [3, Proposition 2.5]. It is enough to notice that the proof given in [3] works in fact for all $\alpha, \beta > -1$, provided that we combine it with Proposition 3.4 and the estimates obtained independently in Section 5 below. \square

In order to complete the proof of Theorem 3.3 we must show that the kernels in question satisfy the so-called standard estimates, see (11)-(14) in Section 5. This is contained in the next statement.

THEOREM 3.6. *Assume that $\alpha, \beta > -1$. The scalar-valued kernels from Proposition 3.5 satisfy the standard estimates (11) and (14) with $\mathbb{B} = \mathbb{C}$. The vector-valued kernels from Proposition 3.5 satisfy the standard estimates (11) and (14) with $\mathbb{B} = L^2(\mathbb{R}_+, t^{2M+2N-1} dt)$.*

The proof of Theorem 3.6 requires an involved analysis and is given in Section 5.

3.2. Results in the symmetrized Jacobi function setting. We now state a counterpart of Theorem 3.1 in the symmetrized Jacobi function context. Actually, it is a consequence of Theorem 3.1 obtained by means of the already announced transference, see the end of Section 2. For the sake of clarity, we restrict here to even double power weights, since this class is invariant under multiplication by powers of $\Psi^{\alpha, \beta}$. Nevertheless, Theorem 3.1 combined with the transference method allows one to conclude results with more general weights as well. This is left to interested readers.

THEOREM 3.7. *Let $\alpha, \beta > -1$ and $1 < p < \infty$. Then the maximal operator $\overline{\mathbb{H}}_*^{\alpha, \beta}$ and the square functions $\overline{\mathbb{G}}_{M, N}^{\alpha, \beta}$, $M, N = 0, 1, 2, \dots$, $M + N > 0$, are bounded on $L^p(w)$, $w \in B_p^{\alpha, \beta}$. Furthermore, the Riesz transforms $\overline{\mathbb{R}}_N^{\alpha, \beta}$, $N = 1, 2, \dots$ and the multipliers $\overline{\mathbb{M}}_m^{\alpha, \beta}$ extend uniquely to bounded linear operators on $L^p(w)$, $w \in B_p^{\alpha, \beta}$.*

PROOF. Consider first $\overline{\mathbb{H}}_*^{\alpha, \beta}$. Observe that

$$\overline{\mathbb{H}}_*^{\alpha, \beta} f = \Psi^{\alpha, \beta} \mathbb{H}_*^{\alpha, \beta} (\Psi^{-\alpha-1, -\beta-1} f).$$

This relation, combined in a straightforward manner with Theorem 3.1 specified to $p > 1$ and even double power weights in $A_p^{\alpha, \beta}$, implies weighted L^p -boundedness for $\overline{\mathbb{H}}_*^{\alpha, \beta}$ with weights belonging to $B_p^{\alpha, \beta}$.

The other operators are treated similarly. The only difference is that in the cases of $\overline{\mathbb{R}}_N^{\alpha, \beta}$ and $\overline{\mathbb{M}}_m^{\alpha, \beta}$ one shows the desired weighted boundedness on the linear span of the $\Theta_n^{\alpha, \beta}$, $n \geq 0$, and then uses a density argument to extend it to the whole $L^p(w)$. \square

We remark that the weak type boundedness results contained in Theorem 3.1 and corresponding to the case $p = 1$ cannot be transferred in a similar spirit to the present context. On the other hand, it is perhaps of interest to specify Theorem 3.7 to the unweighted situation. Notice that a restriction on p comes into play when α or β is less than $-1/2$, and this is due to the so-called pencil phenomenon occurring in the Jacobi function settings, see [4–6].

COROLLARY 3.8. *Let $\alpha, \beta > -1$ and $1 < p < \infty$. Then the operators from Theorem 3.7 are bounded on $L^p(-\pi, \pi)$ or extend to such operators provided that either $\alpha, \beta \geq -1/2$ or $\min(\alpha, \beta) < -1/2$ and p is restricted by the condition*

$$-\min(\alpha, \beta) - \frac{1}{2} < \frac{1}{p} < \min(\alpha, \beta) + \frac{3}{2}.$$

Finally, we note that Theorem 3.7 generalizes [6, Lemma 3.6], where $\overline{\mathbb{R}}_N^{\alpha, \beta}$, $N = 1, 2, \dots$, $\alpha + \beta \neq -1$, were proved to be bounded on unweighted L^p by completely different methods.

3.3. Results in the non-symmetrized Jacobi settings. In this subsection we gather new results in the non-symmetrized Jacobi situations, most of which can be seen as consequences of Theorems 3.1 and 3.7.

Let $\{\mathcal{H}_t^{\alpha, \beta}\}_{t>0}$ be the Jacobi–Poisson semigroup corresponding to the polynomial system $\{\mathcal{P}_n^{\alpha, \beta} : n \geq 0\}$. This semigroup and objects based upon it

were investigated in [2,5,7,8,10], among others. In particular, various square functions involving $\{\mathcal{H}_t^{\alpha,\beta}\}$ were studied in these papers. Here we consider another square function, defined via the interlaced higher-order derivatives $\delta_N^{\text{even}} = \dots \delta \delta_{\alpha,\beta}^* \delta \delta_{\alpha,\beta}^* \delta$ (N components).

PROPOSITION 3.9. *Let $\alpha, \beta > -1$. Then, for each $M, N = 0, 1, 2, \dots$ such that $M + N > 0$, the square function*

$$f \longmapsto \left\| \partial_t^M \delta_N^{\text{even}} \mathcal{H}_t^{\alpha,\beta} f \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}$$

is bounded on $L^p(w d\mu_{\alpha,\beta}^+)$, $w \in (A_p^{\alpha,\beta})^+$, $1 < p < \infty$, and from $L^1(w d\mu_{\alpha,\beta}^+)$ to weak $L^1(w d\mu_{\alpha,\beta}^+)$, $w \in (A_1^{\alpha,\beta})^+$.

PROOF. It is enough to observe that the square function defined in Proposition 3.9 is, up to the factor $1/2$, the ‘restricted’ operator $(G_{M,N}^{\alpha,\beta})^+$. Thus the conclusion follows immediately from Theorem 3.2. \square

Alternatively, to prove Proposition 3.9 one could argue via Theorem 3.1 by considering the action of $\mathbb{G}_{M,N}^{\alpha,\beta}$ on even functions.

We now pass to the less explored non-symmetrized Jacobi function setting. Let $\{\overline{\mathcal{H}}_t^{\alpha,\beta}\}_{t>0}$ be the Jacobi–Poisson semigroup in this context, see Sections 2 in [4,5]. We have

$$\overline{\mathcal{H}}_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} \exp\left(-t\sqrt{\lambda_n^{\alpha,\beta}}\right) \langle f, \phi_n^{\alpha,\beta} \rangle \phi_n^{\alpha,\beta},$$

the series being convergent pointwise on $(0, \pi)$ and defining a smooth function of $(t, \theta) \in (0, \infty) \times (0, \pi)$ provided that $f \in L^p(w)$ for some $1 \leq p < \infty$ and $w = w_{r,s}$ is a double power weight on $(0, \pi)$ satisfying $r < p - 1 + (\alpha + 1/2)p$ and $s < p - 1 + (\beta + 1/2)p$, with the last two inequalities weakened in case $p = 1$; see [13, Proposition 3.1]. Further, we set $D_N^{\text{even}} = \dots D_{\alpha,\beta} D_{\alpha,\beta}^* D_{\alpha,\beta} D_{\alpha,\beta}^* D_{\alpha,\beta}$ (N components).

We consider the following operators related to the system $\{\phi_n^{\alpha,\beta} : n \geq 0\}$ on $(0, \pi)$.

(a) The Riesz–Jacobi transforms of orders $N = 1, 2, \dots$

$$f \longmapsto \sum_{n=1}^{\infty} (\lambda_n^{\alpha,\beta})^{-N/2} \langle f, \phi_n^{\alpha,\beta} \rangle D_{\alpha,\beta}^N \phi_n^{\alpha,\beta}, \quad f \in L^2(0, \pi).$$

(b) The interlaced Riesz–Jacobi transforms of orders $N = 1, 2, \dots$

$$f \longmapsto \sum_{n=1}^{\infty} (\lambda_n^{\alpha,\beta})^{-N/2} \langle f, \phi_n^{\alpha,\beta} \rangle D_N^{\text{even}} \phi_n^{\alpha,\beta}, \quad f \in L^2(0, \pi).$$

(c) Multipliers of Laplace and Laplace–Stieltjes transform type

$$f \mapsto \sum_{n=0}^{\infty} m \left(\sqrt{\lambda_n^{\alpha, \beta}} \right) \langle f, \phi_n^{\alpha, \beta} \rangle \phi_n^{\alpha, \beta}, \quad f \in L^2(0, \pi),$$

where $m = m_\phi$ or $m = m_\nu$, with m_ϕ and m_ν as in Section 2.2.

(d) The Jacobi–Poisson semigroup maximal operator

$$f \mapsto \left\| \overline{\mathcal{H}}_t^{\alpha, \beta} f \right\|_{L^\infty(\mathbb{R}_+, dt)}.$$

(e) Mixed square functions of arbitrary orders M, N

$$f \mapsto \left\| \partial_t^M D_{\alpha, \beta}^N \overline{\mathcal{H}}_t^{\alpha, \beta} f \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)},$$

where $M, N = 0, 1, 2, \dots$ and $M + N > 0$.

(f) Interlaced mixed square functions of arbitrary orders M, N

$$f \mapsto \left\| \partial_t^M D_N^{\text{even}} \overline{\mathcal{H}}_t^{\alpha, \beta} f \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)},$$

where $M, N = 0, 1, 2, \dots$ and $M + N > 0$.

The operators in (a)–(c) are well defined in $L^2(0, \pi)$, with the series being convergent there. The remaining operators are well defined pointwise for $f \in L^p(w)$, where $1 \leq p < \infty$ and w is any weight from $(B_p^{\alpha, \beta})^+$. The latter class consists of restrictions of weights $w_{r, s}$ in $B_p^{\alpha, \beta}$ to the interval $(0, \pi)$, so the definition in terms of ranges of r and s remains the same.

PROPOSITION 3.10. *Let $\alpha, \beta > -1$ and $1 < p < \infty$. Then the operators (d)–(f) are bounded on $L^p(w)$, $w \in (B_p^{\alpha, \beta})^+$. Further, the operators (a)–(c) extend uniquely to bounded linear operators on $L^p(w)$, $w \in (B_p^{\alpha, \beta})^+$.*

PROOF. Because of a transference argument analogous to that from the proof of Theorem 3.7, see the comment at the end of Section 2.1, it is sufficient to check that the counterparts of the operators (a)–(f) in the Jacobi polynomial setting are (or extend to) bounded operators on $L^p(w d\mu_{\alpha, \beta}^+)$, $w \in (A_p^{\alpha, \beta})^+$, $1 < p < \infty$. This, in turn, is done with the aid of the already known results. Indeed, counterparts of (a) and (c)–(e) related to $\{\mathcal{P}_n^{\alpha, \beta} : n \geq 0\}$ are covered by [10, Corollary 5.2], the counterpart of (b) is taken care of by [2, Proposition 3.7], and finally, in case of (f), Proposition 3.9 does the job. \square

Alternatively (but less directly), to prove Proposition 3.10 in cases of (b)–(d) and (f) one could argue by means of Theorem 3.7, restricting the

action of the operators appearing there to even functions. We leave details to interested readers.

We point out that L^p -boundedness of some of the operators in Proposition 3.10 was studied earlier. In particular, double power weighted L^p -boundedness of the first order Riesz–Jacobi transform, (a) with $N = 1$, was obtained in [13, Theorem 4.3] by means of Muckenhoupt’s general multiplier-transplantation theorem. Unweighted L^p -boundedness of the interlaced Riesz–Jacobi transforms (b) and the maximal operator (d) was shown in [4, Proposition 4.2] and [4, Proposition 2.2], respectively. We also mention that [4] contains L^p results for variants of Riesz–Jacobi transforms that are different from the operators in (a) and (b), and in [5] one can find L^p results for variants of square functions different from those in (e) and (f). Weighted L^p boundedness results for the variants just mentioned were obtained recently in [1], though under the restriction $\alpha, \beta \geq -1/2$.

4. Preparatory results

In this section we collect some technical results that will be needed in the proof of Theorem 3.6. Some of these results are taken from other papers, nevertheless we state them here for the sake of the reader’s convenience.

We point out that a crucial role in our developments is played by technical results from [10]. There the authors established an integral formula for the Jacobi–Poisson kernel that is valid for the full range of parameters $\alpha, \beta > -1$, extending an analogous result from [8] burdened by the restriction $\alpha, \beta \geq -1/2$. The representation of the Jacobi–Poisson kernel enabled the authors of [8,10] to elaborate concise and elegant techniques of estimating integral kernels of various harmonic analysis operators. We will take advantage of these methods in Section 5.

In the sequel we shall use the notation from [10]. Thus $\mathbb{N} = \{0, 1, 2, \dots\}$, $d\Pi_\alpha$ and $d\Pi_{\alpha,K}$ are certain measures we do not need to define explicitly here (see [10] for the definitions) and q is a function of $\theta, \varphi \in (0, \pi)$ and $u, v \in [-1, 1]$ valued in $[0, 2]$ (in particular, non-negative and bounded) and given by

$$q(\theta, \varphi, u, v) = 1 - u \sin \frac{\theta}{2} \sin \frac{\varphi}{2} - v \cos \frac{\theta}{2} \cos \frac{\varphi}{2}.$$

For the sake of brevity we shall omit the arguments and write shortly

$$\mathfrak{q} := q(\theta, \varphi, u, v).$$

Furthermore, we write the parameter \mathbf{N} in bold in order to avoid a confusion with another parameter N coming from objects defined in Sections 2 and 3, for instance $R_N^{\alpha,\beta}$.

The following crucial estimates allow one to control mixed derivatives of the Jacobi–Poisson kernel by suitable double-integral expressions.

LEMMA 4.1 ([10, Corollary 3.5]). *Let $M, \mathbf{N} \in \mathbb{N}$ and $L \in \{0, 1\}$ be fixed. The following estimates hold uniformly in $t \in (0, 1]$ and $\theta, \varphi \in (0, \pi)$.*

(i) *If $\alpha, \beta \geq -1/2$, then*

$$|\partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha, \beta}(\theta, \varphi)| \lesssim \iint \frac{d\Pi_\alpha(u) d\Pi_\beta(v)}{(t^2 + \mathbf{q})^{\alpha+\beta+3/2+(L+\mathbf{N}+M)/2}}.$$

(ii) *If $-1 < \alpha < -1/2 \leq \beta$, then*

$$\begin{aligned} & |\partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha, \beta}(\theta, \varphi)| \\ \lesssim & 1 + \sum_{K=0,1} \sum_{k=0,1,2} \left(\sin \frac{\theta}{2} + \sin \frac{\varphi}{2} \right)^{Kk} \iint \frac{d\Pi_{\alpha,K}(u) d\Pi_\beta(v)}{(t^2 + \mathbf{q})^{\alpha+\beta+3/2+(L+\mathbf{N}+M+Kk)/2}}. \end{aligned}$$

(iii) *If $-1 < \beta < -1/2 \leq \alpha$, then*

$$\begin{aligned} & |\partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha, \beta}(\theta, \varphi)| \\ \lesssim & 1 + \sum_{R=0,1} \sum_{r=0,1,2} \left(\cos \frac{\theta}{2} + \cos \frac{\varphi}{2} \right)^{Rr} \iint \frac{d\Pi_\alpha(u) d\Pi_{\beta,R}(v)}{(t^2 + \mathbf{q})^{\alpha+\beta+3/2+(L+\mathbf{N}+M+Rr)/2}}. \end{aligned}$$

(iv) *If $-1 < \alpha, \beta < -1/2$, then*

$$\begin{aligned} & |\partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha, \beta}(\theta, \varphi)| \\ \lesssim & 1 + \sum_{K,R=0,1} \sum_{k,r=0,1,2} \left(\sin \frac{\theta}{2} + \sin \frac{\varphi}{2} \right)^{Kk} \left(\cos \frac{\theta}{2} + \cos \frac{\varphi}{2} \right)^{Rr} \\ & \times \iint \frac{d\Pi_{\alpha,K}(u) d\Pi_{\beta,R}(v)}{(t^2 + \mathbf{q})^{\alpha+\beta+3/2+(L+\mathbf{N}+M+Kk+Rr)/2}}. \end{aligned}$$

Given $\theta \in (0, \pi)$ and $r > 0$, denote by $B(\theta, r)$ the standard ball on the real line restricted to the interval $(0, \pi)$, i.e. $B(\theta, r) = (\theta - r, \theta + r) \cap (0, \pi)$. The next result establishes a bridge between double integrals like those above and the standard estimates we must prove. We remark that only the cases $p \in \{1, 2, \infty\}$ will be needed for our purposes.

LEMMA 4.2 ([10, Lemma 3.7]). *Let $K, R \in \{0, 1\}$, $k, r \in \{0, 1, 2\}$, $W \geq 1$, $s \geq 0$ and $1 \leq p \leq \infty$ be fixed. Consider a function $\Upsilon_s^{\alpha, \beta}(t, \theta, \varphi)$ defined on $(0, 1) \times (0, \pi) \times (0, \pi)$ in the following way.*

(i) For $\alpha, \beta \geq -1/2$,

$$\Upsilon_s^{\alpha, \beta}(t, \theta, \varphi) := \iint \frac{d\Pi_\alpha(u) d\Pi_\beta(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+s/2}}.$$

(ii) For $-1 < \alpha < -1/2 \leq \beta$,

$$\Upsilon_s^{\alpha, \beta}(t, \theta, \varphi) := \left(\sin \frac{\theta}{2} + \sin \frac{\varphi}{2} \right)^{Kk} \iint \frac{d\Pi_{\alpha, K}(u) d\Pi_\beta(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+Kk/2+s/2}}.$$

(iii) For $-1 < \beta < -1/2 \leq \alpha$,

$$\Upsilon_s^{\alpha, \beta}(t, \theta, \varphi) := \left(\cos \frac{\theta}{2} + \cos \frac{\varphi}{2} \right)^{Rr} \iint \frac{d\Pi_\alpha(u) d\Pi_{\beta, R}(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+Rr/2+s/2}}.$$

(iv) For $-1 < \alpha, \beta < -1/2$,

$$\begin{aligned} \Upsilon_s^{\alpha, \beta}(t, \theta, \varphi) &:= \left(\sin \frac{\theta}{2} + \sin \frac{\varphi}{2} \right)^{Kk} \left(\cos \frac{\theta}{2} + \cos \frac{\varphi}{2} \right)^{Rr} \\ &\times \iint \frac{d\Pi_{\alpha, K}(u) d\Pi_{\beta, R}(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+Kk/2+Rr/2+s/2}}. \end{aligned}$$

Then the estimate

$$\|1 + \Upsilon_s^{\alpha, \beta}(t, \theta, \varphi)\|_{L^p((0,1), t^{W-1} dt)} \lesssim \frac{1}{|\theta - \varphi|^s} \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}$$

holds uniformly in $\theta, \varphi \in (0, \pi)$, $\theta \neq \varphi$.

We shall need as well a long-time counterpart of Lemma 4.2.

LEMMA 4.3 ([10, Corollary 3.9]). *Let $\alpha, \beta > -1$, $M, \mathbf{N} \in \mathbb{N}$, $L \in \{0, 1\}$, $W \geq 1$ and $1 \leq p \leq \infty$ be fixed. Then*

$$\left\| \sup_{\theta, \varphi \in (0, \pi)} \left| \partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha, \beta}(\theta, \varphi) \right| \right\|_{L^p((1, \infty), t^{W-1} dt)} < \infty,$$

excluding the cases when simultaneously $\alpha + \beta + 1 = 0$ and $M = \mathbf{N} = L = 0$ and $p < \infty$.

The following strengthened special case of Lemma 4.3 will be used to treat kernels associated with the Laplace–Stieltjes type multipliers.

LEMMA 4.4 ([10, Corollary 3.10]). *Let $\alpha, \beta > -1$ and $L, \mathbf{N} \in \{0, 1\}$ be fixed. Then*

$$\left\| e^{t \frac{\alpha+\beta+1}{2}} \sup_{\theta, \varphi \in (0, \pi)} \left| \partial_\varphi^L \partial_\theta^{\mathbf{N}} H_t^{\alpha, \beta}(\theta, \varphi) \right| \right\|_{L^\infty((1, \infty), dt)} < \infty.$$

The above tools from [10] are yet insufficient to deal with all expressions arising in the analysis of Section 5. The next two lemmas eventually complete the toolbox. Observe that in their proofs, because of the shift in the type parameters, one does not run into the cases excepted in Lemma 4.3.

LEMMA 4.5. *Let $\alpha, \beta > -1$, $M, \mathbf{N} \in \mathbb{N}$, $L, \gamma_1, \gamma_2 \in \{0, 1\}$, $\gamma_1 + \gamma_2 \geq 1$, $W \geq 1$ and $1 \leq p \leq \infty$ be fixed and such that $W/p + \gamma_1 + \gamma_2 - L - \mathbf{N} - M \geq 2$. Then the estimate*

$$\begin{aligned}
 & (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} \left\| \partial_{\varphi}^L \partial_{\theta}^{\mathbf{N}} \partial_t^M H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^p(\mathbb{R}_+, t^{W-1} dt)} \\
 & \lesssim \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}
 \end{aligned}$$

holds uniformly in $\theta, \varphi \in (0, \pi)$, $\theta \neq \varphi$.

LEMMA 4.6. *Let $\alpha, \beta > -1$, $M, \mathbf{N} \in \mathbb{N}$, $L, \gamma_1, \gamma_2 \in \{0, 1\}$, $W \geq 1$ and $1 \leq p \leq \infty$ be fixed and such that $W/p + \gamma_1 + \gamma_2 - L - \mathbf{N} - M \geq 1$. Then the estimate*

$$\begin{aligned}
 & (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} \left\| \partial_{\varphi}^L \partial_{\theta}^{\mathbf{N}} \partial_t^M H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^p(\mathbb{R}_+, t^{W-1} dt)} \\
 & \lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}
 \end{aligned}$$

holds uniformly in $\theta, \varphi \in (0, \pi)$, $\theta \neq \varphi$.

To prove Lemmas 4.5 and 4.6 we need to invoke some auxiliary results. The first of them describes how the measure of a ball changes when one increases the parameters of $\mu_{\alpha, \beta}^+$.

LEMMA 4.7 (see [8, Lemma 4.2]). *Let $\alpha, \beta > -1$. Given any $\xi \geq 0$, we have*

$$\mu_{\alpha+\xi, \beta+\xi}^+(B(\theta, |\theta - \varphi|)) \simeq (\theta + \varphi)^{2\xi} (\pi - \theta + \pi - \varphi)^{2\xi} \mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|)),$$

uniformly in $\theta, \varphi \in (0, \pi)$.

The next necessary result provides some elementary bounds, their proof is a simple exercise. For aesthetic reasons, the constants appearing on the right-hand sides are optimal, but for our purposes any constants would be suitable. Obviously, the bound (a) holds with the roles of θ and φ exchanged.

LEMMA 4.8. *For $\theta, \varphi \in (0, \pi)$ we have the estimates*

$$\text{(a) } \frac{|\theta - \varphi| \varphi (\pi - \varphi)}{(\theta + \varphi)^2 (\pi - \theta + \pi - \varphi)^2} \leq \frac{1}{4\pi},$$

$$(b) \frac{\theta \varphi (\pi - \theta)(\pi - \varphi)}{(\theta + \varphi)^2(\pi - \theta + \pi - \varphi)^2} \leq \frac{1}{16},$$

$$(c) \frac{|\theta - \varphi|}{(\theta + \varphi)(\pi - \theta + \pi - \varphi)} \leq \frac{1}{\pi}.$$

The last auxiliary result is a special case of the generalization of [8, Lemma 4.3] established in [3]. Notice that this result is restricted to the range $\alpha, \beta \geq -1/2$.

LEMMA 4.9 (see [3, Lemma 4.4]). *Let $\alpha, \beta \geq -1/2$ and $\kappa \geq 0$. Then*

$$\iint \frac{d\Pi_{\alpha+\kappa}(u) d\Pi_{\beta+\kappa}(v)}{\mathfrak{q}^{\alpha+\beta+3/2+\kappa}} \lesssim \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}$$

uniformly in $\theta, \varphi \in (0, \pi)$, $\theta \neq \varphi$.

We are now prepared to give proofs of Lemmas 4.5 and 4.6.

PROOF OF LEMMA 4.5. Let

$$A_0 := \left\| \partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^p((0,1), t^{W-1} dt)},$$

$$A_\infty := \left\| \partial_\varphi^L \partial_\theta^{\mathbf{N}} \partial_t^M H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^p((1,\infty), t^{W-1} dt)}.$$

Clearly, it is enough to estimate $(\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_0$ and the analogous expression with A_∞ replacing A_0 . By Lemma 4.3 we have

$$(\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_\infty \lesssim 1 \lesssim \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))},$$

thus it remains to estimate the expression related to A_0 .

To begin with, assume that $p < \infty$. Using sequentially Lemma 4.1 (i), Minkowski's inequality and then the assumption $W/p + \gamma_1 + \gamma_2 - L - \mathbf{N} - M \geq 2$ together with the boundedness of \mathfrak{q} , we get

$$A_0 \lesssim \iint d\Pi_{\alpha+1}(u) d\Pi_{\beta+1}(v) \left(\int_0^1 \frac{t^{W-1} dt}{(t^2 + \mathfrak{q})^{p[\alpha+\beta+5/2+W/(2p)+(\gamma_1+\gamma_2)/2]}} \right)^{1/p}.$$

Changing the variable of integration $t \mapsto \sqrt{\mathfrak{q}} t$ and enlarging the upper limit of integration in the resulting integral to infinity (this integral converges), we see that

$$A_0 \lesssim \iint \frac{d\Pi_{\alpha+1}(u) d\Pi_{\beta+1}(v)}{\mathfrak{q}^{\alpha+\beta+5/2+(\gamma_1+\gamma_2)/2}}.$$

The last estimate is valid also for $p = \infty$, as can be seen directly by using the assumption $\gamma_1 + \gamma_2 - L - \mathbf{N} - M \geq 2$ and the trivial bound

$$\frac{1}{(t^2 + \mathfrak{q})^{\alpha+\beta+5/2+(\gamma_1+\gamma_2)/2}} \leq \frac{1}{\mathfrak{q}^{\alpha+\beta+5/2+(\gamma_1+\gamma_2)/2}}.$$

To proceed, it is convenient to distinguish two cases. Combined, they complete the proof.

Case 1: $\gamma_1 = \gamma_2 = 1$. Applying sequently Lemma 4.9 (with $\kappa = 0$), Lemma 4.7 (with $\xi = 1$) and Lemma 4.8 (b) we get

$$\begin{aligned} (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_0 &\lesssim \theta(\pi - \theta)\varphi(\pi - \varphi) \iint \frac{d\Pi_{\alpha+1}(u) d\Pi_{\beta+1}(v)}{\mathfrak{q}^{(\alpha+1)+(\beta+1)+3/2}} \\ &\lesssim \frac{\theta(\pi - \theta)\varphi(\pi - \varphi)}{\mu_{\alpha+1,\beta+1}^+(B(\theta, |\theta - \varphi|))} \simeq \frac{\theta(\pi - \theta)\varphi(\pi - \varphi)}{(\theta + \varphi)^2(\pi - \theta + \pi - \varphi)^2} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))} \\ &\leq \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}. \end{aligned}$$

Case 2: $\gamma_1 + \gamma_2 = 1$. By Lemma 4.9 (with $\kappa = 1/2$) and Lemma 4.7 (with $\xi = 1/2$)

$$\begin{aligned} &(\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_0 \\ &\lesssim [\theta(\pi - \theta)]^{\gamma_1} [\varphi(\pi - \varphi)]^{\gamma_2} \iint \frac{d\Pi_{(\alpha+1/2)+1/2}(u) d\Pi_{(\beta+1/2)+1/2}(v)}{\mathfrak{q}^{(\alpha+1/2)+(\beta+1/2)+3/2+1/2}} \\ &\lesssim \frac{[\theta(\pi - \theta)]^{\gamma_1} [\varphi(\pi - \varphi)]^{\gamma_2}}{\mu_{\alpha+1/2,\beta+1/2}^+(B(\theta, |\theta - \varphi|))} \simeq \frac{[\theta(\pi - \theta)]^{\gamma_1} [\varphi(\pi - \varphi)]^{\gamma_2}}{(\theta + \varphi)(\pi - \theta + \pi - \varphi)} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))} \\ &\leq \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}. \quad \square \end{aligned}$$

PROOF OF LEMMA 4.6. Arguing in the same way as in the corresponding part of the proof of Lemma 4.5 we are reduced to estimating $(\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_0$ and then arrive at the estimate

$$A_0 \lesssim \iint \frac{d\Pi_{\alpha+1}(u) d\Pi_{\beta+1}(v)}{\mathfrak{q}^{\alpha+\beta+3+(\gamma_1+\gamma_2)/2}}.$$

Now, similarly as before, we distinguish two cases which jointly complete the proof.

Case 1: $\gamma_1 = \gamma_2 = 0$. Using sequentially Lemma 4.9 (with $\kappa = 1/2$), Lemma 4.7 (with $\xi = 1/2$) and Lemma 4.8 (c) we get

$$\begin{aligned} (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_0 &\lesssim \iint \frac{d\Pi_{(\alpha+1/2)+1/2}(u) d\Pi_{(\beta+1/2)+1/2}(v)}{\mathfrak{q}^{(\alpha+1/2)+(\beta+1/2)+3/2+1/2}} \\ &\lesssim \frac{1}{\mu_{\alpha+1/2, \beta+1/2}^+(B(\theta, |\theta - \varphi|))} \\ &\simeq \frac{|\theta - \varphi|}{(\theta + \varphi)(\pi - \theta + \pi - \varphi)} \frac{1}{|\theta - \varphi| \mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))} \\ &\leq \frac{1}{|\theta - \varphi| \mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}. \end{aligned}$$

Case 2: $\gamma_1 + \gamma_2 \geq 1$. Due to the bound $\mathfrak{q} \geq |\theta - \varphi|^2/\pi^2$ we have

$$\begin{aligned} (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} A_0 &\lesssim (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} \iint \frac{d\Pi_{\alpha+1}(u) d\Pi_{\beta+1}(v)}{\mathfrak{q}^{\alpha+\beta+3+(\gamma_1+\gamma_2)/2}} \\ &\lesssim \frac{1}{|\theta - \varphi|} (\sin \theta)^{\gamma_1} (\sin \varphi)^{\gamma_2} \iint \frac{d\Pi_{\alpha+1}(u) d\Pi_{\beta+1}(v)}{\mathfrak{q}^{\alpha+\beta+5/2+(\gamma_1+\gamma_2)/2}} \\ &\lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}, \end{aligned}$$

where the last bound follows from the estimates obtained in the proof of Lemma 4.5. \square

5. Kernel estimates

Let \mathbb{B} be a Banach space and let $K(\theta, \varphi)$ be a kernel defined on $(0, \pi) \times (0, \pi) \setminus \{(\theta, \varphi) : \theta = \varphi\}$ and taking values in \mathbb{B} . We say that $K(\theta, \varphi)$ satisfies the so-called *standard estimates* in the sense of the space of homogeneous type $((0, \pi), d\mu_{\alpha, \beta}^+, |\cdot|)$ if it satisfies the growth estimate

$$(11) \quad \|K(\theta, \varphi)\|_{\mathbb{B}} \lesssim \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}, \quad \theta \neq \varphi,$$

and the smoothness estimates

$$(12) \quad \|K(\theta, \varphi) - K(\theta', \varphi)\|_{\mathbb{B}} \lesssim \frac{|\theta - \theta'|}{|\theta - \varphi|} \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}, \quad |\theta - \varphi| > 2|\theta - \theta'|,$$

$$(13) \quad \|K(\theta, \varphi) - K(\theta, \varphi')\|_{\mathbb{B}} \lesssim \frac{|\varphi - \varphi'|}{|\theta - \varphi|} \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}, \quad |\theta - \varphi| > 2|\varphi - \varphi'|.$$

Notice that in these formulas the interval

$$B(\theta, |\theta - \varphi|) = (\theta - |\theta - \varphi|, \theta + |\theta - \varphi|) \cap (0, \pi)$$

can be replaced by $B(\varphi, |\theta - \varphi|)$, in view of the doubling property of $\mu_{\alpha, \beta}^+$.

When $K(\theta, \varphi)$ is scalar-valued, i.e. $\mathbb{B} = \mathbb{C}$, it is well known that the bounds (12) and (13) follow from the more convenient gradient estimate

$$(14) \quad \|\partial_{\theta} K(\theta, \varphi)\|_{\mathbb{B}} + \|\partial_{\varphi} K(\theta, \varphi)\|_{\mathbb{B}} \lesssim \frac{1}{|\theta - \varphi| \mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}, \quad \theta \neq \varphi.$$

The same holds also in the vector-valued cases we consider, see [10, Section 4], the derivatives in (14) being then taken in the weak sense. The latter means that for any functional $v \in \mathbb{B}^*$

$$(15) \quad v(\partial_{\theta} K(\theta, \varphi)) = \partial_{\theta} v(K(\theta, \varphi))$$

and similarly for ∂_{φ} . If these weak derivatives $\partial_{\theta} K(\theta, \varphi)$ and $\partial_{\varphi} K(\theta, \varphi)$ exist as elements of \mathbb{B} and their norms satisfy (14), the scalar-valued case applies and (12) and (13) follow.

Now we are in a position to prove Theorem 3.6. In what follows always $(\theta, \varphi) \in (0, \pi) \times (0, \pi)$ and $\theta \neq \varphi$. Further, we tacitly assume that changing orders of certain differentiations and integrations is legitimate. In fact, such manipulations can be easily justified with the aid of the estimates obtained along the proof of Theorem 3.6 and the dominated convergence theorem.

5.1. Proof of Theorem 3.6, the part related to Riesz transforms.

PROOF OF THEOREM 3.6; THE CASE OF $\tilde{R}_N^{\alpha, \beta}(\theta, \varphi)$. We consider two cases, depending on whether $N \geq 1$ is even or odd.

Case 1: N is even. Let $N = 2k_0$ with $k_0 \geq 1$. We first proceed as in [3, p. 281]. Term by term differentiation of the series defining $\tilde{H}_t^{\alpha, \beta}(\theta, \varphi)$ allows one to verify that this kernel satisfies in the strip $(t, \theta) \in (0, \infty) \times (0, \pi)$ the Laplace equation based on the modified Jacobi operator $\delta \delta_{\alpha, \beta}^* + \lambda_0^{\alpha, \beta}$. This can be written as

$$(16) \quad \delta_2^{\text{odd}} \tilde{H}_t^{\alpha, \beta}(\theta, \varphi) = \partial_t^2 \tilde{H}_t^{\alpha, \beta}(\theta, \varphi) - \lambda_0^{\alpha, \beta} \tilde{H}_t^{\alpha, \beta}(\theta, \varphi).$$

Iterating this relation we get

$$(17) \quad \delta_N^{\text{odd}} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) = \sum_{j=0}^{k_0} c_j \partial_t^{2(k_0-j)} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi),$$

with some constants c_j . Consequently, it suffices to show that for each $0 \leq j \leq k_0$ the kernel

$$\tilde{S}_j^{\alpha,\beta}(\theta, \varphi) := \int_0^\infty \partial_t^{2j} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) t^{2k_0-1} dt$$

satisfies conditions (11) and (14) with $\mathbb{B} = \mathbb{C}$.

First we show the growth estimate (11). We have

$$|\tilde{S}_j^{\alpha,\beta}(\theta, \varphi)| \leq \sin \theta \sin \varphi \int_0^\infty |\partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0-1} dt.$$

From here the growth bound follows from Lemma 4.5 (applied with $L = \mathbf{N} = 0$, $M = 2j$, $W = 2k_0$, $\gamma_1 = \gamma_2 = 1$, $p = 1$).

We pass to the gradient estimate (14). For symmetry reasons, it is enough to consider only the derivative in θ . A simple computation shows that

$$\begin{aligned} |\partial_\theta \tilde{S}_j^{\alpha,\beta}(\theta, \varphi)| &\leq \sin \theta \sin \varphi \int_0^\infty |\partial_\theta \partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0-1} dt \\ &+ \sin \varphi \int_0^\infty |\partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0-1} dt. \end{aligned}$$

Then both the terms on the right-hand side are treated directly by Lemma 4.6 (applied with $L = 0$, $\mathbf{N} = 1$, $M = 2j$, $W = 2k_0$, $\gamma_1 = \gamma_2 = 1$, $p = 1$ in case of the first term and $L = \mathbf{N} = 0$, $M = 2j$, $W = 2k_0$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = 1$ in case of the second one). Thus the reasoning for the case of N even is finished.

Case 2: N is odd. Let now $N = 2k_0 + 1$ for some $k_0 \geq 0$. In view of (17), we have

$$\tilde{R}_N^{\alpha,\beta}(\theta, \varphi) = \int_0^\infty \sum_{j=0}^{k_0} c_j \delta_{\alpha,\beta}^* \partial_t^{2(k_0-j)} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) t^{2k_0} dt.$$

Again, we observe that it is enough to show that, for each $0 \leq j \leq k_0$, the kernel

$$\tilde{S}_j^{\alpha,\beta}(\theta, \varphi) := \int_0^\infty \delta_{\alpha,\beta}^* \partial_t^{2j} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) t^{2k_0} dt$$

satisfies the standard estimates (here and elsewhere $\delta_{\alpha,\beta}^*$ acts always on the θ variable).

First we show the growth bound. A direct computation reveals that

$$\begin{aligned} |\tilde{\mathcal{S}}_j^{\alpha,\beta}(\theta, \varphi)| &\lesssim \sin \theta \sin \varphi \int_0^\infty |\partial_\theta \partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt \\ &\quad + \sin \varphi \int_0^\infty |\partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt. \end{aligned}$$

Both the above terms fall under the scope of Lemma 4.5 (specified to $L = 0$, $\mathbf{N} = 1$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = \gamma_2 = 1$, $p = 1$ in case of the first summand and $L = \mathbf{N} = 0$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = 1$ in case of the second one), hence (11) follows.

For the gradient condition we use (16) to write

$$\begin{aligned} |\partial_\theta \tilde{\mathcal{S}}_j^{\alpha,\beta}(\theta, \varphi)| &\lesssim \sin \theta \sin \varphi \int_0^\infty |\partial_t^{2j+2} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt \\ &\quad + \sin \theta \sin \varphi \int_0^\infty |\partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt. \end{aligned}$$

Now a double application of Lemma 4.6 (with $L = \mathbf{N} = 0$, $M = 2j + 2$, $W = 2k_0 + 1$, $\gamma_1 = \gamma_2 = 1$, $p = 1$ in case of the first component and $L = \mathbf{N} = 0$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = \gamma_2 = 1$, $p = 1$ in case of the second one) gives

$$|\partial_\theta \tilde{\mathcal{S}}_j^{\alpha,\beta}(\theta, \varphi)| \lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}.$$

It remains to estimate the derivative in φ . We have

$$\begin{aligned} |\partial_\varphi \tilde{\mathcal{S}}_j^{\alpha,\beta}(\theta, \varphi)| &\lesssim \sin \theta \sin \varphi \int_0^\infty |\partial_\varphi \partial_\theta \partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt \\ &\quad + \sin \varphi \int_0^\infty |\partial_\varphi \partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt \\ &\quad + \sin \theta \int_0^\infty |\partial_\theta \partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt + \int_0^\infty |\partial_t^{2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi)| t^{2k_0} dt. \end{aligned}$$

All the terms on the right-hand side can be treated by means of Lemma 4.6 (taken, respectively, with $L = \mathbf{N} = 1$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = \gamma_2 = 1$, $p = 1$; $L = 1$, $\mathbf{N} = 0$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = 1$; $L = 0$, $\mathbf{N} = 1$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = 1$, $\gamma_2 = 0$, $p = 1$ and $L = \mathbf{N} = 0$, $M = 2j$, $W = 2k_0 + 1$, $\gamma_1 = \gamma_2 = 0$, $p = 1$). The smoothness bound follows. \square

5.2. Proof of Theorem 3.6, the part related to square functions.

PROOF OF THEOREM 3.6; THE CASE OF $\{\partial_t^M \delta_N^{\text{even}} H_t^{\alpha,\beta}(\theta, \varphi)\}_{t>0}$. We consider two cases.

Case 1: N is even. Let $k_0 = N/2 \geq 0$. Term by term differentiation of the defining series reveals that $H_t^{\alpha,\beta}(\theta, \varphi)$ satisfies in the strip $(t, \theta) \in (0, \infty) \times (0, \pi)$ the Laplace equation based on the Jacobi Laplacian, which can be written as (see [3, p. 278])

$$\delta_2^{\text{even}} H_t^{\alpha,\beta}(\theta, \varphi) = \partial_t^2 H_t^{\alpha,\beta}(\theta, \varphi) - \lambda_0^{\alpha,\beta} H_t^{\alpha,\beta}(\theta, \varphi).$$

Iterating this identity, we get

$$(18) \quad \delta_N^{\text{even}} H_t^{\alpha,\beta}(\theta, \varphi) = \sum_{j=0}^{k_0} c_j \partial_t^{2(k_0-j)} H_t^{\alpha,\beta}(\theta, \varphi)$$

with some constants c_j . Consequently, it is enough to verify that for each $0 \leq j \leq k_0$ the vector-valued kernel

$$T_j^{\alpha,\beta}(\theta, \varphi) := \{ \partial_t^{M+2j} H_t^{\alpha,\beta}(\theta, \varphi) \}_{t>0}$$

satisfies the standard estimates (11) and (14) with $\mathbb{B} = L^2(\mathbb{R}_+, t^{2M+2N-1} dt)$.

The growth condition (11) for $T_j^{\alpha,\beta}(\theta, \varphi)$ follows directly from Lemma 4.1 (with $L = \mathbf{N} = 0$) combined with the boundedness of \mathfrak{q} , Lemma 4.2 (taken with $W = 2N + 2M, s = 0, p = 2$) and Lemma 4.3 (specified to $L = \mathbf{N} = 0, W = 2N + 2M, p = 2$). More precisely, here Lemma 4.3 cannot be applied directly when $M = j = 0$ and $\alpha + \beta = -1$. However, in the singular case $\alpha + \beta = -1$ the decomposition (18) reduces to $\delta_N^{\text{even}} H_t^{\alpha,\beta}(\theta, \varphi) = \partial_t^N H_t^{\alpha,\beta}(\theta, \varphi)$ and so we need to estimate only the kernel $T_{N/2}^{\alpha,\beta}(\theta, \varphi)$. In this situation Lemma 4.3 again applies (recall that $M + N > 0$) and the growth bound follows.

To prove the smoothness condition, because of the symmetry, it suffices to consider the derivative in θ . It is not hard to check that the weak derivative $\partial_\theta T_j^{\alpha,\beta}(\theta, \varphi)$ in the sense of (15) equals $\{ \partial_\theta \partial_t^{M+2j} H_t^{\alpha,\beta}(\theta, \varphi) \}_{t>0}$, cf. [10, Proof of Theorem 4.1]. Therefore we need to show that

$$\| \partial_\theta \partial_t^{M+2j} H_t^{\alpha,\beta}(\theta, \varphi) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}.$$

This, however, follows again from Lemma 4.1 (specified to $L = 0$ and $\mathbf{N} = 1$), the boundedness of \mathfrak{q} combined with Lemma 4.2 (applied with $W =$

$2M + 2N$, $s = 1$, $p = 2$) and Lemma 4.3 (with $L = 0$, $\mathbf{N} = 1$, $W = 2M + 2N$, $p = 2$).

Case 2: N is odd, say $N = 2k_0 + 1$. We take into account (18) and observe that it suffices, for each $0 \leq j \leq k_0$, to verify the standard estimates for the vector-valued kernel

$$\mathcal{T}_j^{\alpha,\beta}(\theta, \varphi) := \{ \partial_\theta \partial_t^{M+2j} H_t^{\alpha,\beta}(\theta, \varphi) \}_{t>0}.$$

This, however, is done by a straightforward repetition of the arguments used for the case of N even. We omit the details. \square

PROOF OF THEOREM 3.6; THE CASE OF $\{ \partial_t^M \delta_N^{\text{odd}} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \}_{t>0}$. Again, we consider two cases.

Case 1: N is even. Let $N = 2k_0$ with $k_0 \geq 0$. In view of (17), it is enough to show that for each $0 \leq j \leq k_0$ the vector-valued kernel

$$\tilde{T}_j^{\alpha,\beta}(\theta, \varphi) := \{ \partial_t^{M+2j} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \}_{t>0}$$

satisfies the standard estimates with $\mathbb{B} = L^2(\mathbb{R}_+, t^{2M+2N-1} dt)$.

The growth bound (11) is straightforward, since by Lemma 4.5 (specified to $L = \mathbf{N} = 0$, $W = 2M + 2N$, $\gamma_1 = \gamma_2 = 1$, $p = 2$) we have

$$\begin{aligned} & \| \tilde{T}_j^{\alpha,\beta}(\theta, \varphi) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ &= \frac{1}{4} \sin \theta \sin \varphi \| \partial_t^{M+2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ &\lesssim \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}. \end{aligned}$$

We pass to the smoothness condition (14). For symmetry reasons, it is enough to treat the derivative in θ . One easily verifies that the weak derivative $\partial_\theta \tilde{T}_j^{\alpha,\beta}(\theta, \varphi)$ in the sense of (15) is given by $\{ \partial_\theta \partial_t^{M+2j} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \}_{t>0}$. Then we have

$$\begin{aligned} & \| \partial_\theta \partial_t^{M+2j} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ &\leq \sin \theta \sin \varphi \| \partial_\theta \partial_t^{M+2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ &\quad + \sin \varphi \| \partial_t^{M+2j} H_t^{\alpha+1,\beta+1}(\theta, \varphi) \|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}. \end{aligned}$$

Each of the above terms can be estimated suitably by means of Lemma 4.6 (applied with $L = 0$, $\mathbf{N} = 1$, $W = 2M + 2N$, $\gamma_1 = \gamma_2 = 1$, $p = 2$ in case of the first summand and $L = \mathbf{N} = 0$, $W = 2M + 2N$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = 2$ in case of the second one). The conclusion follows.

Case 2: N is odd. Let $N = 2k_0 + 1$ with $k_0 \geq 0$. The kernel we need to estimate is, see (17),

$$\left\{ \partial_t^M \delta_N^{\text{odd}} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \right\}_{t>0} = \left\{ \sum_{j=0}^{k_0} c_j \delta_{\alpha,\beta}^* \partial_t^M \partial_t^{2(k_0-j)} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \right\}_{t>0}.$$

Thus it is enough to show that for each $0 \leq j \leq k_0$ the kernel

$$\tilde{\mathcal{T}}_j^{\alpha,\beta}(\theta, \varphi) := \left\{ \delta_{\alpha,\beta}^* \partial_t^{M+2j} \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \right\}_{t>0}$$

satisfies the standard estimates.

A direct computation reveals that

$$\begin{aligned} & \left\| \tilde{\mathcal{T}}_j^{\alpha,\beta}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ & \lesssim \sin \theta \sin \varphi \left\| \partial_\theta \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ & \quad + \sin \varphi \left\| \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}. \end{aligned}$$

Then the growth bound (11) is a consequence of Lemma 4.5 applied twice (with $L = 0$, $\mathbf{N} = 1$, $W = 2M + 2N$, $\gamma_1 = \gamma_2 = 1$, $p = 2$ in case of the first term and with $L = \mathbf{N} = 0$, $W = 2M + 2N$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = 2$ in case of the second one).

To show the smoothness condition (14) we begin with the derivative in θ . A simple computation involving (16) reveals that

$$\begin{aligned} & \left\| \partial_\theta \tilde{\mathcal{T}}_j^{\alpha,\beta}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ & \lesssim \sin \theta \sin \varphi \left\| \partial_t^{M+2j+2} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ & \quad + \sin \theta \sin \varphi \left\| \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}. \end{aligned}$$

Both of the above terms are estimated with the aid of Lemma 4.6 (applied with $L = \mathbf{N} = 0$, $W = 2M + 2N$, $\gamma_1 = \gamma_2 = 1$, $p = 2$).

It remains to consider the derivative in φ . A straightforward computation leads to the bound

$$\begin{aligned} & \left\| \partial_\varphi \tilde{\mathcal{T}}_j^{\alpha,\beta}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ & \lesssim \sin \theta \sin \varphi \left\| \partial_\varphi \partial_\theta \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\ & \quad + \sin \varphi \left\| \partial_\varphi \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \end{aligned}$$

$$\begin{aligned}
 & + \sin \theta \left\| \partial_\theta \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)} \\
 & + \left\| \partial_t^{M+2j} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^2(\mathbb{R}_+, t^{2M+2N-1} dt)}.
 \end{aligned}$$

All of the above terms are controlled by the right-hand side of (14), which follows by applying repeatedly Lemma 4.6 (with $L = \mathbf{N} = 1, W = 2M + 2N, \gamma_1 = \gamma_2 = 1, p = 2$ in case of the first summand; with $L = 1, \mathbf{N} = 0, W = 2M + 2N, \gamma_1 = 0, \gamma_2 = 1, p = 2$ in case of the second one; with $L = 0, \mathbf{N} = 1, W = 2M + 2N, \gamma_1 = 1, \gamma_2 = 0, p = 2$ in case of the third one and with $L = \mathbf{N} = 0, W = 2M + 2N, \gamma_1 = \gamma_2 = 0, p = 2$ in case of the last one). This finishes the reasoning. \square

5.3. Proof of Theorem 3.6, the part related to spectral multipliers.

PROOF OF THEOREM 3.6; THE CASE OF $\widetilde{M}_\phi^{\alpha, \beta}(\theta, \varphi)$. Since ϕ is bounded, we have

$$\widetilde{M}_\phi^{\alpha, \beta}(\theta, \varphi) \lesssim \sin \theta \sin \varphi \int_0^\infty \left| \partial_t H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right| dt.$$

Now to get the growth bound (11) with $\mathbb{B} = \mathbb{C}$ it is enough to apply Lemma 4.5 (specified to $L = \mathbf{N} = 0, M = 1, W = 1, \gamma_1 = \gamma_2 = 1, p = 1$).

To show the smoothness estimate (14) we can restrict, for symmetry reasons, to the derivative in θ . Taking into account the boundedness of ϕ , we get

$$\begin{aligned}
 \left| \partial_\theta \widetilde{M}_\phi^{\alpha, \beta}(\theta, \varphi) \right| & \lesssim \sin \theta \sin \varphi \int_0^\infty \left| \partial_\theta \partial_t H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right| dt \\
 & + \sin \varphi \int_0^\infty \left| \partial_t H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right| dt.
 \end{aligned}$$

Both the terms on the right-hand side satisfy the desired estimate by Lemma 4.6 (specified to $L = 0, \mathbf{N} = M = 1, W = 1, \gamma_1 = \gamma_2 = 1, p = 1$ in case of the first summand and $L = \mathbf{N} = 0, M = 1, W = 1, \gamma_1 = 0, \gamma_2 = 1, p = 1$ in case of the second one). \square

PROOF OF THEOREM 3.6; THE CASE OF $\widetilde{M}_\nu^{\alpha, \beta}(\theta, \varphi)$. Taking into account the condition (8) it is clear that proving the growth estimate for $\widetilde{M}_\nu^{\alpha, \beta}(\theta, \varphi)$ reduces to showing that

$$\begin{aligned}
 \sin \theta \sin \varphi \left\| H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^\infty((0,1), dt)} & \lesssim \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}, \\
 \sin \theta \sin \varphi \left\| e^{t \frac{\alpha + \beta + 1}{2}} H_t^{\alpha+1, \beta+1}(\theta, \varphi) \right\|_{L^\infty((1, \infty), dt)} & \lesssim \frac{1}{\mu_{\alpha, \beta}^+(B(\theta, |\theta - \varphi|))}.
 \end{aligned}$$

The first bound here follows by Lemma 4.5 (applied with $L = \mathbf{N} = M = 0$, $W = 1$, $\gamma_1 = \gamma_2 = 1$, $p = \infty$). The second one is a consequence of Lemma 4.4 (taken with $L = \mathbf{N} = 0$); when applying Lemma 4.4 here and below we use implicitly the fact that $|\frac{\alpha+\beta+1}{2}| < |\frac{(\alpha+1)+(\beta+1)+1}{2}|$.

For symmetry reasons, verification of the gradient bound (14) amounts to checking that

$$B_0 := \left\| \partial_\theta \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \right\|_{L^\infty((0,1),dt)} \lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))},$$

$$B_\infty := \left\| e^{t|\frac{\alpha+\beta+1}{2}|} \partial_\theta \tilde{H}_t^{\alpha,\beta}(\theta, \varphi) \right\|_{L^\infty((1,\infty),dt)} \lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}.$$

An easy calculation reveals that

$$|\partial_\theta \tilde{H}_t^{\alpha,\beta}(\theta, \varphi)| \leq \sin \theta \sin \varphi |\partial_\theta H_t^{\alpha+1,\beta+1}(\theta, \varphi)| + \sin \varphi H_t^{\alpha+1,\beta+1}(\theta, \varphi).$$

Therefore

$$B_0 \leq \sin \theta \sin \varphi \left\| \partial_\theta H_t^{\alpha+1,\beta+1}(\theta, \varphi) \right\|_{L^\infty((0,1),dt)} + \sin \varphi \left\| H_t^{\alpha+1,\beta+1}(\theta, \varphi) \right\|_{L^\infty((0,1),dt)}$$

and each of the above terms can be estimated by means of Lemma 4.6 (applied with $L = 0$, $\mathbf{N} = 1$, $M = 0$, $W = 1$, $\gamma_1 = \gamma_2 = 1$, $p = \infty$ in case of the first summand and with $L = \mathbf{N} = M = 0$, $W = 1$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = \infty$ in case of the second one). Thus the smoothness bound for B_0 follows. Considering B_∞ , With the aid of Lemma 4.4 (applied twice: with $L = 0$, $\mathbf{N} = 1$ and with $L = \mathbf{N} = 0$) we infer that

$$B_\infty \leq \left\| e^{t|\frac{\alpha+\beta+1}{2}|} \partial_\theta H_t^{\alpha+1,\beta+1}(\theta, \varphi) \right\|_{L^\infty((1,\infty),dt)} + \left\| e^{t|\frac{\alpha+\beta+1}{2}|} H_t^{\alpha+1,\beta+1}(\theta, \varphi) \right\|_{L^\infty((1,\infty),dt)}$$

$$\lesssim 1 \lesssim \frac{1}{|\theta - \varphi|} \frac{1}{\mu_{\alpha,\beta}^+(B(\theta, |\theta - \varphi|))}. \quad \square$$

The proof of Theorem 3.6 is finished.

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