Acta Math. Hungar., **150** (1) (2016), 228–233
DOI: 10.1007/s10474-016-0626-x
First published online June 20, 2016

# ON THE PRINCIPAL IDEAL THEOREM AND SPECTRAL SYNTHESIS ON DISCRETE ABELIAN GROUPS*

L. SZÉKELYHIDI

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Hungary
e-mail: lszekelyhidi@gmail.com

*(Received March 1, 2016; accepted April 18, 2016)*

**Abstract.** We prove a generalization of the discrete version of the Principal Ideal Theorem.

The following result of B. Malgrange [1] is referred to as the Principal Ideal Theorem:

Theorem 1 (Principal Ideal Theorem). *For any nonzero linear partial differential operator $P(D)$ in $\mathbb{R}^n$ the linear hull of the exponential monomial solutions of the partial differential equation $P(D)f = 0$ is dense in the set of all solutions.*

This theorem was generalized by L. Ehrenpreis [2] to the following:

Theorem 2. *If the annihilator of a variety in $\mathcal{E}(\mathbb{C}^n)$ is a principal ideal, then the variety is the closed linear hull of the exponential monomials which are contained in it.*

Here $\mathcal{E}(\mathbb{C}^n)$ denotes the space of holomorphic functions on $\mathbb{C}^n$ equipped with the uniform convergence on compact sets. Later R. J. Elliott in [4] proved the following extension:

Theorem 3. *If $G$ is a locally compact Abelian group, then in $\mathcal{C}(G)$ any variety whose annihilator ideal is a principal ideal is the closed linear hull of the exponential monomials which are contained in it.*

Here, and in the forthcoming paragraphs, $\mathcal{C}(G)$ will denote the space of all continuous complex valued functions on $G$ equipped with the topology

* The research was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. K111651.

*Key words and phrases:* spectral synthesis, principal ideal, convolution equation.

*Mathematics Subject Classification:* 43A45, 39A06.

0236-5294/$20.00 © 2016 Akadémiai Kiadó, Budapest, Hungary

of uniform convergence on compact sets. In particular, in case of discrete Abelian groups this topology coincides with the topology of pointwise convergence. We recall that if $G$ is an Abelian group then by an *exponential monomial* we mean a function of the form

$$x \mapsto P\big(a_1(x), a_2(x), \ldots, a_n(x)\big)\, m(x)$$

where $P$ is an ordinary complex polynomial in $n$ variables, $m : G \to \mathbb{C}$ is a complex *exponential*, that is, a homomorphism of $G$ into the multiplicative group of nonzero complex numbers, and the functions $a_i : G \to \mathbb{C}$ are complex *additive functions,* that is, homomorphisms of $G$ into the additive group of complex numbers. If $G$ is a topological group then all these functions are supposed to be continuous. Linear combinations of exponential monomials are called *exponential polynomials.* A *variety* on $G$ is a translation invariant linear space of complex valued functions which is closed with respect to pointwise convergence.

In the long and complicated proof of Theorem 3 the author uses the structure theory of locally compact Abelian groups. We note that the same author published another result in the same volume (see [5]) which states that if $G$ is any discrete Abelian group then, in fact, every variety on $G$ possesses the given property, which is called *spectral synthesis.* Unfortunately, it turned out (see [9]) that this latter statement is false. Nevertheless, the relation of the two results has never been clarified. In this paper we give an independent proof for Theorem 3 on discrete Abelian groups, more precisely, we prove a generalization of that statement for varieties whose annihilator is finitely generated. In fact, we consider the following functional equation system on the discrete Abelian group $G$:

$$f * \mu_i = 0, \quad i = 1, 2, \ldots, n. \tag{1}$$

Here $\mu_i$ is a finitely supported complex valued function on $G$ and $f : G \to \mathbb{C}$ is a function. We recall that if $G$ is discrete, then the space of all finitely supported complex functions on $G$ can be identified with the dual of the topological vector space $\mathcal{C}(G)$. The elements of this dual can be considered as finitely supported measures on $G$, hence the dual of $\mathcal{C}(G)$ is identified with $\mathcal{M}_c(G)$, the convolution algebra of all finitely supported complex measures on $G$. The duality between $\mathcal{C}(G)$ and $\mathcal{M}_c(G)$ is given by

$$\langle f, \mu \rangle = \int f \, d\mu = \sum_{y \in G} f(y)\mu(y)$$

for each $f$ in $\mathcal{C}(G)$ and $\mu$ in $\mathcal{M}_c(G)$. The annihilator of a variety $V$ in $\mathcal{C}(G)$, resp. of an ideal $I$ in $\mathcal{M}_c(G)$ is defined as usual by

$$V^{\perp} = \big\{ \mu : \langle f, \mu \rangle = 0 \text{ for each } f \in V \big\}$$

and

$$I^{\perp} = \left\{ f : \langle f, \mu \rangle = 0 \text{ for each } \mu \in I \right\}.$$

It is obvious that $V^{\perp}$ is an ideal and $I^{\perp}$. It is less obvious that we have $V^{\perp\perp} = V$ and $I^{\perp\perp} = I$ for each variety $V$ and ideal $I$ (see [11], p. 4.).

We shall use the notation $\check{f}(x) = f(-x)$ and $\langle f, \check{\mu} \rangle = \langle \check{f}, \mu \rangle$ for each $f$ in $\mathcal{C}(G)$ and $\mu$ in $\mathcal{M}_c(G)$.

Now we prove the following result which generalizes Theorem 3 in the discrete case.

THEOREM 4. *The exponential monomial solutions of system* (1) *span a dense subspace in the solution space of* (1).

PROOF. Let $F$ denote the (finitely generated) subgroup generated by the supports of the measures $\mu_i$, $i = 1, 2, \dots, n$. We suppose that a linearly independent set of additive functions on $F$ is given and all exponential monomials on $F$ are built up from this set. Given a finitely generated subgroup $H \supseteqq F$ in $G$ let $E_H$ denote all finite linear combinations of those exponential monomials on $H$ which are solutions of the system (1) on $H$. In other words, $E_H$ is the set of all *exponential polynomial* solutions of (1) on the group $H$. By the theorem of Lefranc about spectral synthesis on finitely generated Abelian groups (see [3], Théoréme, p. 1953 and [10], Theorem 22.3, p. 21.), the set $E_H$ is dense in the set of all solutions of (1) on $H$. It is obvious (see e.g. [8]) that every exponential monomial on $H$ can be extended to an exponential monomial on $G$: the set of all such possible extensions of the exponential polynomials in the sets $E_H$ will be denoted by $E$, where $H$ runs through all finitely generated subgroups of $G$ containing $F$. We show that every function in $E$ is a solution of (1) on $G$. In fact, it is enough to prove the following statement: let $\varphi : G \to \mathbb{C}$ be an exponential monomial. Then $\varphi$ is a solution of (1) on $G$ if and only if the restriction of $\varphi$ to $F$ is a solution of (1) on the group $F$. The necessity of this condition is obvious. To prove the sufficiency suppose that $\varphi$ has the following form:

$$\varphi(x) = P\big(a_1(x), a_2(x), \dots, a_k(x)\big)\, m(x)$$

for each $x$ in $G$, where $m : G \to \mathbb{C}$ is an exponential, the functions $a_i : G \to \mathbb{C}$ are additive for $i = 1, 2, \dots, k$, their restrictions to $F$ are linearly independent, and $P : \mathbb{C}^k \to \mathbb{C}$ is an ordinary complex polynomial in $k$ variables. Suppose that $\varphi$ is a solution of (1) on $F$, that is, we have for each $x$ in $F$ and for $j = 1, 2, \dots, n$:

$$0 = \varphi * \mu_j(x) = \sum_{y \in G} \varphi(x - y)\mu_j(y) \tag{2}$$

$$= m(x) \sum_{y \in F} P\big(a_1(x-y), a_2(x-y), \dots, a_k(x-y)\big)\, m(-y)\mu_j(y).$$

For the sake of simplicity we shall use standard multi-index notation and $a : G \to \mathbb{C}^k$ will denote the vector valued function $a = (a_1, a_2, \dots, a_k)$, further

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k, \quad \partial = (\partial_1, \partial_2, \dots, \partial_k), \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_k^{\alpha_k},$$

$$a(x)^\alpha = a_1(x)^{\alpha_1} a_2(x)^{\alpha_2} \cdots a_k(x)^{\alpha_k}.$$

Using Taylor's Formula we have

$$P\big(a(x-y)\big) = \sum_{|\alpha| \leqq \deg P} \frac{1}{\alpha!} \partial^\alpha P\big(a(-y)\big)\, a(x)^\alpha.$$

We insert this into (2) to obtain

$$\sum_{|\alpha| \leqq \deg P} \frac{1}{\alpha!} a(x)^\alpha \sum_{y \in F} \partial^\alpha P\big(a(-y)\big)\, m(-y)\mu_j(y) = 0$$

for each $x$ in $G$. As the additive functions $a_1, a_2, \dots, a_k$ are linearly independent on $F$, hence so are the functions $a^\alpha$ for different multi-indices $\alpha$ (see e.g. [7], Lemma 2.7, p. 29.). Consequently, the above equation implies that

$$\sum_{y \in F} \partial^\alpha P\big(a(-y)\big)\, m(-y)\mu_j(y) = 0 \tag{3}$$

holds for $j = 1, 2, \dots, n$. Multiplying these equations by $\frac{1}{\alpha!} m(x) a(x)^\alpha$ for any $x$ in $G$ and summing up for $|\alpha| \leqq \deg P$ we obtain (2) for every $x$ in $G$, which means that the exponential monomial $\varphi$ is a solution of the system (1) on $G$.

We have proved that every function in $E$ is an exponential polynomial solution of the system (2). Now let $f$ be any solution of (2), further let $x_1, x_2, \dots, x_k$ be arbitrary elements in $G$ and $\varepsilon > 0$ a given number. If $H$ denotes the subgroup generated by $x_1, x_2, \dots, x_k$ and $F$, then $H$ is finitely generated and the restriction of $f$ is a solution of the system (2) on $H$. By assumption, there exists an exponential polynomial $\varphi_0 : H \to \mathbb{C}$ in $E_H$ such that $\big|f(x_j) - \varphi_0(x_j)\big| < \varepsilon$ for $j = 1, 2, \dots, k$. On the other hand, any extension $\varphi$ of $\varphi_0$ to an exponential monomial on $G$ belongs to $E$, and it satisfies the same inequalities, as $\varphi = \varphi_0$ on $H$. This proves that the exponential polynomial solutions of the system (2) form a dense subset in the set of all solutions of (2), and our theorem is proved. □

We can formulate this theorem in the following equivalent way.

Theorem 5. *Let $G$ be a discrete Abelian group. Then spectral synthesis holds for every variety whose annihilator ideal is finitely generated.*

Proof. Suppose that the annihilator $I = V^{\perp}$ of the variety $V$ is generated by the measures $\mu_1, \mu_2, \dots, \mu_n$. We know form the previous theorem that in the solution space of the system of functional equations

$$\langle f, \check{\mu}_i \rangle = 0, \quad i = 1, 2, \dots, n$$

the exponential monomial solutions span a dense subspace. On the other hand, it is clear that the solution space of this system coincides with the variety $V$. $\square$

This proof shows that Theorem 4 implies Theorem 5. Conversely, if Theorem 5 holds true and we are given the system of functional equations (1), then let $I$ be the ideal generated by the measures $\check{\mu}_i$, $i = 1, 2, \dots, n$. Let $V$ denote the solution space of the system (1), then $V$ is clearly a variety. As $I$ is finitely generated, and it is equal to the annihilator of $V$, by the statement of our present theorem the exponential monomial solutions of the system (1) span a dense subspace in $V$. Hence Theorem 4 and Theorem 5 are equivalent.

We note that, in fact, we proved the following statement: spectral synthesis holds for every variety on $G$ whose annihilator has a generating set of functions with supports in a finitely generated subgroup. We note that in his paper [6] D. I. Gurevich gave an example for a similar functional equation system with $n = 2$ on $G = \mathbb{R}^2$ with compactly supported measures $\mu_1, \mu_2$ such that the exponential monomial solutions do not span a dense subspace in the space of continuous solutions of the system in the sense of the topology of uniform convergence on compact sets. This shows that an analogue of Theorem 4, resp. Theorem 5 does not hold in the non-discrete case.

## References

[1] B. Malgrange, Equations aux dérivées partielles à coefficients constants. II. Equations avec second membre, *C.R. Acad. Sci. Paris,* **238** (1954), 196–198.

[2] L. Ehrenpreis, Mean periodic functions. I. Varieties whose annihilator ideals are principal, *Amer. J. Math.,* **77** (1955), 293–328.

[3] M. Lefranc, Analyse spectrale sur $Z_n$, *C.R. Acad. Sci. Paris,* **246** (1958), 1951–1953.

[4] R. J. Elliott, Some results in spectral synthesis, *Proc. Cambridge Phil. Soc.,* **61** (1965), 395–424.

[5] R. J. Elliott, Two notes on spectral synthesis for discrete Abelian groups, *Proc. Cambridge Phil. Soc.,* **61** (1965), 617–620.

[6] D. I. Gurevič, Counterexamples to a problem of L. Schwartz, *Funkcional. Anal. i Priložen.,* **9** (1975), 29–35.

[7] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups,* World Scientific Publishing Co. Inc. (Teaneck, NJ, 1991).

[8] L. Székelyhidi, On the extension of exponential polynomials, *Math. Bohemica,* **125** (2000), 365–370.
[9] L. Székelyhidi, The failure of spectral synthesis on some types of discrete abelian groups, *J. Math. Anal. Appl.,* **291** (2004), 757–763.
[10] L. Székelyhidi, *Discrete Spectral Synthesis and its Applications,* Springer (Dordrecht, 2006).
[11] M. Laczkovich and L. Székelyhidi, Spectral synthesis on discrete abelian groups, *Math. Proc. Cambridge Phil. Soc.,* **143** (2007), 103–120.