

AN EXTENSION OF VAN VLECK'S FUNCTIONAL EQUATION FOR THE SINE

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Abstract. In [7] H. Stetkær obtained the solutions of Van Vleck's functional equation

$$f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in S$$

for the sine where S is a semigroup, τ is an involution of S and z_0 is a fixed element in the center of S . The purpose of this paper is to determine the complex-valued solutions of the following extension of Van Vleck's functional equation for the sine

$$\mu(y)f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in S$$

where $\mu : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$. Furthermore, we obtain the solutions of a variant of Van Vleck's functional equation

$$\mu(y)f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in M$$

for the sine on a monoid M , where σ is an involutive automorphism of M .

1. Introduction

Van Vleck [11,12] studied the continuous solutions $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \neq 0$ of the functional equation

$$(1.1) \quad f(x - y + z_0) - f(x + y + z_0) = 2f(x)f(y), \quad x, y \in \mathbb{R},$$

where $z_0 > 0$ is fixed. He showed first that all solutions are periodic with period $4z_0$, and then he selected for his study any continuous solution with

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minimal period $4z_0$. He proved that any such solution has to be the sine function $f(x) = \sin\left(\frac{\pi}{2z_0}x\right)$, $x \in \mathbb{R}$. We refer also to [4].

In [6], Sahoo studied the generalization

$$(1.2) \quad f(x - y + z_0) + g(x + y + z_0) = 2f(x)f(y) \quad x, y \in G$$

of the functional equation (1.1). He determined the general solutions of this equation on an abelian group G . Stetkær [9, Exercise 9.18] found the complex-valued solution of equation

$$(1.3) \quad f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G,$$

when G is a not necessarily abelian group and z_0 is a fixed element in the center of G . Recently, Perkins and Sahoo [5] replaced the group inversion by an involution $\tau : G \rightarrow G$ and they obtained the abelian, complex-valued solutions of equation

$$(1.4) \quad f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G.$$

Stetkær [7] extended the results of Perkins and Sahoo [5] about equation (1.4) to the more general case where G is a semigroup and the solutions are not assumed to be abelian.

The first purpose of this paper is to extend the results of Stetkær [7] to the following generalization of Van Vleck's functional equation for the sine:

$$(1.5) \quad \mu(y)f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G,$$

where μ is a multiplicative function of a semigroup G such that $\mu(x\tau(x)) = 1$ for all $x \in G$. As in the previous results the main idea is to relate the functional equation (1.4) to the corresponding d'Alembert's functional equation. In our case we shall relate (1.5) to the following version of d'Alembert's functional equation

$$(1.6) \quad g(xy) + \mu(y)g(x\tau(y)) = 2g(x)g(y), \quad x, y \in G$$

and we apply the crucial propositions [9, Proposition 9.17(c)] and [9, Proposition 8.14(a)].

Replacing f by $-F$ in (1.5) we arrive at

$$F(xyz_0) - \mu(y)F(x\tau(y)z_0) = 2F(x)F(y), \quad x, y \in G$$

which shows the similarity between (1.5) and (1.6).

The second purpose of this paper is to obtain the solutions of a variant of Van Vleck's functional equation

$$(1.7) \quad \mu(y)f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G$$

for the sine where G is a monoid (semigroup with identity element e), and σ is an involutive automorphism: That is $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all x, y . In this case the main idea is to relate the functional equation (1.7) to the following variant of d'Alembert's functional equation

$$(1.8) \quad g(xy) + \mu(y)g(\sigma(y)x) = 2g(x)g(y), \quad x, y \in G$$

and apply the result obtained by Elqorachi and Redouani in [2, Lemma 3.2]. We refer also to [10] in which the solutions of equation (1.8) with $\mu = 1$ are obtained on semigroups.

The new feature of our paper is the introduction of the multiplicative function μ in (1.5) and (1.7).

2. Solutions of equation (1.5) on semigroups

Throughout this section S denotes a semigroup and $\tau : S \rightarrow S$ is an involution of S . That is $\tau(xy) = \tau(y)\tau(x)$ and $\tau(\tau(x)) = x$ for all $x, y \in S$. The element z_0 denotes a fixed element in the center of S . Finally $\mu : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$.

We first prove the following lemmas which are generalizations of the useful lemmas obtained by Stetkær [7].

LEMMA 2.1. *Let $f \neq 0$ be a solution of (1.5). Then for all $x \in S$ we have*

$$(2.1) \quad f(x) = -\mu(x)f(\tau(x)),$$

$$(2.2) \quad f(z_0) \neq 0,$$

$$(2.3) \quad f(z_0^2) = 0,$$

$$(2.4) \quad f(x\tau(z_0)z_0) = \mu(\tau(z_0))f(x)f(z_0),$$

$$(2.5) \quad f(xz_0^2) = -f(z_0)f(x),$$

$$(2.6) \quad \mu(x)f(\tau(x)z_0) = f(xz_0).$$

Let G be a group and $\tau(x) = x^{-1}$ for all $x \in G$. Then $f(z_0) = \mu(z_0)$ for any solution $f \neq 0$ of (1.5) and $f(xz_0^4) = \mu(z_0)^2f(x)$ for all $x \in G$.

PROOF. If we replace y by $\tau(y)$ in (1.5) we obtain

$$(2.7) \quad \mu(\tau(y))f(xyz_0) - f(x\tau(y)z_0) = 2f(x)f(\tau(y)).$$

By multiplying (2.7) by $\mu(y)$ and using the assumption that $\mu(y\tau(y)) = 1$ we get

$$(2.8) \quad \mu(y)f(x\tau(y)z_0) - f(xy z_0) = -2f(x)\mu(y)f(\tau(y)).$$

By comparing (1.5) with (2.8) we get $-f(x)\mu(y)f(\tau(y)) = f(x)f(y)$ for all $x, y \in S$. Since $f \neq 0$, then we have (2.1).

Setting $x = \tau(z_0)$ in (1.5) and using (2.1) we see that

$$(2.9) \quad \begin{aligned} \mu(y)f(\tau(z_0)\tau(y)z_0) - f(\tau(z_0)y z_0) \\ = 2f(\tau(z_0))f(y) = -2\mu(\tau(z_0))f(z_0)f(y). \end{aligned}$$

On the other hand by using (2.1) and $\mu(x\tau(x)) = 1$ for all $x \in G$ we get

$$\mu(y)f(\tau(z_0)\tau(y)z_0) = -\mu(y)\mu(\tau(z_0)\tau(y)z_0)f(\tau(z_0)y z_0) = -f(\tau(z_0)y z_0).$$

So, equation (2.9) implies that

$$(2.10) \quad f(\tau(z_0)y z_0) = \mu(\tau(z_0))f(z_0)f(y)$$

for all $y \in S$. Since $z_0, \tau(z_0)$ are in the center of S then we obtain (2.4).

Putting $y = z_0$ in (1.5) and using (2.4) and $\mu(z_0\tau(z_0)) = 1$ we get

$$\begin{aligned} \mu(z_0)f(x\tau(z_0)z_0) - f(xz_0^2) \\ = \mu(z_0)\mu(\tau(z_0))f(z_0)f(x) - f(xz_0^2) = 2f(x)f(z_0). \end{aligned}$$

So, we have $f(xz_0^2) = -f(x)f(z_0)$ for all $x \in S$, which proves (2.5).

By replacing x by xz_0 in the functional equation (1.5) and using (2.5) we obtain

$$\begin{aligned} 2f(xz_0)f(y) &= \mu(y)f(x\tau(y)z_0^2) - f(xy z_0^2) \\ &= -\mu(y)f(z_0)f(x\tau(y)) + f(z_0)f(xy). \end{aligned}$$

If $f(z_0) = 0$, then $f(y)f(xz_0) = 0$ for all $x, y \in S$. Since $f \neq 0$ then $f(xz_0) = 0$ for all $x \in S$, so we have $\mu(y)f(x\tau(y)z_0) - f(xy z_0) = 0 = 2f(x)f(y)$ for all $x, y \in S$ from which we deduce that $f(x) = 0$ for all $x \in S$. This contradicts the assumption that $f \neq 0$ and it follows that $f(z_0) \neq 0$.

By replacing x by $z\tau(z)$ in (2.1) and using $\mu(z\tau(z)) = 1$ we get $f(z\tau(z)) = 0$ for all $z \in S$.

From (2.5) and (2.1) we have

$$0 = f(\tau(z_0^2)z_0^2) = -f(z_0)f(\tau(z_0^2)) = \mu(\tau(z_0^2))f(z_0^2)f(z_0).$$

Since $f(z_0) \neq 0$ we get $\mu(\tau(z_0^2))f(z_0^2) = 0$. It follows from $1 = \mu(x\tau(x)) = \mu(x)\mu(\tau(x))$, that $\mu(x) \neq 0$ for all $x \in S$. It is thus immediate that $f(z_0^2) = 0$. Putting $x = z_0^2$ in (1.5) and using (2.3) to get $\mu(y)f(z_0^2\tau(y)z_0) = f(z_0^2yz_0)$ and from (2.5) we obtain $-\mu(y)f(z_0)f(\tau(y)z_0) = -f(z_0)f(yz_0)$. Since $f(z_0) \neq 0$ then we get $\mu(y)f(\tau(y)z_0) = f(yz_0)$ for all $y \in S$.

The statements for the group case are consequences of the formulas (2.4) and (2.5). \square

LEMMA 2.2. *Let $f \neq 0$ be a solution of equation (1.5). Then*
 (a) *the function*

$$g(x) := \frac{f(xz_0)}{f(z_0)} \quad \text{for } x \in S$$

is a non-zero abelian solution of d'Alembert's functional equation (1.6).

(b) *We have $g = \frac{\chi + \mu\chi \circ \tau}{2}$, where $\chi : S \rightarrow \mathbb{C}$, $\chi \neq 0$, is a multiplicative function.*

PROOF. By using (2.4) and (2.5) we get

$$\begin{aligned} f(z_0)^2 [g(xy) + \mu(y)g(x\tau(y))] &= \mu(y)f(z_0)f(x\tau(y)z_0) + f(z_0)f(xyz_0) \\ &= \mu(y)\mu(z_0)f(x\tau(y)z_0\tau(z_0)z_0) - f(xyz_0z_0^2) \\ &= \mu(yz_0)f((xz_0)\tau(yz_0)z_0) - f((xz_0)(yz_0)z_0) = 2f(xz_0)f(yz_0), \end{aligned}$$

which implies that g satisfies (1.6). Furthermore, $g(z_0^2) = f(z_0^2z_0)/f(z_0) = -f(z_0)f(z_0)/f(z_0) = -f(z_0) \neq 0$, so $g \neq 0$.

As g is a solution of equation (1.6) then by [8, Proposition 9.17(c)] g is a pre-d'Alembert function.

Note that $g(z_0) = f(z_0^2)/f(z_0) = 0$,

$$d(z_0) = 2g(z_0)^2 - g(z_0^2) = 0 - (-f(z_0)) = f(z_0) \neq 0.$$

So, we have $g(z_0)^2 \neq d(z_0)$ and according to [9, Proposition 8.14(a)] g is abelian.

(b) is immediate from [9, Proposition 9.31]. \square

Now, we are ready to prove the first main result of the present paper.

THEOREM 2.3. *The non-zero solutions $f : S \rightarrow \mathbb{C}$ of the functional equation (1.5) are the functions of the form*

$$(2.11) \quad f = \mu(z_0)\chi(\tau(z_0)) \frac{\chi - \mu\chi \circ \tau}{2} = \chi(z_0) \frac{\mu\chi \circ \tau - \chi}{2},$$

where $\chi : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\tau(z_0)) = -\chi(z_0)$. Furthermore, $f(z_0) = \mu(z_0)\chi(z_0\tau(z_0))$.

If S is a group and τ is the group inversion, then $\chi(z_0^2) = -\mu(z_0)$ and $f(z_0) = \mu(z_0)$ for any non-zero solution of equation (1.5).

If S is a topological semigroup and $\tau : S \rightarrow S, \mu : S \rightarrow \mathbb{C}$ are continuous, then the non-zero solution f of equation (1.5) is continuous, if and only if χ is continuous.

PROOF. Let $f : S \rightarrow \mathbb{C}$ be a non-zero solution of equation (1.5). Then we get

$$f(x) = \frac{\mu(z_0)f(x\tau(z_0)z_0) - f(xz_0z_0)}{2f(z_0)} = \frac{\mu(z_0)g(x\tau(z_0)) - g(xz_0)}{2}$$

for all $x \in S$ and where g is the function defined in Lemma 2.2. So, from Lemma 2.2 we have $g = \frac{\chi + \mu\chi \circ \tau}{2}$ and then after an easy computation we obtain

$$(2.12) \quad f = \frac{\chi(z_0) - \mu(z_0)\chi(\tau(z_0))}{2} \frac{\mu\chi \circ \tau - \chi}{2}.$$

From (2.6) we have $\mu(x)f(\tau(x)z_0) = f(xz_0)$ for all $x \in S$. Substituting (2.12) into (2.6) we get after an elementary computation that

$$[\mu(z_0)\chi(\tau(z_0)) + \chi(z_0)][\chi - \mu\chi \circ \tau] = 0.$$

The rest of the proof is similar to the one by Stetkær [7]. \square

Under the hypotheses of Theorem 2.3 any solution f of (1.5) is abelian.

3. Solutions of equation (1.7) on monoids

Throughout this section M denotes a monoid and $\sigma : M \rightarrow M$ is an involutive automorphism of M . The element z_0 denotes a fixed element in the center of M . Finally $\mu : M \rightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in M$. The following useful lemma will be used later.

LEMMA 3.1. *Let $f \neq 0$ be a solution of (1.7). Then*
 (a) *For all $x \in M$ we have*

$$(3.1) \quad f(x) = -\mu(x)f(\sigma(x)),$$

$$(3.2) \quad f(z_0) \neq 0,$$

$$(3.3) \quad f(x\sigma(z_0)z_0) = \mu(\sigma(z_0))f(x)f(z_0),$$

$$(3.4) \quad f(xz_0^2) = -f(z_0)f(x),$$

$$(3.5) \quad \mu(x)f(\sigma(x)z_0) = f(xz_0).$$

(b) the function

$$g(x) := \frac{f(xz_0)}{f(z_0)} \quad \text{for } x \in M$$

is a non-zero solution of a variant of d'Alembert's functional equation

$$(3.6) \quad g(xy) + \mu(y)g(\sigma(y)x) = 2g(x)g(y), \quad x, y \in M.$$

(c) We have $g = \frac{\chi + \mu\chi\circ\sigma}{2}$, where $\chi : M \rightarrow \mathbb{C}$, $\chi \neq 0$, is a multiplicative function.

PROOF. (a) Putting $y = e$ in (1.7) we get $f(xz_0) - f(xz_0) = 2f(x)f(e) = 0$. Since $f \neq 0$ then we have $f(e) = 0$.

Taking $x = e$ in (1.7) and using $f(e) = 0$ we get (3.5).

By replacing x by $\sigma(x)$ in (1.7) we obtain

$$\mu(y)f(\sigma(y)\sigma(x)z_0) - f(\sigma(x)yz_0) = 2f(\sigma(x))f(y).$$

Since from (3.5) we have

$$\begin{aligned} \mu(y)f(\sigma(y)\sigma(x)z_0) &= \mu(y)f(\sigma(yx)z_0) \\ &= \mu(y)\mu(\sigma(yx))f(yxz_0) = \mu(\sigma(x))f(yxz_0), \end{aligned}$$

it follows that

$$\begin{aligned} \mu(x)[\mu(\sigma(x))f(yxz_0) - f(\sigma(x)yz_0)] &= \mu(x)2f(\sigma(x))f(y) \\ &= f(yxz_0) - \mu(x)f(\sigma(x)yz_0). \end{aligned}$$

Since

$$f(yxz_0) - \mu(x)f(\sigma(x)yz_0) = -[\mu(x)f(\sigma(x)yz_0) - f(yxz_0)] = -2f(y)f(x),$$

we conclude that

$$-2f(x)f(y) = 2\mu(x)f(\sigma(x))f(y)$$

for all $x, y \in M$. Since $f \neq 0$ then we get (3.1).

Putting $x = \sigma(z_0)$ in (1.7) and using (3.1) we get

$$\mu(y)f(\sigma(y)\sigma(z_0)z_0) - f(\sigma(z_0)yz_0)$$

$$= 2f(\sigma(z_0))f(y) = -2\mu(\sigma(z_0))f(z_0)f(y).$$

On the other hand we have

$$\mu(y)f(\sigma(y)\sigma(z_0)z_0) = -\mu(y)\mu(\sigma(y)\sigma(z_0)z_0)f(yz_0\sigma(z_0)) = -f(yz_0\sigma(z_0)).$$

Then, since $z_0, \sigma(z_0)$ are in the center of M we get (3.3).

By replacing y by z_0 in (1.7) and using (3.3) we get

$$\begin{aligned} & \mu(z_0)f(\sigma(z_0)xz_0) - f(xz_0^2) \\ &= 2f(x)f(z_0) = \mu(z_0)\mu(\sigma(z_0))f(x)f(z_0) - f(xz_0^2) = f(x)f(z_0) - f(xz_0^2). \end{aligned}$$

So, we deduce equation (3.4).

By replacing x by xz_0 in (1.7) and using (3.4) we get

$$\begin{aligned} & \mu(y)f(\sigma(y)xz_0^2) - f(xz_0yz_0) = 2f(xz_0)f(y) \\ &= -\mu(y)f(\sigma(y)x)f(z_0) + f(xy)f(z_0). \end{aligned}$$

If $f(z_0) = 0$, then $f(xz_0)f(y) = 0$ for all $x, y \in M$, since $f \neq 0$ then $f(xz_0) = 0$ for all $x \in M$ and from (1.7) we obtain $2f(x)f(y) = 0$ for all $x, y \in M$. This contradicts the assumption that $f \neq 0$ and this proves (3.2).

(b) For all $x, y \in M$ we have

$$\begin{aligned} & f(z_0)^2[g(xy) + \mu(y)g(\sigma(y)x)] = f(z_0)\mu(y)f(\sigma(y)xz_0) + f(z_0)f(xyz_0) \\ &= \mu(z_0)\mu(y)f(\sigma(y)xz_0\sigma(z_0)z_0) - f(xyz_0z_0^2) \\ &= \mu(yz_0)f(\sigma(yz_0)xz_0z_0) - f((xz_0)(yz_0)z_0) = 2f(xz_0)f(yz_0). \end{aligned}$$

Dividing by $f(z_0)^2$ we get (3.6). Furthermore, $g(e) = 1$ so $g \neq 0$.

(c) Now, from [2, Lemma 3.2] we get (c) and this completes the proof.

□

The second main result of this paper is:

THEOREM 3.2. *The non-zero solutions $f : M \rightarrow \mathbb{C}$ of the functional equation (1.7) are the functions of the form*

$$(3.7) \quad f = \mu(z_0)\chi(\sigma(z_0)) \frac{\chi - \mu\chi \circ \sigma}{2} = \chi(z_0) \frac{\mu\chi \circ \sigma - \chi}{2},$$

where $\chi : M \rightarrow \mathbb{C}$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\sigma(z_0)) = -\chi(z_0)$. Furthermore, $f(z_0) = \mu(z_0)\chi(z_0\sigma(z_0))$.

If M is a topological monoid and that $\sigma : M \rightarrow M$, $\mu : M \rightarrow \mathbb{C}$ are continuous, then the non-zero solution f of equation (1.7) is continuous, if and only if χ is continuous.

PROOF. Simple computations show that f defined by (3.7) is a solution of (1.7). To see the converse, let $f : M \rightarrow \mathbb{C}$ be a non-zero solution of the functional equation (1.7). Putting $y = z_0$ in (1.7) we get

$$(3.8) \quad f(x) = \frac{\mu(z_0)f(\sigma(z_0)xz_0) - f(xz_0z_0)}{2f(z_0)} = \frac{1}{2}(\mu(z_0)g(\sigma(z_0)x) - g(xz_0))$$

for all $x \in M$ and where g is the function defined in Lemma 3.1. So, from Lemma 3.1 we have $g = \frac{\chi + \mu\chi \circ \sigma}{2}$, where $\chi : M \rightarrow \mathbb{C}$, $\chi \neq 0$, is a multiplicative function. Substituting this into (3.8) we find that f has the form

$$(3.9) \quad f = \frac{\chi(z_0) - \mu(z_0)\chi(\sigma(z_0))}{2} \frac{\mu\chi \circ \sigma - \chi}{2}.$$

We have from (3.5) that f satisfies $\mu(x)f(\sigma(x)z_0) = f(xz_0)$ for all $x \in M$. Applying the last expression of f in (3.5) gives after elementary computations that

$$[\mu(z_0)\chi(\sigma(z_0)) + \chi(z_0)][\chi - \mu\chi \circ \sigma] = 0.$$

Since $\chi \neq \mu\chi \circ \sigma$, we deduce that $\mu(z_0)\chi(\sigma(z_0)) + \chi(z_0) = 0$. So, (3.9) can be written as follows

$$f = \mu(z_0)\chi(\sigma(z_0)) \frac{\chi - \mu\chi \circ \sigma}{2} = \chi(z_0) \frac{\mu\chi \circ \sigma - \chi}{2}.$$

For the topological statement we use [9, Theorem 3.18(d)]. \square

If $\mu = 1$ and $z_0 = e$ the identity element of M , we get the following elementary result obtained by Ebanks and Stetkær [3].

COROLLARY 3.3 [3]. *The only solution $f : M \rightarrow \mathbb{C}$ of the functional equation*

$$(3.10) \quad f(\sigma(y)x) - f(xy) = 2f(x)f(y), \quad x, y \in M$$

is $f = 0$.

REMARK 3.4. On groups the solutions of the functional equation

$$\mu(y)f(\sigma(y)xz_0) + g(xyz_0) = h(x)h(y), \quad x, y \in G,$$

where z_0 is an arbitrary element of G (not necessarily in the center of G) can be found by putting $f_1(x) = f(xz_0)$; $f_2(x) = g(xz_0)$, so f_1, f_2 satisfy the functional equation

$$\mu(y)f_1(\sigma(y)x) + f_2(xy) = h(x)h(y), \quad x, y \in G$$

which was solved by the authors in [1].

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