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CONVERGENCE IN p-MEAN FOR ARRAYS OF ROW-WISE EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. Some convergence results in mean of order p for arrays of row-wise extended negatively dependent random variables are presented under asymptotic integrability conditions. A Rosenthal type inequality for these dependent structures is also announced playing a central role in our approach to this issue. As consequence, well-known results about convergence in p-mean for random variables will be extended.

1. Introduction

The convergence in mean of order p has been studied in the last decades by several authors. In the final sixties, Pyke and Root showed in the classical paper [13] that, for each $0 < p < 2$,

$$n^{-1/p} \sum_{k=1}^{n} X_k \xrightarrow{\mathscr{L}_p} 0$$

for every sequence $\{X_n, n \geqslant 1\}$ of independent and identically distributed random variables satisfying $\mathbb{E}\,|X_1|^p < \infty$ (and $\mathbb{E}\,X_1 = 0$ when $p \geqslant 1$). Since then, many extensions of this result have been performed either relaxing the assumptions on the random variables or going towards to arrays and weighted arrays of random variables (see [1,2,8,15–18], among others). Our purpose in this sequel is to establish the convergence in mean of order p for

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arrays of random variables in a wide scenario of dependence. In fact, assumptions involving dependent random variables make statistical models be more suitable (and realistic), which is a stimulus to develop further results in this direction.

We begin by recovering the notion of extended negative dependence for triangular arrays of random variables recently introduced in [11]. A triangular array $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ of random variables is said to be *row-wise upper extended negatively dependent* (row-wise UEND) if for each $n \geqslant 1$, there exists a positive finite number M_n such that

$$\mathbb{P}\big(X_{n,1} > x_1, X_{n,2} > x_2, \dots, X_{n,n} > x_n\big) \leqslant M_n \prod_{k=1}^{n} \mathbb{P}(X_{n,k} > x_k) \tag{1.1}$$

holds for all real numbers $x_1, \dots, x_n$. A triangular array $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ of random variables is said to be *row-wise lower extended negatively dependent* (row-wise LEND) if for each $n \geqslant 1$, there exists a positive finite number M_n such that

$$\mathbb{P}\big(X_{n,1} \leqslant x_1, X_{n,2} \leqslant x_2, \dots, X_{n,n} \leqslant x_n\big) \leqslant M_n \prod_{k=1}^{n} \mathbb{P}(X_{n,k} \leqslant x_k) \tag{1.2}$$

holds for all real numbers $x_1, \dots, x_n$ (see [11]). A triangular array $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ of random variables is said to be *row-wise extended negatively dependent* (row-wise END) if it is both row-wise UEND and row-wise LEND. The sequence $\{M_n,\ n \geqslant 1\}$ in (1.1) and (1.2) is called a *dominating sequence* of $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$. We emphasize that the above definition covers, in particular, the concept of widely orthant dependent random sequence (see, for instance, [4, p. 116]) taking, indeed $X_{n,k} = \xi_k$ and $M_n := \max\{g_U(n), g_L(n)\}$ in (1.1) and (1.2) with $\{\xi_k,\ k \geqslant 1\}$, $g_U(n)$ and $g_L(n)$ as in [4, Definition 1.1]. Note also that the auxiliary lemmata used in [17] are no more helpful for random arrays satisfying (1.1) and (1.2), thus implying their complete reformulation.

Associated to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall consider the space $\mathscr{L}_p$ $(p > 0)$ of all measurable functions X (necessarily random variables) for which $\mathbb{E}\,|X|^p < \infty$. Throughout, the letter C will denote a positive constant, which is not necessarily the same one in each appearance; the symbol $C(p)$ has identical meaning with the additional information that the constant depends on p.

2. Main results

The first main result of this paper states the convergence in mean of order p $(1 \leqslant p < 2)$ for triangular arrays of random variables having dependent structure and prescribed norming constants.

Theorem 1. *Let* $1 \leqslant p < 2$ *and* $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be a triangular array of row-wise END random variables with dominating sequence* $\{M_n,\ n \geqslant 1\}$. *If* $\{b_n\}$ *is a sequence of positive constants such that*

(a)
$$\sum_{k=1}^{n} \int_0^{\varepsilon b_n^p} \mathbb{P}\{|X_{n,k}|^p > t\}\, dt = O\Big(\frac{b_n^p}{1+M_n}\Big) \quad \text{as } n \to \infty \text{ for any } \varepsilon > 0,$$

(b)
$$\sum_{k=1}^{n} \int_{\varepsilon b_n^p}^{\infty} \mathbb{P}\{|X_{n,k}|^p > t\}\, dt = o\Big(\frac{b_n^p}{1+M_n}\Big) \quad \text{as } n \to \infty \text{ for any } \varepsilon > 0$$

when $1 < p < 2$, *or*

(b′)
$$\sum_{k=1}^{n} \int_{\varepsilon b_n}^{\infty} \mathbb{P}\{|X_{n,k}| > t\}\, dt = o\Big(\frac{b_n}{1+M_n}\Big) \quad \text{as } n \to \infty$$

and

$$\sum_{k=1}^{n} \mathbb{P}\{|X_{n,k}| > \varepsilon b_n\} = o(1) \quad \text{as } n \to \infty \text{ for any } \varepsilon > 0$$

whenever $p = 1$, *then*

$$\frac{1}{b_n} \sum_{k=1}^{n} (X_{n,k} - \mathbb{E}\, X_{n,k}) \xrightarrow{\mathscr{L}_p} 0.$$

The p-mean convergence holds true for $0 < p < 1$ under no assumptions of dependence (or independence) for the random variables.

Theorem 2. *Let* $0 < p < 1$ *and* $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be a triangular array of random variables. If* $\{b_n\}$ *is a sequence of positive constants satisfying*

(i)
$$\sum_{k=1}^{n} \int_0^{b_n} \mathbb{P}\{|X_{n,k}| > t\}\, dt = o(b_n) \quad \text{as } n \to \infty,$$

(ii)
$$\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p\, I_{\{|X_{n,k}| > b_n\}} = o\left(b_n^p\right) \quad \text{as } n \to \infty,$$

then

$$\frac{1}{b_n} \sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathscr{L}_p} 0.$$

Considering $\{u_n\}$ and $\{v_n\}$ any two (finite) sequences of integers such that $u_n < v_n$ for all $n \geqslant 1$ and $v_n - u_n \to \infty$ as $n \to \infty$, it can be stated that Theorem 1 is valid for general arrays $\{X_{n,k},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ of row-wise END random variables with dominating sequence $\{M_n,\ n \geqslant 1\}$, i.e. for $1 \leqslant p < 2$, $\sum_{k=u_n}^{v_n} (X_{n,k} - \mathbb{E}\, X_{n,k})/b_n \xrightarrow{\mathscr{L}_p} 0$ provided that $\{b_n\}$ is a sequence of positive constants such that

(A)

$$\sum_{k=u_n}^{v_n} \int_0^{\varepsilon b_n^p} \mathbb{P}\{|X_{n,k}|^p > t\}\, dt = O\Big(\frac{b_n^p}{1+M_n}\Big) \quad \text{as } n \to \infty$$

for any $\varepsilon > 0$,

(B)

$$\sum_{k=u_n}^{v_n} \int_{\varepsilon b_n^p}^{\infty} \mathbb{P}\{|X_{n,k}|^p > t\}\, dt = o\Big(\frac{b_n^p}{1+M_n}\Big) \quad \text{as } n \to \infty$$

for any $\varepsilon > 0$ when $1 < p < 2$, or

(B′)

$$\sum_{k=u_n}^{v_n} \int_{\varepsilon b_n}^{\infty} \mathbb{P}\{|X_{n,k}| > t\}\, dt = o\Big(\frac{b_n}{1+M_n}\Big) \quad \text{as } n \to \infty$$

and

$$\sum_{k=u_n}^{v_n} \mathbb{P}\{|X_{n,k}| > \varepsilon b_n\} = o(1) \quad \text{as } n \to \infty$$

for any $\varepsilon > 0$ whenever $p = 1$.

Moreover, Theorem 2 still holds for $\{X_{n,k},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$, that is, if $0 < p < 1$ and $\{b_n\}$ is a sequence of positive constants satisfying

(I)

$$\sum_{k=u_n}^{v_n} \int_0^{b_n} \mathbb{P}\{|X_{n,k}| > t\}\, dt = o(b_n) \quad \text{as } n \to \infty,$$

(II)

$$\sum_{k=u_n}^{v_n} \mathbb{E}\, |X_{n,k}|^p\, I_{\{|X_{n,k}| > b_n\}} = o(b_n^p) \quad \text{as } n \to \infty,$$

then

$$\sum_{k=u_n}^{v_n} (X_{n,k} - \mathbb{E}\, X_{n,k})/b_n \xrightarrow{\mathscr{L}_p} 0.$$

The proofs can be performed *mutatis mutandis* the presented ones.

Let us point out that for arrays of row-wise pairwise negative quadrant dependent random variables the covariance of any two distinct elements in the same row is nonpositive (see [9] or [15, Lemma 2.1]). In general, this implication is not true for arrays of row-wise END random variables. Therefore, the approach employed here is essentially different from [15]. Furthermore, in cases where the arrays of random variables are both row-wise pairwise negative quadrant dependent and row-wise END having constant dominating sequence $M_n = M > 0$ for every n, condition (b) in Theorem 1 improves assumption (ii) of [15, Theorem 2.1] for $1 < p < 2$. In fact, taking $u_n = 1$, $v_n = n$, $a_{n,k} = 1/b_n = o(1)$ as $n \to \infty$ and the identical distributed array $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ such that the tail distribution of $|X_{1,1}|^p$ is rapidly varying with index $-\infty$, i.e. $\overline{F}_{|X_{1,1}|^p} \in \mathcal{R}_{-\infty}$ (e.g. $|X_{1,1}|^p$ having exponential distribution), assumption (ii) in [15, Theorem 2.1] is

$$\int_{\varepsilon b_n^p}^{\infty} \overline{F}_{|X_{1,1}|^p}(t)\,dt + \varepsilon b_n^p \overline{F}_{|X_{1,1}|^p}(\varepsilon b_n^p) = o(b_n^p/n), \quad n \to \infty$$

and, for each $\varepsilon > 0$,

$$\frac{\int_{\varepsilon b_n^p}^{\infty} \overline{F}_{|X_{1,1}|^p}(t)\,dt}{\varepsilon b_n^p \overline{F}_{|X_{1,1}|^p}(\varepsilon b_n^p)} = o(1), \quad n \to \infty$$

(see [6, p. 570]), whence the numerator is of smaller order than the denominator.

The next result gives the convergence in p-mean for weighted sums of random variables. To this purpose, we recuperate the notion of extended negative dependence for a random sequence (see [3]), as well as the concept of stochastic dominance (see, for instance, [10]).

Corollary 1. *If $\{X_n,\ n \geqslant 1\}$ is a sequence of END random variables stochastically dominated by a random variable $X \in \mathscr{L}_p$ for some $1 \leqslant p < 2$, and $\{c_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is an array of constants such that*

$$\max_{1 \leqslant k \leqslant n} |c_{n,k}| = O(1), \quad n \to \infty$$

then

$$n^{-1/p} \sum_{k=1}^{n} c_{n,k}(X_k - \mathbb{E}\,X_k) \xrightarrow{\mathscr{L}_p} 0.$$

Retrieving the definition of weakly mean domination for triangular arrays due to Gut (see [7, p. 54]) and assuming it in previous Theorem 2, we are lead to a much shorter result.

Corollary 2. *Let $0 < p < 1$ and $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of random variables weakly mean dominated by a random variable $X \in \mathscr{L}_p$. If $\{b_n\}$ is a sequence of positive constants satisfying $n = O(b_n^p)$, $n \to \infty$ then*

$$\frac{1}{b_n} \sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathscr{L}_p} 0.$$

Remark 1. Let us observe that Pyke and Root's classical result is now a particular case of Corollaries 1 and 2. Furthermore, Theorem 1 and Corollary 2 clearly extend [7, Lemma 2.2] to a general scenario of dependence. Additionally, if the dominating sequence is a positive constant (i.e. $M_n = M > 0$ for all n) then assumptions (a) and (b) of Theorem 1 are less restrictive than hypothesis [17, (14)] when $\Psi(t) = |t|^q$, leading to convergence (15) of this paper and, thereby, improving the corresponding statement.

3. Lemmas and proofs

The auxiliary result below shows that triangular arrays of row-wise extended negatively dependent random variables preserve their dependence structure (and particularly, their dominating sequence) under nondecreasing or nonincreasing transformations. Since all assertions can be demonstrated in the same way, i.e. as in [11, Lemma 1], the proof will be omitted.

Lemma 1. *Let $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of random variables and $\{f_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ a triangular array of real functions.*

(i) *If $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is row-wise UEND, LEND, or END with dominating sequence $\{M_n,\ n \geqslant 1\}$ and the functions $f_{n,k}$, $1 \leqslant k \leqslant n$, $n \geqslant 1$ are all nondecreasing then $\{f_{n,k}(X_{n,k}),\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is still row-wise UEND, LEND, or END, respectively. Moreover, if the functions $f_{n,k}$, $1 \leqslant k \leqslant n$, $n \geqslant 1$ are also positive then for each $n \geqslant 1$,*

$$\mathbb{E}\Bigg[\prod_{k=1}^{n} f_{n,k}(X_{n,k})\Bigg] \leqslant M_n \prod_{k=1}^{n} \mathbb{E}\, f_{n,k}(X_{n,k}).$$

(ii) *If $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is row-wise UEND, LEND, or END with dominating sequence $\{M_n,\ n \geqslant 1\}$ and the functions $f_{n,k}$, $1 \leqslant k \leqslant n$, $n \geqslant 1$ are all nonincreasing then $\{f_{n,k}(X_{n,k}),\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is row-wise LEND, UEND, or END, respectively.*

For each case, the dominating sequence $\{M_n,\ n \geqslant 1\}$ remains unchanged.

The next Lemma is a Rosenthal type inequality (see [12, p. 59]) for triangular arrays of row-wise extended negatively dependent random variables with dominating sequence $\{M_n,\ n \geqslant 1\}$ and extends [14, Corollary 3.2] or [17, Lemma 7].

Lemma 2. *If $p \geqslant 2$ and $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is an array of zero-mean row-wise END random variables with dominating sequence $\{M_n, n \geqslant 1\}$ such that $\mathbb{E}\,|X_{n,k}|^p < \infty$, for all $1 \leqslant k \leqslant n$, $n \geqslant 1$ then*

$$\mathbb{E}\left|\sum_{k=1}^{n} X_{n,k}\right|^p \leqslant C(p)(1+M_n)\left[\sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}|^p + \left(\sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}|^2\right)^{p/2}\right]$$

where $C(p)$ is a positive constant depending only on p.

Proof. Let $\{\delta_n\}$ be a sequence of positive constants and consider the random variables $T_{n,k} := \min(X_{n,k}, \delta_n)$, $1 \leqslant k \leqslant n$, $n \geqslant 1$. Hence,

$$\left\{\omega : \sum_{k=1}^{n} X_{n,k} > \varepsilon\right\} \subset \left\{\omega : \sum_{k=1}^{n} T_{n,k} \neq \sum_{k=1}^{n} X_{n,k}\right\} \cup \left\{\omega : \sum_{k=1}^{n} T_{n,k} > \varepsilon\right\}$$

and for all $t_n > 0$,

$$\begin{aligned} \mathbb{P}\left\{\sum_{k=1}^{n} X_{n,k} > \varepsilon\right\} &\leqslant \mathbb{P}\left\{\sum_{k=1}^{n} T_{n,k} \neq \sum_{k=1}^{n} X_{n,k}\right\} + \mathbb{P}\left\{\sum_{k=1}^{n} T_{n,k} > \varepsilon\right\} \\ &\leqslant \sum_{k=1}^{n} \mathbb{P}\{X_{n,k} > \delta_n\} + \exp(-\varepsilon t_n)\mathbb{E}\exp\left(t_n \sum_{k=1}^{n} T_{n,k}\right) \\ &\leqslant \sum_{k=1}^{n} \mathbb{P}\{X_{n,k} > \delta_n\} + M_n \exp(-\varepsilon t_n) \prod_{k=1}^{n} \mathbb{E}\exp(t_n T_{n,k}) \end{aligned} \tag{3.1}$$

provided that $\{T_{n,k},\ 1 \leqslant k \leqslant n\ \ n \geqslant 1\}$ is row-wise END (see Lemma 1). Thus,

$$\begin{aligned} \mathbb{E}\exp(t_n T_{n,k}) &\leqslant 1 + t_n \mathbb{E}X_{n,k} + \int_{-\infty}^{\delta_n} (e^{t_n x} - 1 - t_n x)\, d\mathbb{P}\{T_{n,k} \leqslant x\} \\ &\quad + \int_{\delta_n}^{\infty} (e^{t_n \delta_n} - 1 - t_n \delta_n)\, d\mathbb{P}\{T_{n,k} \leqslant x\} \\ &\leqslant 1 + \frac{e^{t_n\delta_n} - 1 - t_n\delta_n}{\delta_n^2} \int_{-\infty}^{\delta_n} x^2\, d\mathbb{P}\{T_{n,k} \leqslant x\} \\ &\quad + \frac{e^{t_n\delta_n} - 1 - t_n\delta_n}{\delta_n^2} \int_{\delta_n}^{\infty} \delta_n^2\, d\mathbb{P}\{T_{n,k} \leqslant x\} \\ &\leqslant 1 + \frac{e^{t_n\delta_n} - 1 - t_n\delta_n}{\delta_n^2} \mathbb{E}T_{n,k}^2 \leqslant \exp\left(\frac{e^{t_n\delta_n} - 1 - t_n\delta_n}{\delta_n^2} \mathbb{E}X_{n,k}^2\right) \end{aligned}$$

since, for each $n \geqslant 1$, the function $x \mapsto \left(e^{t_n x} - 1 - t_n x\right)/x^2$ is nondecreasing. From the previous inequality and (3.1), we obtain

$$\mathbb{P}\Big\{\sum_{k=1}^{n} X_{n,k} > \varepsilon\Big\} \leqslant \sum_{k=1}^{n} \mathbb{P}\{X_{n,k} > \delta_n\} + M_n \exp\Big[-\varepsilon t_n + \frac{e^{t_n\delta_n} - 1 - t_n\delta_n}{\delta_n^2}\sum_{k=1}^{n}\mathbb{E}X_{n,k}^2\Big]. \tag{3.2}$$

Setting $s_n := \sum_{k=1}^{n}\mathbb{E}X_{n,k}^2$ and taking $t_n = \log\left(1 + \varepsilon\delta_n/\sum_{k=1}^{n}\mathbb{E}X_{n,k}^2\right)^{1/\delta_n}$ in (3.2), we get

$$\mathbb{P}\Big\{\sum_{k=1}^{n} X_{n,k} > \varepsilon\Big\} \leqslant \sum_{k=1}^{n} \mathbb{P}\{X_{n,k} > \delta_n\} + M_n \exp\Big[\frac{\varepsilon}{\delta_n} - \Big(\frac{\varepsilon}{\delta_n} + \frac{s_n}{\delta_n^2}\Big)\log\Big(1 + \frac{\varepsilon\delta_n}{s_n}\Big)\Big]$$
$$\leqslant \sum_{k=1}^{n} \mathbb{P}\{X_{n,k} > \delta_n\} + M_n \exp\Big[\frac{\varepsilon}{\delta_n} - \frac{\varepsilon}{\delta_n}\log\Big(1 + \frac{\varepsilon\delta_n}{s_n}\Big)\Big].$$

Replacing $X_{n,k}$ by $-X_{n,k}$ and noting that $\{-X_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ is still an array of zero-mean row-wise END random variables with dominating sequence $\{M_n,\, n \geqslant 1\}$ according to Lemma 1, satisfying $\mathbb{E}\,|X_{n,k}|^p < \infty$, for all $1 \leqslant k \leqslant n$, $n \geqslant 1$, we have

$$\mathbb{P}\Big\{-\sum_{k=1}^{n} X_{n,k} > \varepsilon\Big\} \leqslant \sum_{k=1}^{n} \mathbb{P}\{-X_{n,k} > \delta_n\} + M_n \exp\Big[\frac{\varepsilon}{\delta_n} - \frac{\varepsilon}{\delta_n}\log\Big(1 + \frac{\varepsilon\delta_n}{s_n}\Big)\Big]$$

and

$$\mathbb{P}\Big\{\Big|\sum_{k=1}^{n} X_{n,k}\Big| > \varepsilon\Big\} \leqslant \sum_{k=1}^{n} \mathbb{P}\{|X_{n,k}| > \delta_n\} + 2M_n \exp\Big[\frac{\varepsilon}{\delta_n} - \frac{\varepsilon}{\delta_n}\log\Big(1 + \frac{\varepsilon\delta_n}{s_n}\Big)\Big]. \tag{3.3}$$

Considering $\delta_n = \varepsilon/p$ in (3.3), yields

$$\mathbb{P}\Big\{\Big|\sum_{k=1}^{n} X_{n,k}\Big| > \varepsilon\Big\} \leqslant \sum_{k=1}^{n} \mathbb{P}\Big\{|X_{n,k}| > \frac{\varepsilon}{p}\Big\} + 2M_n e^p\Big(1 + \frac{\varepsilon^2}{p s_n}\Big)^{-p},$$

which implies

$$\mathbb{E}\Big|\sum_{k=1}^{n} X_{n,k}\Big|^{p} \leqslant p^{p}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|^{p} + 2M_{n}pe^{p}\int_{0}^{\infty} x^{p-1}\Big(1+\frac{x^{2}}{ps_{n}}\Big)^{-p}\,dx.$$

Since

$$\begin{aligned}\int_{0}^{\infty} x^{p-1}\Big(1+\frac{x^{2}}{ps_{n}}\Big)^{-p}\,dx &= \int_{0}^{\infty} x^{p-1}\Big(1-\frac{x^{2}}{ps_{n}+x^{2}}\Big)^{p}\,dx \\ &= \frac{p^{p/2}s_{n}^{p/2}}{2}\int_{0}^{1} y^{\frac{p}{2}-1}(1-y)^{\frac{p}{2}-1}\,dy = \frac{p^{p/2}s_{n}^{p/2}}{2}\,B\Big(\frac{p}{2},\frac{p}{2}\Big),\end{aligned}$$

where $B(p,q)=\int_{0}^{1} x^{p-1}(1-x)^{q-1}\,dx$, $p,q>0$ is the Beta function, it follows

$$\mathbb{E}\Big|\sum_{k=1}^{n} X_{n,k}\Big|^{p} \leqslant \Big[p^{p}+p^{(p+2)/2}M_{n}e^{p}B\Big(\frac{p}{2},\frac{p}{2}\Big)\Big]\Big(\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|^{p}+s_{n}^{p/2}\Big)$$

and the thesis is established with $C(p)=\max\big\{p^{p},p^{(p+2)/2}e^{p}B(\frac{p}{2},\frac{p}{2})\big\}$. □

REMARK 2. If $\{X_{n,k},\ 1\leqslant k\leqslant n,\ n\geqslant 1\}$ is an array of row-wise END random variables with dominating sequence $\{M_{n},\ n\geqslant 1\}$ then

$$\sum_{k=1}^{n}\frac{(X_{n,k}-\mathbb{E}\,X_{n,k})}{b_{n}} \xrightarrow{\mathscr{L}_{2}} 0,$$

provided that $\{b_{n}\}$ is a sequence of positive constants such that

$$\sum_{k=1}^{n}\mathbb{E}\,X_{n,k}^{2} = o\Big(\frac{b_{n}^{2}}{1+M_{n}}\Big),\quad n\to\infty.$$

Indeed, according to Lemma 2 we have

$$\begin{aligned}\mathbb{E}\Big|\frac{1}{b_{n}}\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})\Big|^{2} &\leqslant \frac{C(p)(1+M_{n})}{b_{n}^{2}}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}-\mathbb{E}\,X_{n,k}|^{2} \\ &\leqslant \frac{C(p)(1+M_{n})}{b_{n}^{2}}\sum_{k=1}^{n}\mathbb{E}\,X_{n,k}^{2}\end{aligned}$$

which leads to $\mathbb{E}\,|\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})/b_{n}|^{2}\longrightarrow 0$ as $n\to\infty$. Recall that Pyke and Root's statement is no longer valid for $p=2$. Indeed, the central limit theorem imposes norming constants b_{n} asymptotically equivalent to

$\sqrt{n}$ as $n \to \infty$ whence, assuming X_k i.i.d. such that $0 < \mathbb{V}(X_1) < \infty$ and $b_n = \sqrt{n}$ we obtain

$$\sum_{k=1}^{n}(X_k - \mathbb{E}\,X_1)/\sqrt{n\mathbb{V}(X_1)} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty.$$

Proof of Theorem 1. Fix $\varepsilon > 0$ and define

$$X'_{n,k} := X_{n,k}I_{\{|X_{n,k}|\leqslant t^{1/p}\}} + t^{1/p}I_{\{X_{n,k}>t^{1/p}\}} - t^{1/p}I_{\{X_{n,k}<-t^{1/p}\}},$$
$$X''_{n,k} := X_{n,k}I_{\{|X_{n,k}|>t^{1/p}\}} + t^{1/p}I_{\{X_{n,k}<-t^{1/p}\}} - t^{1/p}I_{\{X_{n,k}>t^{1/p}\}}.$$

Thus, $X'_{n,k} + X''_{n,k} = X_{n,k}$ and

$$\mathbb{E}\Big|\frac{1}{b_n}\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\Big|^p = \frac{1}{b_n^p}\int_0^\infty \mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\Big|^p > t\Big\}\,dt$$
$$\leqslant \varepsilon + \frac{1}{b_n^p}\int_{\varepsilon b_n^p}^\infty \mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\Big| > t^{1/p}\Big\}\,dt$$
$$\leqslant \varepsilon + \frac{1}{b_n^p}\int_{\varepsilon b_n^p}^\infty \mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X'_{n,k} - \mathbb{E}\,X'_{n,k})\Big| > \frac{t^{1/p}}{2}\Big\}\,dt$$
$$+ \frac{1}{b_n^p}\int_{\varepsilon b_n^p}^\infty \mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X''_{n,k} - \mathbb{E}\,X''_{n,k})\Big| > \frac{t^{1/p}}{2}\Big\}\,dt.$$

The triangular array $\{X'_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ is row-wise END with dominating sequence $\{M_n,\, n \geqslant 1\}$ since the function $g_\ell(u) = \max(\min(u,\ell),-\ell)$, which describes the truncation at level ℓ, is nondecreasing (Lemma 1). Moreover, $\{X'_{n,k} - \mathbb{E}\,X'_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ is also row-wise END with dominating sequence $\{M_n,\, n \geqslant 1\}$. From Chebyshev inequality and Lemma 2, we have for any $1 \leqslant p < 2$,

$$\int_{\varepsilon b_n^p}^\infty \mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X'_{n,k} - \mathbb{E}\,X'_{n,k})\Big| > \frac{t^{1/p}}{2}\Big\}\,dt \tag{3.4}$$
$$\leqslant C\int_{\varepsilon b_n^p}^\infty t^{-2/p}\mathbb{E}\Big|\sum_{k=1}^{n}(X'_{n,k} - \mathbb{E}\,X'_{n,k})\Big|^2\,dt$$
$$\leqslant C(p)\,(1+M_n)\int_{\varepsilon b_n^p}^\infty t^{-2/p}\sum_{k=1}^{n}\mathbb{E}\,\big|X'_{n,k} - \mathbb{E}\,X'_{n,k}\big|^2\,dt$$

$$\leqslant C(p)(1+M_n)\int_{\varepsilon b_n^p}^{\infty} t^{-2/p}\sum_{k=1}^{n}\Big[\mathbb{E}X_{n,k}^2 I_{\{|X_{n,k}|\leqslant t^{1/p}\}} + t^{2/p}\mathbb{P}\big\{\,|X_{n,k}|>t^{1/p}\big\}\Big]\,dt$$

$$= C(p)(1+M_n)\sum_{k=1}^{n}\int_{\varepsilon b_n^p}^{\infty} t^{-2/p}\int_0^{t^{1/p}} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds\,dt$$

$$= C(p)(1+M_n)\sum_{k=1}^{n}\int_0^{\infty} s\,\mathbb{P}\{|X_{n,k}|>s\}\int_{\max(\varepsilon b_n^p,s^p)}^{\infty} t^{-2/p}\,dt\,ds$$

$$= C(p)(1+M_n)\sum_{k=1}^{n}\bigg[\frac{p\varepsilon^{1-2/p}b_n^{p-2}}{2-p}\int_0^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

$$+\frac{p}{2-p}\int_{\varepsilon^{1/p}b_n}^{\infty} s^{p-1}\mathbb{P}\{|X_{n,k}|>s\}\,ds\bigg].$$

Setting

$$A_{n,p}(\varepsilon) := \int_0^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

it follows, for any $0<\varepsilon<1$,

$$A_{n,p}(\varepsilon) = \int_0^{\varepsilon^{2/p}b_n} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds + \int_{\varepsilon^{2/p}b_n}^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

$$\leqslant \varepsilon^{4/p-2}b_n^{2-p}\int_0^{\varepsilon^{2/p}b_n} s^{p-1}\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

$$+\varepsilon^{2/p-1}b_n^{2-p}\int_{\varepsilon^{2/p}b_n}^{\varepsilon^{1/p}b_n} s^{p-1}\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

and (3.4) yields

$$\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\bigg\{\bigg|\sum_{k=1}^{n}(X'_{n,k}-\mathbb{E}\,X'_{n,k})\bigg|>\frac{t^{1/p}}{2}\bigg\}\,dt \tag{3.5a}$$

$$\leqslant C(p)(1+M_n)\sum_{k=1}^{n}\bigg[\frac{p\varepsilon^{(2-p)/p}}{2-p}\int_0^{\varepsilon^{2/p}b_n} s^{p-1}\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

$$+\frac{p}{2-p}\int_{\varepsilon^{2/p}b_n}^{\infty} s^{p-1}\mathbb{P}\{|X_{n,k}|>s\}\;ds\bigg]$$

$$\leqslant C(p)(1+M_n)\sum_{k=1}^{n}\left[\frac{\varepsilon^{(2-p)/p}}{2-p}\int_0^{\varepsilon^2 b_n^p}\mathbb{P}\{|X_{n,k}|>t^{1/p}\}\,dt\right.$$

$$\left.+\frac{1}{2-p}\int_{\varepsilon^2 b_n^p}^{\infty}\mathbb{P}\{|X_{n,k}|>t^{1/p}\}\,dt\right].$$

On the other hand, if $\varepsilon \geqslant 1$ then

$$A_{n,p}(\varepsilon)\leqslant\int_0^{\varepsilon^{2/p}b_n} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds$$

$$\leqslant \varepsilon^{4/p-2}b_n^{2-p}\int_0^{\varepsilon^{2/p}b_n} s^{p-1}\,\mathbb{P}\{|X_{n,k}|>s\}\,ds,$$

and (3.4) gives

$$\text{(3.5b)}\qquad \int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X'_{n,k}-\mathbb{E}\,X'_{n,k})\Big|>\frac{t^{1/p}}{2}\Big\}\,dt$$

$$\leqslant C(p)(1+M_n)\sum_{k=1}^{n}\left[\frac{p\varepsilon^{(2-p)/p}}{2-p}\int_0^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\{|X_{n,k}|>s\}\,ds\right.$$

$$\left.+\frac{p}{2-p}\int_{\varepsilon^{1/p}b_n}^{\infty} s^{p-1}\mathbb{P}\{|X_{n,k}|>s\}\,ds\right]$$

$$= C(p)(1+M_n)\sum_{k=1}^{n}\left[\frac{\varepsilon^{(2-p)/p}}{2-p}\int_0^{\varepsilon^2 b_n^p}\mathbb{P}\{|X_{n,k}|>t^{1/p}\}\,dt\right.$$

$$\left.+\frac{1}{2-p}\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\{|X_{n,k}|>t^{1/p}\}\,dt\right].$$

Since $|X''_{n,k}|\leqslant |X_{n,k}|\,I_{\{|X_{n,k}|>t^{1/p}\}}$, we obtain for every $1<p<2$

$$\text{(3.6)}\qquad \int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\Big\{\Big|\sum_{k=1}^{n}(X''_{n,k}-\mathbb{E}\,X''_{n,k})\Big|>\frac{t^{1/p}}{2}\Big\}\,dt$$

$$\leqslant C\int_{\varepsilon b_n^p}^{\infty} t^{-1/p}\sum_{k=1}^{n}\mathbb{E}\,\big|X''_{n,k}\big|\,dt\leqslant C\sum_{k=1}^{n}\int_{\varepsilon b_n^p}^{\infty} t^{-1/p}\mathbb{E}\,|X_{n,k}|\,I_{\{|X_{n,k}|>t^{1/p}\}}\,dt$$

$$= C\sum_{k=1}^{n}\bigg(\int_{\varepsilon b_n^p}^{\infty} t^{-1/p}\int_{t^{1/p}}^{\infty}\mathbb{P}\{|X_{n,k}|>s\}\,ds\,dt+\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\{\,|X_{n,k}|>t^{1/p}\}\,dt\bigg)$$

$$= C\sum_{k=1}^{n}\left(\int_{b_n\varepsilon^{1/p}}^{\infty}\mathbb{P}\{|X_{n,k}|>s\}\int_{\varepsilon b_n^p}^{s^p}t^{-1/p}\,dt\,ds+\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\{\,|X_{n,k}|>t^{1/p}\}\,dt\right)$$

$$\leqslant C\sum_{k=1}^{n}\left(\frac{p}{p-1}\int_{b_n\varepsilon^{1/p}}^{\infty}s^{p-1}\mathbb{P}\{|X_{n,k}|>s\}\,ds+\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\{\,|X_{n,k}|>t^{1/p}\}\,dt\right)$$

$$\leqslant \frac{p\,C}{p-1}\sum_{k=1}^{n}\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\{\,|X_{n,k}|>t^{1/p}\}\,dt.$$

Hence, inequalities (3.5a), (3.5b), (3.6) and the arbitrariness of ε guarantee

$$\mathbb{E}\left|\frac{1}{b_n}\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})\right|^p \longrightarrow 0$$

as $n\to\infty$ completing the proof for the case $1<p<2$. It suffices to prove

$$\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})/b_n \xrightarrow{\mathscr{L}_1} 0.$$

For each $\varepsilon>0$, we have

$$\sup_{t\geqslant\varepsilon b_n}\left|\frac{1}{t}\sum_{k=1}^{n}\mathbb{E}\,X''_{n,k}\right| \leqslant \sup_{t\geqslant\varepsilon b_n}\frac{1}{t}\sum_{k=1}^{n}\mathbb{E}|X''_{n,k}|$$

$$\leqslant \sup_{t\geqslant\varepsilon b_n}\frac{1}{t}\sum_{k=1}^{n}\mathbb{E}|X_{n,k}|I_{\{|X_{n,k}|>t\}} \leqslant \frac{1}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}|X_{n,k}|I_{\{|X_{n,k}|>\varepsilon b_n\}} \longrightarrow 0$$

as $n\to\infty$ from assumption (b$'$). Hence, for n large we obtain

$$\left|\sum_{k=1}^{n}\mathbb{E}\,X''_{n,k}\right| \leqslant \frac{t}{4}$$

and

$$\int_{\varepsilon b_n}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X''_{n,k}-\mathbb{E}\,X''_{n,k})\right|>\frac{t}{2}\right\}dt \leqslant \int_{\varepsilon b_n}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}X''_{n,k}\right|>\frac{t}{4}\right\}dt \tag{3.7}$$

$$\leqslant \int_{\varepsilon b_n}^{\infty}\mathbb{P}\left\{\sum_{k=1}^{n}|X''_{n,k}|>\frac{t}{4}\right\}dt \leqslant \int_{\varepsilon b_n}^{\infty}\mathbb{P}\left(\bigcup_{k=1}^{n}\{|X_{n,k}|>t\}\right)dt$$

$$\leqslant \int_{\varepsilon b_n}^{\infty}\sum_{k=1}^{n}\mathbb{P}\Big\{|X_{n,k}|>t\Big\}\,dt \leqslant \sum_{k=1}^{n}\mathbb{E}|X_{n,k}|I_{\{|X_{n,k}|>\varepsilon b_n\}}.$$

According to inequalities (3.5a), (3.5b), (3.7) and the arbitrariness of ε, it follows

$$\mathbb{E}\left|\frac{1}{b_n}\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})\right|\longrightarrow 0$$

as $n\to\infty$ finishing the proof. $\square$

Proof of Theorem 2. Putting

$$Y'_{n,k}:=X_{n,k}I_{\{|X_{n,k}|\leqslant b_n\}}+b_nI_{\{X_{n,k}>b_n\}}-b_nI_{\{X_{n,k}<-b_n\}},$$
$$Y''_{n,k}:=X_{n,k}I_{\{|X_{n,k}|>b_n\}}+b_nI_{\{X_{n,k}<-b_n\}}-b_nI_{\{X_{n,k}>b_n\}}$$

we have $X_{n,k}=Y'_{n,k}+Y''_{n,k}$ and

$$\mathbb{E}\left|\sum_{k=1}^{n}X_{n,k}\right|^p\leqslant\mathbb{E}\left|\sum_{k=1}^{n}Y'_{n,k}\right|^p+\mathbb{E}\left|\sum_{k=1}^{n}Y''_{n,k}\right|^p\leqslant\mathbb{E}^p\left|\sum_{k=1}^{n}Y'_{n,k}\right|+\mathbb{E}\left|\sum_{k=1}^{n}Y''_{n,k}\right|^p$$

via Jensen's inequality (see [5, p. 104]). Thus, $\left|Y''_{n,k}\right|^p\leqslant|X_{n,k}|^p\,I_{\{|X_{n,k}|>b_n\}}$ and

$$\frac{1}{b_n^p}\mathbb{E}\left|\sum_{k=1}^{n}Y''_{n,k}\right|^p\leqslant\frac{1}{b_n^p}\mathbb{E}\left[\sum_{k=1}^{n}\left|Y''_{n,k}\right|^p\right]=\frac{1}{b_n^p}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|^p\,I_{\{|X_{n,k}|>b_n\}}\longrightarrow 0 \tag{3.8}$$

as $n\to\infty$ from assumption (ii). Furthermore,

$$\begin{aligned}\mathbb{E}^p\left|\sum_{k=1}^{n}Y'_{n,k}\right|&=\mathbb{E}^p\left|\sum_{k=1}^{n}X_{n,k}I_{\{|X_{n,k}|\leqslant b_n\}}+b_nI_{\{X_{n,k}>b_n\}}-b_nI_{\{X_{n,k}<-b_n\}}\right|\\&\leqslant\mathbb{E}^p\sum_{k=1}^{n}\big(\,|X_{n,k}|\,I_{\{|X_{n,k}|\leqslant b_n\}}+b_nI_{\{|X_{n,k}|>b_n\}}\big)\\&=\left[\sum_{k=1}^{n}\big(\mathbb{E}\,|X_{n,k}|\,I_{\{|X_{n,k}|\leqslant b_n\}}+b_n\mathbb{P}\{|X_{n,k}|>b_n\}\big)\right]^p\\&=\left(\sum_{k=1}^{n}\int_0^{b_n}\mathbb{P}\{|X_{n,k}|>t\}\,dt\right)^p,\end{aligned}$$

whence

$$\frac{1}{b_n^p}\,\mathbb{E}^p\Big|\sum_{k=1}^{n} Y'_{n,k}\Big| \longrightarrow 0 \tag{3.9}$$

as $n \to \infty$ according to (i). The thesis is now a consequence of (3.8) and (3.9). □

Proof of Corollary 1. Without loss of generality, we shall assume $c_{n,k} \geqslant 0$ for all $1 \leqslant k \leqslant n$, $n \geqslant 1$ since

$$\sum_{k=1}^{n} c_{n,k} X_k = \sum_{k=1}^{n} c_{n,k}^{+} X_k - \sum_{k=1}^{n} c_{n,k}^{-} X_k\,,$$

where $c_{n,k}^{+} = \max\{c_{n,k}, 0\} \geqslant 0$ and $c_{n,k}^{-} = \max\{-c_{n,k}, 0\} \geqslant 0$. The triangular array $\{c_{n,k}X_k,\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is row-wise END with constant dominating sequence. From [10, Lemma 1] and the dominated convergence theorem, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} \mathbb{E}\,|c_{n,k}X_k|^p\, I_{\{|c_{n,k}X_k|^p > \varepsilon n\}} \\
&\leqslant C^p \sum_{k=1}^{n} \mathbb{E}\,|X_k|^p\, I_{\{C^p|X_k|^p > \varepsilon n\}} \leqslant C(p) \sum_{k=1}^{n} \mathbb{E}\,|X|^p\, I_{\{C^p|X|^p > \varepsilon n\}} \\
&= C(p) n\, \mathbb{E}\,|X|^p\, I_{\{C^p|X|^p > \varepsilon n\}} = o(n), \quad n \to \infty
\end{aligned}$$

for each $\varepsilon > 0$. For any $\varepsilon > 0$, we still have

$$\sum_{k=1}^{n} \int_0^{\varepsilon n} \mathbb{P}\big\{\, |c_{n,k}X_k|^p > t \big\}\, dt \leqslant \sum_{k=1}^{n} \int_0^{\infty} \mathbb{P}\big\{\, C^p\,|X_k|^p > t \big\}\, dt \leqslant C(p)\, n\, \mathbb{E}\,|X|^p\,,$$

so that assumptions (a), (b) and (b′) of Theorem 1 are fulfilled with $b_n = n^{1/p}$ and $X_{n,k} = c_{n,k}X_k$. Thus, $\sum_{k=1}^{n} c_{n,k}(X_k - \mathbb{E}\,X_k)/n^{1/p} \xrightarrow{\mathscr{L}_p} 0$ and the thesis is established. □

Proof of Corollary 2. Since $n = O(b_n^p)$, $n \to \infty$ and $\mathbb{E}\,|X|^p\, I_{\{|X| > b_n\}} \longrightarrow 0$ as $n \to \infty$ (dominated convergence theorem), we obtain

$$\frac{1}{b_n^p} \sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}|^p\, I_{\{|X_{n,k}| > b_n\}} \leqslant \frac{n\,C}{b_n^p}\, \mathbb{E}\,|X|^p\, I_{\{|X| > b_n\}} \longrightarrow 0$$

as $n \to \infty$ using [7, Lemma 2.1]. Hence, condition (ii) in Theorem 2 is satisfied. On the other hand,

$$\begin{aligned}\sum_{k=1}^{n}\int_0^{b_n}\mathbb{P}\{|X_{n,k}|>t\}\,dt &= \sum_{k=1}^{n}\bigl(\mathbb{E}\,|X_{n,k}|\,I_{\{|X_{n,k}|\leqslant b_n\}}+b_n\,\mathbb{P}\{|X_{n,k}|>b_n\}\bigr)\\ &\leqslant Cn\,\mathbb{E}\,|X|\,I_{\{|X|\leqslant b_n\}}+Cnb_n\,\mathbb{P}\{|X|>b_n\}\\ &= Cnb_n^{1-p}\,\mathbb{E}\,|X|^p\,|X/b_n|^{1-p}\,I_{\{|X|\leqslant b_n\}}+Cnb_n\,\mathbb{P}\{|X|>b_n\}.\end{aligned}$$

According to condition $n=O(b_n^p)$, $n\to\infty$ we have $b_n\to\infty$ as $n\to\infty$ and $X/b_n \xrightarrow{\text{a.s.}} 0$, which yields

$$n\mathbb{P}\left\{|X|>b_n\right\}\leqslant Cb_n^p\,\mathbb{P}\left\{|X|>b_n\right\}\leqslant C\,\mathbb{E}\,|X|^p\,I_{\{|X|>b_n\}}\longrightarrow 0$$

as $n\to\infty$ and $\mathbb{E}\,|X|^p\,|X/b_n|^{1-p}\,I_{\{|X|\leqslant b_n\}}\longrightarrow 0$ as $n\to\infty$ by the dominated convergence theorem. Thus,

$$\frac{1}{b_n}\sum_{k=1}^{n}\int_0^{b_n}\mathbb{P}\left\{|X_{n,k}|>t\right\}\,dt\longrightarrow 0$$

as $n\to\infty$ and condition (i) in Theorem 2 is verified. □

References

[1] A. Adler, A. Rosalsky and A. I. Volodin, A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces, *Statist. Probab. Lett.*, **32** (1997), 167–174.

[2] M. O. Cabrera and A. I. Volodin, Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability, *J. Math. Anal. Appl.*, **305** (2005), 644–658.

[3] Y. Chen, A. Chen and K. W. Ng, The strong law of large numbers for extended negatively dependent random variables, *J. Appl. Probab.*, **47** (2010), 908–922.

[4] Y. Chen, L. Wang and Y. Wang, Uniform asymptotics for the finite-time ruin probabilities of two kinds of nonstandard bidimensional risk models, *J. Math. Anal. Appl.*, **401** (2013), 114–129.

[5] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed., Springer-Verlag (New York, 1997).

[6] P. Embrechts, C. Klüppelberg and T. Mikosch, *Modelling Extremal Events*, Springer-Verlag (Berlin–Heidelberg, 1997).

[7] A. Gut, Complete convergence for arrays, *Period. Math. Hungar.*, **25** (1992), 51–75.

[8] A. Gut, The weak law of large numbers for arrays, *Statist. Probab. Lett.*, **14** (1992), 49–52.

[9] E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.*, **37** (1966), 1137–1153.

[10] J. Lita da Silva, Almost sure convergence for weighted sums of extended negatively dependent random variables, *Acta Math. Hungar.*, **146** (2015), 56–70.

[11] J. Lita da Silva, Limiting behavior for arrays of row-wise upper extended negatively dependent random variables, *Acta Math. Hungar.*, **148** (2016), 481–492.

[12] V. V. Petrov, *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*, Oxford Studies in Probability, Vol. 4, Clarendon Press (Oxford, 1995).

[13] R. Pyke and D. Root, On convergence in r-mean of normalized partial sums, *Ann. Math. Statist.*, **39** (1968), 379–381.

[14] A. Shen, Probability inequalities for END sequence and their applications, *J. Inequal. Appl.*, **98** (2011), pp. 12.

[15] S. H. Sung, Convergence in r-mean of weighted sums of NQD random variables, *Appl. Math. Lett.*, **26** (2013), 18–24.

[16] Y. Wu and M. Guan, Convergence properties of the partial sums for sequences of END random variables, *J. Korean Math. Soc.*, **49** (2012), 1097–1110.

[17] Y. Wu, M. Song and C. Wang, Complete moment convergence and mean convergence for arrays of rowwise extended negatively dependent random variables. *The Scientific World Journal*, Article ID 478612, Vol. 2014, pp. 7.

[18] D. Yuan and B. Tao, Mean convergence theorems for weighted sums of arrays of residually h-integrable random variables concerning the weights under dependence assumptions, *Acta Appl. Math.*, **103** (2008), 221–234.